Week 5 Notes, Math 865, Tanveer

1. Rigorous Navier-Stokes Analysis: Mathematical Preliminaries

1.1. Basic Definitions.

Definition 1.1. *For $v \in \mathbb{R}^N$,*

$$|v| = \left(\sum_{j=1}^{N} v_j^2 \right)^{1/2} \tag{1.1}$$

For a function $f(x)$, with $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$,

$$Df = \left(\partial_{x_1} f, .., \partial_{x_N} f \right), \tag{1.2}$$

with each component $\partial_{x_j} f \in \mathbb{R}^N$.

$$|Df| = \left(\sum_{i,j=1}^{N} [\partial_{x_j} f_i]^2 \right)^{1/2} \tag{1.3}$$

Analogously, higher order tensors $D^2 f$, $D^3 f$ and their absolute values are defined. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..\alpha_N)$, each being non-negative integers, we define

$$D^\alpha f = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} .. \partial_{x_N}^{\alpha_N} f \tag{1.4}$$

We define the norm of the multi-index α:

$$|\alpha| = \alpha_1 + \alpha_2 + .. + \alpha_N \tag{1.5}$$

We also consider norms

$$\|f\|_0 \equiv \|f\|_{\mathbf{L}^2(\Omega)} = \left(\int_\Omega |f(x)|^2 dx \right)^{1/2} \tag{1.6}$$

The corresponding $\mathbf{L}^2$ inner product will be denoted by

$$(f, g)_0 = \int_{\Omega^N} f(x) g(x) dx \tag{1.7}$$

We define higher order energy norms $\|.\|_m$[(1)]:

$$\|f\|_m \equiv \|f\|_{\mathbf{H}^m(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha f\|_0^2 \right)^{1/2} \tag{1.8}$$

The corresponding inner-product in $\mathbf{H}^m$ will be denoted by $(., .)_m$

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(1) Note that through a Fourier-representation, $\|.\|_m$ can be generalized to nonintegral or negative m. We will use such generalizations later.

1.2. **Hodge Projection.**

Theorem 1.1. *Assume $u \in \mathbf{L}^2(\Omega)$ for a simply connected domain with smooth boundaries. Then, there exists a unique decomposition*

$$u = w + v, \tag{1.9}$$

where $\nabla \cdot w = 0$, $\nabla \times v = 0$[2] in Ω and $w \cdot n = 0$ on $\partial\Omega$. Further,

$$\|u\|_0^2 = \|w\|_0^2 + \|v\|_0^2$$

Also for any scalar function $\Psi \in \mathbf{H}^1(\Omega)$, $(w, \nabla\Psi)_0 = 0$.

Proof. We assume first $u \in \mathbf{C}_c^\infty(\Omega_1)$, where Ω_1 is some domain[3] with $\overline{\Omega} \subset \Omega_1$. First consider solution ϕ to the following elliptic problem

$$\Delta\phi = \nabla \cdot u \ , \text{ for } x \in \Omega; \ \frac{\partial\phi}{\partial n} = \mathbf{u} \cdot \mathbf{n} \tag{1.10}$$

Elliptic theory gives a smooth solution ϕ that is is unique upto an additive constant. Then, define

$$w = u - \nabla\phi$$

Then, $\nabla \cdot w = \nabla \cdot u - \Delta\phi = 0$ in Ω and on $\partial\Omega$, $w \cdot n = u \cdot n - \frac{\partial\phi}{\partial n} = 0$. So, if we define $v = \nabla\phi$, then $u = w + v$, where $\nabla \cdot w = 0$ and $\nabla \times v = 0$. This decomposition is unique since the only solution to $\Delta\psi = 0$ with $\frac{\partial\psi}{\partial n} = 0$ is $\psi = \text{const.}$. Further, we note that

$$(w, v) = (w, \nabla\phi)_0 = \int_\Omega w_j(x)\phi_{,j}(x)dx = \int_{\partial\Omega} \nabla \cdot (w\phi)dx = 0$$

since $w \cdot n = 0$ on $\partial\Omega$. Therefore, (1.9) is indeed an orthogonal decomposition of u and Pythagorus theorem gives the result $\|u\|_0^2 = \|w\|_0^2 + \|v\|_0^2$ as desired. Note an arbitrary function $u \in \mathbf{L}^2(\Omega)$ can be written as an $\mathbf{L}^2$ limit of a Cauchy sequence $u_m \in \mathbf{C}_c^\infty(\Omega_1)$. However, the Pythagorus theorem above implies that the corresponding w_m and v_m, have corresponding limits $w, v \in L^2(\Omega)$. Further, since $\nabla \cdot w_m = 0$, $\nabla \times v_m = 0$ the limit w satisfies $\nabla \cdot w = 0$, $\nabla \times v = 0$ in the weak sense.

Further, we have for any scalar function $\Psi \in \mathbf{H}^1(\Omega)$,

$$(w, \nabla\Psi)_0 = \int_\Omega w_j \cdot \Psi_{,j} dx = \int_{\partial\Omega} n_j w_j \Psi dx = \int_{\partial\Omega} (n \cdot w)\Psi dx = 0$$

∎

Definition 1.2. *We define Hodge Projection $\mathcal{P}$ so that $w = \mathcal{P}[u]$ in Theorem 1.1. This is a projection to the set of divergence free vector fields. From Theorem above, it follows that $\|\mathcal{P}[u]\|_0 \le \|u\|_0$.*

Lemma 1.3. *The operator $\mathcal{P}$ is self-adjoint in $\mathbf{L}^2(\Omega)$: for arbitrary vector fields $u, v \in \mathbb{L}^2(\Omega)$, $(u, \mathcal{P}v)_0 = (\mathcal{P}u, v)_0$.*

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(2) The derivative is understood in the weak sense, i.e. for any $\phi \in \mathbf{C}_c(\Omega)$, $\int_\Omega \phi \partial_{x_j} w dx = -\int_\Omega w \partial_{x_j} \phi dx$

(3) Domain will always be understood to be an open connected set

Proof. We express $u = \mathcal{P}u + (I - \mathcal{P})u$. Since $(I - \mathcal{P})u = \nabla \Psi$, it follows from previous theorem that

$$(u, \mathcal{P}v)_0 = (\mathcal{P}u, \mathcal{P}v)_0$$

Again from previous theorem, $(\mathcal{P}u, (I - \mathcal{P})v) = 0$. So,

$$(u, \mathcal{P}v)_0 = (\mathcal{P}u, v)_0$$

$\blacksquare$

If $\Omega = \mathbb{R}^N$ for $N = 2$ or $N = 3$, it is readily checked from that

$$\mathcal{F}\left\{-\nabla[\Delta^{-1}][\nabla \cdot u]\right\}(k) = \frac{ik}{|k|^2}[ik \cdot \hat{u}] = -\frac{k(k \cdot \hat{u})}{|k|^2} \ , \text{ where } \hat{u}(k) = \mathcal{F}[u](k) \tag{1.11}$$

Applying this to the construction of w in Theorem 1.1, it follows that $\mathcal{P}[u]$ for $u \in \mathbf{L}^2(\mathbb{R}^N)$, for $N = 2, 3$, has the following representation in the Fourier Space:

$$\mathcal{F}\left\{\mathcal{P}[u]\right\}(k) = P_k \hat{u}(k) \ , \text{ where } P_k = \left(I - \frac{k(k \cdot)}{|k|^2}\right), \text{ and } \hat{u}(k) = \mathcal{F}[u](k) \tag{1.12}$$

where $\mathcal{F}$ denotes the Fourier-Transform operator and $k \in \mathbb{R}^N$ is the Fourier dual of $x \in \mathbb{R}^N$. The same representation is valid when $k \in \mathbb{Z}^3$ for $k \neq 0$, when we consider Navier-Stokes equation for $x \in \mathbb{T}[0, 2\pi]^N$, *i.e.* space periodic problem.

Formally applying the Hodge Projection operator to the NS equation

$$u_t - \nu \Delta u = -\nabla p - (u \cdot \nabla)u + f \ , \ \nabla \cdot u = 0 \ , \text{ with } u(x, 0) = u_0(x)$$

we obtain an alternate form of Navier-Stokes equation

$$u_t + \nu \mathcal{A} u = -\mathcal{P}\left[(u \cdot \nabla)u\right] + \mathcal{P}f, \text{ with } u(x, 0) = u_0(x) \tag{1.13}$$

where $\mathcal{A} = \mathcal{P}(-\Delta)$ is called the Stokes operator. Note also that in deriving (1.13), we used the fact that operators ∂_t and $\mathcal{P}$ commute in appropriate space of functions. Note that $-\Delta$ and $\mathcal{P}$ do not necessarily commute because of the boundary condition since $w \cdot n = 0$ on $\partial\Omega$ does not generally imply $(\Delta w) \cdot n = 0$ on $\partial\Omega$. However, if $\Omega = \mathbb{R}^N$ or $\mathbb{T}[0, 2\pi]^N$, then $\mathcal{A} = -\Delta$. Equation (1.13) is supplemented by appropriate boundary condition on $\partial\Omega$ (no-slip condition for solid boundaries); for $\Omega = \mathbb{R}^N$, we require appropriate decay at ∞, while if $\Omega = \mathbb{T}[0, 2\pi]^N$, no boundary condition is required since periodicity suffices.

The abstract representation (1.13) is widely used in the mathematical analysis of Navier-Stokes. We may, without loss of generality, replace the forcing term $\mathcal{P}f$ by f by assuming f to be divergence free since since $(I - \mathcal{P})f$ is a conservative force that can be absorbed in the term $-\nabla p$, by redefining p.

1.3. **N-S Equation in the Fourier-Space for $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{T}[0, 2\pi]^N$ and integral representation.** We formally apply Fourier Transform operator on the Navier Stokes equation in the form (1.13), noting that $\mathcal{A} = -\Delta$ in this case, while

$$\mathcal{F}[-\Delta u](k) = |k|^2 \mathcal{F}[u](k) \ , \ \mathcal{F}[\nabla u](k) = ik\mathcal{F}[u](k) \ , \ \mathcal{F}[gh] = \mathcal{F}[g] \hat{*} \mathcal{F}[h] \ ,$$

where for $k \in \mathbb{R}^N$, we have the convolution operation $\hat{*}$ defined as

$$\left[\hat{g} \hat{*} \hat{h}\right](k) = \int_{k' \in \mathbb{R}^N} \hat{g}(k - k') \hat{h}(k') dk'$$

while for $k \in \mathbb{Z}^3$, $\hat{*}$ is defined by

$$\left[\hat{g}\hat{*}\hat{h}\right](k) = \sum_{k' \in \mathbb{Z}^3} \hat{g}(k-k')\hat{h}(k')$$

The Navier-Stokes equation in the Fourier-Space becomes

$$\hat{u}_t + \nu |k|^2 \hat{u} = -ik_j P_k \left[\hat{u}_j \hat{*} \hat{u}\right] + \hat{f} \text{ with } \hat{u}(k,0) = \hat{u}_0(k) \tag{1.14}$$

where $\hat{u}(k,t) = \mathcal{F}[u(.,t)](k)$. Note in (1.14) we used divergence free condition $k \cdot \hat{u}(k) = 0$ to conclude that convolution term

$$ik'_j \hat{u}_j(k-k') = i(k'_j - k_j)u_j(k-k') + ik_j u_j(k-k') = ik_j \hat{u}_j(k-k')$$

Inverting the left hand side of (1.14), we obtain the integral equation:

$$\hat{u}(k,t) = \int_0^t e^{-\nu|k|^2(t-\tau)} H(k,\tau) d\tau + \hat{u}^{(0)}(k,t) =: \mathcal{N}[\hat{u}](k,t) \tag{1.15}$$

where

$$\hat{H}(k,t) = -ik_j \mathcal{P}_k \left[\hat{u}_j \hat{*} \hat{u}\right] \tag{1.16}$$

and

$$\hat{u}^{(0)}(k,t) = \hat{u}_0(k) e^{-\nu|k|^2 t} + \int_0^t e^{-\nu|k|^2(t-\tau)} \hat{f}(k,\tau) d\tau \tag{1.17}$$

Remark 1.4. *Note that in the Stokes limit the nonlinear term H is absent, and we have an exact representation of solution to the Initial Value problem for the Stokes Problem. In general, (1.15) is an integral equation to determine $\hat{u}(k,t)$, which can be abstractly written as $\hat{u} = \mathcal{N}[\hat{u}]$ We will use contraction mapping theorem to determine solution to this integral equation in the right space of function.*

1.3.1. *Solution to $\hat{u} = \mathcal{N}[\hat{u}]$.*

Definition 1.5. *Define norm $\|.\|$ in the space of N-dimensional vector functions of (k,t), for $k \in \mathbb{R}^N$ $t \in [0,T]$, equipped with norm*

$$\|\hat{u}\| = \int_{\mathbb{R}^N} \sup_{t\in[0,T]} |\hat{u}(k,.)| d\hat{k}$$

For $k \in \mathbb{Z}^N$, we define

$$\|\hat{u}\| = \sum_{k \in \mathbb{Z}^N} \sup_{t\in[0,T]} |\hat{u}(k,.)|$$

It is also convenient to define

$$\hat{w}(k) = \sup_{t\in[0,T]} |\hat{u}(k,t)|$$

$$\hat{w}^{(0)}(k) = \sup_{t\in[0,T]} |\hat{u}^{(0)}(k,t)|$$

$$\hat{w}^{(1,2)}(k) = \sup_{t\in[0,T]} |\hat{u}^{(1)}(k,t) - \hat{u}^{(2)}(k,t)|$$

Remark 1.6. *The space $\mathcal{S}$ of functions $\hat{u}(k,t)$ that are continous in $t \in [0,T]$ for each k and for which $\|u\| < \infty$ is a Banach Space. It is not difficult to argue that $u(x,t) \equiv \mathcal{F}^{-1}[\hat{u}(.,t)](x)$ defines a continuous function in x and t.*

Lemma 1.7. *If the forcing term $\hat{f}(k,t) = \hat{f}(k)$, then the inhomogeneous term $\hat{u}^{(0)}(k,t)$ in the integral equation satisfies the following bound for $k \in \mathbb{R}^N$:*

$$\|\hat{u}^{(0)}\| \leq \|\hat{u}_0\|_{\mathbf{L}^1(\mathbb{R}^N)} + c_2 T \|\hat{f}(k)\|\|_{\mathbf{L}^1(\mathbb{R}^N)} \ , \text{ where } c_2 = \sup_{\gamma>0} \left(\frac{1-e^{-\gamma}}{\gamma} \right) \tag{1.18}$$

The same result, except with $\mathbf{L}^1(\mathbb{R}^N)$ replaced by $\mathbf{l}^1(\mathbb{Z}^N)$, holds when $k \in \mathbb{Z}^N$.

Proof. We note that in this case, calculation based on (1.17) shows that

$$\hat{u}^{(0)}(k,t) = \hat{u}_0(k) e^{-\nu|k|^2 t} + \hat{f}(k) \left(\frac{1-e^{-\nu|k|^2 t}}{\nu |k|^2} \right) \tag{1.19}$$

The Lemma follows immediately result on using the definition of $\|.\|$. ∎

Lemma 1.8. *For $\hat{H}$ defined in (1.16),*

$$\sup_{t\in[0,T]} |\hat{H}(k,t)| \leq 2|k| \hat{w} \hat{*} \hat{w} \tag{1.20}$$

Further, if $\hat{H}^{(1)}$ and $\hat{H}^{(2)}$ corresponds to $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$, then

$$\sup_{t\in[0,T]} |\hat{H}^{(1)}(k,t) - \hat{H}^{(2)}(k,t)| \leq 2|k| \left(\hat{w}^{(1)} + \hat{w}^{(2)} \right) \hat{*} \hat{w}^{(1,2)} \tag{1.21}$$

Proof. Since $|P_k \hat{v}| \leq 2|\hat{v}|$, it follows that follows from noting that $t \in [0,T]$,

$$|ik_j P_k \left[\hat{u}_j \hat{*} \hat{u} \right]| \leq 2|k| \hat{w} \hat{*} \hat{w}$$

Further, note that

$$|ik_j P_k \left[\hat{u}_j^{(1)} \hat{*} \hat{u}^{(1)} - \hat{u}_j^{(2)} \hat{*} \hat{u}^{(2)} \right]| \leq 2|k| \left(|\hat{u}^{(1)}| + |\hat{u}^{(2)}| \right) \hat{*} |\hat{u}^{(1)} - \hat{u}^{(2)}| \leq 2|k| \left(|\hat{w}^{(1)}| + |\hat{w}^{(2)}| \right) \hat{*} |\hat{w}^{(1,2)}|$$

Taking supremum of the left side over $t \in [0,T]$, the second part of the Lemma follows. ∎

Lemma 1.9. *In the space of functions $\hat{u}(k,t)$ for which $\|\hat{u}\|$, $\|\hat{u}^{(1)}\|$ and $\|\hat{u}^{(2)}\|$ are finite,*

$$\|\mathcal{N}[\hat{u}]\| \leq \|\hat{u}^{(0)}\| + 2c_1 \sqrt{\frac{T}{\nu}} \|\hat{u}\|^2 \tag{1.22}$$

where $c_1 = \sup_{\gamma\geq 0} \frac{1-e^{-\gamma}}{\sqrt{\gamma}}$. Further,

$$\|\mathcal{N}\left[\hat{u}^{(1)}\right] - \mathcal{N}\left[\hat{u}^{(2)}\right]\| \leq 2c_1 \sqrt{\frac{T}{\nu}} \left(\|\hat{u}_1\| + \|\hat{u}^{(2)}\| \right) \|\hat{u}_1 - \hat{u}_2\| \tag{1.23}$$

Proof. We note that for $t \in [0,T]$, we note by using (1.20),

$$|\mathcal{N}[\hat{u}](k,t)| \leq \sup_{t\in[0,T]} |\hat{H}(k,t)| \int_0^t e^{-\nu|k|^2(t-\tau)} d\tau + \hat{w}^{(0)}(k) \leq 2\hat{w} \hat{*} \hat{w} \left(\frac{1-e^{-\nu|k|^2 t}}{\nu|k|} \right) + \hat{w}^{(0)}(k)$$

Therefore, it follows that

$$\sup_{t\in[0,T]} |\mathcal{N}[\hat{u}](k,t)| \leq 2c_1 \sqrt{\frac{T}{\nu}} \hat{w} \hat{*} \hat{w} + \hat{w}^{(0)}$$

Therefore, using

$$\|\hat{g} \hat{*} \hat{h}\|_{\mathbf{L}^1(\mathbb{R}^N)} \leq \|\hat{g}\|_{\mathbf{L}^1(\mathbb{R}^N)} \|\hat{h}\|_{\mathbf{L}^1(\mathbb{R}^N}$$

or in the discrete case,

$$\|\hat{g}\hat{*}\hat{h}\|_{\mathbf{l}^1(\mathbb{Z}^N)} \leq \|\hat{g}\|_{\mathbf{l}^1(\mathbb{Z}^N)}\|\hat{h}\|_{\mathbf{l}^1(\mathbb{Z}^N}$$

we obtain from definition of $\|.\|$ that

$$\|\mathcal{N}[\hat{u}]\| \leq 2c_1\sqrt{\frac{T}{\nu}}\|\hat{u}\|^2 + \|\hat{u}^{(0)}\|$$

and (1.22) follows. Again, since

$$|\mathcal{N}[\hat{u}^{(1)}](k,t) - \mathcal{N}[\hat{u}^{(2)}](k,t)| \leq \sup_{t\in[0,T]} |\hat{H}^{(1)}(k,t) - \hat{H}^{(2)}(k,t)| \int_0^t e^{-\nu|k|^2(t-\tau)} d\tau$$

Using (1.21) and going through the same steps as above, (1.23) follows. ∎

Proposition 1.10. *There exists unique solution to the equation* $\hat{u} = \mathcal{N}[\hat{u}]$ *(in (1.15) in the ball* $\mathcal{B}$*:* $\|\hat{u}| \leq 2\|\hat{u}^{(0)}\|$ *for* T *small enough to satisfy*

$$8c_1\sqrt{\frac{T}{\nu}}\|\hat{u}^{(0)}\| = \beta < 1 \tag{1.24}$$

Proof. It is clear that condition (1.24) implies from (1.22) that $\mathcal{N}$ maps the ball $\mathcal{B}$ back to itself. Further, from (1.23), condition (1.24) implies that $\mathcal{N}$ is a contraction map in the ball $\mathcal{B}$, *i.e.*

$$\|\hat{N}[u^{(1)}] - \mathcal{N}[u^{(2)}]\| \leq \beta\|u^{(1)} - u^{(2)}\|$$

Hence from contraction mapping theorem applied to the Banach space of N-dimensional vector functions of (k,t) with finite $\|.\|$, the proposition follows. ∎

Remark 1.11. *Since the solution to the integral equation (1.15) is in the space where* $\|.\| < \infty$*, it implies that* $\hat{u}(k,t)$ *is continuous in* t *and* $\|\hat{u}(.,t)\|_{\mathbf{L}^1(\mathbb{R}^N)}$ *(or* $\|\hat{u}(.,t)\|_{\mathbf{l}^1(\mathbb{Z}^N)}$ *for the discrete case) exists and is continuous in* t*. This implies that* $\hat{u}(k,t)$ *satisfies the Navier-Stokes equation (1.14) in the Fourier-Space.*

1.4. **Instantaneous smoothing.** The following result shows that the inverse Fourier-Transform of the solution $\hat{u}(k,t)$ to (1.15) corresponds to a classical solution of Navier-Stokes, with appropriate derivatives existing for $t \in (0,T]$, *i.e.* there is instantaneous smoothing due to viscous effects. For simplicity of presentation, we take $\hat{f}(k,t) = \hat{f}(k)$, though results for more general $\hat{f}(k,t)$ that are bounded in t follow in a similar manner. We also limit ourselves to $k \in \mathbb{R}^N$. The periodic box case for which $k \in \mathbb{Z}^N$ is treated in a similar manner.

Lemma 1.12. *Assume* $\hat{u}_0, \hat{f} \in \mathbf{L}^1(\mathbb{R}^N)$*. Assume the integral equation* $\hat{u} = \mathcal{N}[\hat{u}]$ *has a solution* $\hat{u}(k,t)$ *with* $\|\hat{u}(\cdot,t)\|_{\mathbf{L}^1(\mathbb{R}^N)} < \infty$ *for* $t \in [0,T]$*. Then* $u(x,t) = \mathcal{F}^{-1}[\hat{u}(\cdot,t)](x)$ *is a classical solution of the Navier-Stokes solution for* $t \in (0,T]$*.*

Proof. We first show show $|k|^2\hat{u}(\cdot,t) \in \mathbf{L}^1(\mathbb{R}^N)$ for $t \in (0,T]$.

Consider the time interval $[\epsilon, T]$ for $\epsilon \geq 0$. Define

$$\hat{w}_\epsilon(k) = \sup_{\epsilon\leq t\leq T} |\hat{u}|(k,t).$$

Since $|\hat{u}(k,t)|$ is finite for $t \in [0,T]$ $\hat{w}_0$ (or $\hat{w}_\epsilon$) satisfies

$$\|\hat{w}_0\|_{\mathbf{L}^1} < \infty$$

On $[\epsilon, T]$ for $\epsilon > 0$, the integral equation implies

$$\hat{u}(k,t) = -ik_j \int_0^t e^{-\nu|k|^2(t-\tau)} P_k \left(\hat{u}_j \hat{*} \hat{u}\right)(k,\tau) d\tau + \hat{u}^{(0)}(k,t) \tag{1.25}$$

Therefore,

$$|k||\hat{u}|(k,t) \leq 2\left\{\hat{w}_0 \hat{*} \hat{w}_0\right\} \int_0^t |k|^2 e^{-\nu|k|^2(t-\tau)} d\tau + |k||\hat{u}^{(0)}|(k,t)$$

Since $\int_0^t \nu|k|^2 e^{-\nu|k|^2(t-\tau)} d\tau \leq 1$, it follows from (1.19) that

$$|k|\hat{w}_{\epsilon/2} \leq \frac{2}{\nu}\left\{\hat{w}_0 \hat{*} \hat{w}_0\right\} + \sqrt{\frac{2}{\nu\epsilon}} \left(\sup_{\gamma>0} \gamma e^{-\gamma^2}\right) |\hat{u}_0| + c_1 |\hat{f}(k)| \sqrt{\frac{T}{\nu}} \tag{1.26}$$

Using now the bounds on $\hat{w}_0$ we get

$$\||k|\hat{w}_{\epsilon/2}\|_{\mathbf{L}^1} \leq \frac{2}{\nu}\|\hat{u}\|^2 + \frac{C}{\epsilon^{1/2}\nu^{1/2}}\|\hat{u}_0\|_{\mathbf{L}^1} + c_1\|\hat{f}\|_{\mathbf{L}^1}\sqrt{\frac{T}{\nu}}$$

The evolution of $\hat{u}$ is autonomous in time, and thus, for $t \in \left[\frac{\epsilon}{2}, T\right]$ we have

$$\begin{aligned} \hat{u}(k,t) = -i \int_{\epsilon/2}^t & e^{-\nu|k|^2(t-\tau)} P_k \left(\hat{u}_j \hat{*} [k_j \hat{u}]\right)(k,\tau) d\tau \\ & + \hat{u}(k,\epsilon/2) e^{-\nu|k|^2(t-\epsilon/2)} + \hat{f}(k) \frac{1 - e^{-\nu|k|^2(t-\epsilon/2)}}{\nu|k|^2}, \end{aligned} \tag{1.27}$$

where we used the divergence condition $k \cdot \hat{u}(k,t) = 0$. Multiplying (1.27) by $|k|^2$ and using (1.26), it follows that for $t \in [\epsilon, T]$ we have

$$\begin{aligned} |k|^2 |\hat{u}(k,t)| \leq 2\hat{w}_{\epsilon/2} \hat{*} \left[|k|\hat{w}_{\epsilon/2}\right] & \int_{\epsilon/2}^t |k|^2 e^{-\nu|k|^2(t-\tau)} d\tau \\ & + \frac{1}{\nu(t-\epsilon/2)} \left(\sup_{\gamma>0} \gamma e^{-\gamma}\right) |\hat{u}(k,\epsilon/2)| + \frac{|\hat{f}|}{\nu}, \end{aligned}$$

implying that

$$\||k|^2 \hat{w}_\epsilon\|_{\mathbf{L}^1} \leq \frac{2}{\nu}\|\hat{w}_{\epsilon/2}\|_{\mathbf{L}^1} \||k|\hat{w}_{\epsilon/2}\|_{\mathbf{L}^1} + \frac{C}{\epsilon\nu}\|\hat{w}_{\epsilon/2}\|_{\mathbf{L}^1} + c_1\|\hat{f}\|_{\mathbf{L}^1}\sqrt{\frac{T}{\nu}}$$

Since $\epsilon > 0$ is arbitrary, it follows that $|k|^2\hat{u}(\cdot,t) \in \mathbf{L}^1$ for $t \in (0,T]$, with $\||k|^2\hat{u}(\cdot,t)\|_{\mathbf{L}^1(\mathbb{Z}^N)}$ continuous in that interval. Applying inverse Fourier-Transform $\mathcal{F}^{-1}$ to (1.14), from Fourier theory, we obtain $u(x,t) = \mathcal{F}^{-1}[\hat{u}(.,t)](x)$ to be a a classical solution to the Navier-Stokes equation (1.15). ∎

1.5. **Kinetic Energy Dissipation:** Consider the incompressible constant density Navier-Stokes equation

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \tag{1.28}$$

$$\nabla \cdot u = 0 \tag{1.29}$$

Then the i-th component of (1.28) is given by

$$\partial_t u_i + u_j \partial_{x_j} u_i = -\partial_{x_i} p + \nu \partial_{x_j}^2 u_i + b_i \tag{1.30}$$

Multiplying (1.30) by u_i and summing over i and using (1.29), we have on using Einstein convention of summation over repeated indices:

$$\partial_t \frac{1}{2} u_i^2 + \partial_{x_j}\left(\frac{1}{2} u_j u_i^2\right) = -\partial_{x_i}(p u_i) + \nu \partial_{x_j}\left(u_i \partial_{x_j} u_i\right) - \nu \partial_{x_j} u_i \partial_{x_j} u_i + b_i u_i \tag{1.31}$$

Assuming that $|u|(|u|^2 + p) + |u||Du| = o(|x|^{-N+1})$, as $|x| \to \infty$, we obtain from (1.31) after integration over $\mathbb{R}^N$ the following identity:

$$\partial_t \frac{1}{2} \|u(.,t)\|_0^2 = -\nu \|Du\|_0^2 + (b, u)_0 \tag{1.32}$$

The physical kinetic energy in the fluid is

$$E(t) = \int_{\mathbb{R}^N} \frac{1}{2} |u(x,t)|^2 dx = \frac{1}{2} \|u(.t)\|_0^2, \tag{1.33}$$

In the absence of any force, *i.e.* $b = 0$, we obtain from (1.32)

$$\frac{d}{dt} E = -\nu \|Du\|_0^2 = -\nu \int_{x \in \mathbb{R}^N} |Du(x,t)|^2 dx = -\epsilon, \tag{1.34}$$

where ϵ is usually referred to as the rate of energy dissipation. For zero viscosity, *i.e.* Euler flow, kinetic energy E is conserved and $E(t) = E(0)$. In the presence of a force b, the term $(b, u)_0$ is the work done by the force b per-unit time. So, (1.32) is a physical statement that *rate of change of Kinetic Energy* is the *rate of work done by the external force* minus the *Energy dissipated /lost due to viscous friction*.

This dissipated energy is actually converted to heat. For *compressible* fluid flow, we have to couple heat energy and thermodynamics with momentum equation to get a complete set of equations. But this is not the case for *incompressible* Navier-Stokes equation, which we are studying here, and we do not have to consider thermodynamics to solve for fluid flow field u.