

Week 6 Notes, Math 865, Tanveer

1. ENERGY METHODS FOR EULER AND NAVIER-STOKES EQUATION

We will consider this week basic energy estimates. These are estimates on the $\mathbf{L}^2$ spatial norms of the solution $u(x, t)$ and its higher derivatives with respect to x . Like other PDE initial value problems, these estimates are most useful in establishing existence and uniqueness of solutions.

For simplicity, we will first take $\Omega = \mathbb{R}^N$, where $N = 2$ or 3 . We will drop the boldfaced notation for vectors since by this time you are sufficiently familiar to recognize which quantities are vectors and scalars. The exposition of this topic follows Bertozzi & Majda (Text), though with some differences in notation.

1.1. Kinetic Energy Dissipation: Consider the incompressible constant density Navier-Stokes equation

$$(1.1) \quad \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + b$$

$$(1.2) \quad \nabla \cdot u = 0$$

Then the i -th component of (1.1) is given by

$$(1.3) \quad \partial_t u_i + u_j \partial_{x_j} u_i = -\partial_{x_i} p + \nu \partial_{x_j}^2 u_i + b_i$$

Multiplying (1.3) by u_i and summing over i and using (1.2), we have on using Einstein convention of summation over repeated indices:

$$(1.4) \quad \partial_t \frac{1}{2} u_i^2 + \partial_{x_j} \left(\frac{1}{2} u_j u_i^2 \right) = -\partial_{x_i} (p u_i) + \nu \partial_{x_j} (u_i \partial_{x_j} u_i) - \nu \partial_{x_j} u_i \partial_{x_j} u_i + b_i u_i$$

Assuming that $|u|(|u|^2 + p) + |u||Du| = o(|x|^{-N+1})$, as $|x| \rightarrow \infty$, we obtain from (1.4) after integration over $\mathbb{R}^N$ the following identity:

$$(1.5) \quad \partial_t \frac{1}{2} \|u(\cdot, t)\|_0^2 = -\nu \|Du\|_0^2 + (b, u)_0$$

The physical kinetic energy in the fluid is

$$(1.6) \quad E(t) = \int_{\mathbb{R}^N} \frac{1}{2} |u(x, t)|^2 dx = \frac{1}{2} \|u(\cdot, t)\|_0^2,$$

In the absence of any force, *i.e.* $b = 0$, we obtain from (1.5)

$$(1.7) \quad \frac{d}{dt} E = -\nu \|Du\|_0^2 = -\nu \int_{x \in \mathbb{R}^N} |Du(x, t)|^2 dx = -\epsilon,$$

where ϵ is usually referred to as the rate of energy dissipation. For zero viscosity, *i.e.* Euler flow, kinetic energy E is conserved and $E(t) = E(0)$. In the presence of a force b , the term $(b, u)_0$ is the work done by the force b per-unit time. So, (1.5) is a physical statement that *rate of change of Kinetic Energy* is the *rate of work done by the external force* minus the *Energy dissipated /lost due to viscous friction*.

This dissipated energy is actually converted to heat. For *compressible* fluid flow, we have to couple heat energy and thermodynamics with momentum equation to get a complete set of equations. But this is not the case for *incompressible* Navier-Stokes equation, which we are studying here, and we do not have to consider thermodynamics to solve for fluid flow field u .

1.2. Energy Estimate, Uniqueness and ν dependence of Smooth Solutions. Let u, w be two Navier-Stokes solution, corresponding to forcing b and c respectively. We assume b and c to be smooth as well and decaying sufficiently fast in x at ∞ . We denote the corresponding pressures by p and q . Then consider the difference $v = u - w$. It is easy to check that v satisfies:

$$(1.8) \quad \partial_t v + v \cdot \nabla v + w \cdot \nabla v + v \cdot \nabla w = -\nabla P + \nu \Delta v + f, \text{ where } f = b - c, P = p - q$$

The i -th component of the above equation may be written as

$$(1.9) \quad \partial_t v_i + v_j \partial_{x_j} v_i + w_j \partial_{x_j} v_i + v_j \partial_{x_j} w_i = -\partial_{x_i} P + \nu \partial_{x_j}^2 v_i + f_i$$

Multiplying above by v_i and integrating we obtain that

$$(1.10) \quad \partial_t \frac{1}{2} v_i^2 + \frac{1}{2} \partial_{x_j} [(v_j + w_j) v_i^2] + v_i v_j \partial_{x_j} w_i = -\partial_{x_i} (u_i P) + \nu \partial_{x_j} (v_i \partial_{x_j} v_i) - \nu (\partial_{x_j} v_i) (\partial_{x_j} v_i) + v_i f_i$$

So integrating over $\mathbb{R}^N$ with usual assumptions on decay of velocity and pressure fields at ∞ , we obtain by using

$$(1.11) \quad |(v, f)_0| \leq \|v\|_0 \|f\|_0, \text{ and } |(v, v \cdot \nabla w)_0| \leq \|Dw(\cdot, t)\|_\infty \|v\|_0^2,$$

$$(1.12) \quad \frac{d}{dt} \frac{1}{2} \|v\|_0^2 + \nu \|Dv\|_0^2 \leq \|Dw\|_\infty \|v\|_0^2 + \|v\|_0 \|f\|_0$$

So, in particular,

$$(1.13) \quad \frac{d}{dt} \|v\|_0 \leq \|\nabla w\|_\infty \|v\|_0 + \|f\|_0$$

Also, on integrating (1.12) between $t = 0$ to $t = T$, we obtain

$$(1.14) \quad \frac{1}{2} \|v(\cdot, T)\|_0^2 + \nu \int_0^T \|Dv(\cdot, t)\|_0^2 dt \leq \frac{1}{2} \|v(\cdot, 0)\|_0^2 + \int_0^T \|Dw(\cdot, t)\|_\infty \|v(\cdot, t)\|_0^2 dt + \int_0^T \|v(\cdot, t)\|_0 \|f(\cdot, t)\|_0 dt$$

Using well-known Gronwall's inequality on (1.13) and the definition of v , we obtain the following Lemma

Lemma 1.1. *Let u and w be two smooth $\mathbf{L}^2(\mathbb{R}^N)$ solutions to the Navier-Stokes equation for $t \in [0, T]$ for the same viscosity ν , but different forcing b and c respectively. Then,*

$$(1.15) \quad \sup_{t \in [0, T]} \|u(\cdot, t) - w(\cdot, t)\|_0 \leq \left\{ \|u(\cdot, 0) - w(\cdot, 0)\|_0 + \int_0^T \|b(\cdot, t) - c(\cdot, t)\|_0 dt \right\} \exp \left[\int_0^T \|\nabla w(\cdot, t)\|_\infty dt \right]$$

Corollary 1.2. *Uniqueness of smooth solutions*

Let $u(\cdot, t)$ and $w(\cdot, t)$ be two smooth $\mathbf{L}^2(\mathbb{R}^N)$ solutions to incompressible constant density Navier-Stokes equation for $t \in [0, T]$ with same initial data and forcing. Then, the solution is unique.

Proof. This simply follows from Lemma ??, since $u(\cdot, 0) - w(\cdot, 0) = 0$ and $b - c = 0$. ■

Remark 1.3. *The energy estimate (1.15) does not explicitly depend on ν and is equally valid for $\nu = 0$, i.e. for the Euler equation.*

The energy estimate (1.15) is also useful in estimating the difference between smooth Euler and Navier-Stokes solution with the same initial data and forcing. Let $u^{[0]}$ be a smooth solution to the Euler equation, *i.e.* $\nu = 0$, while $u^{[\nu]}$ is a solution to Navier-Stokes equation with the same initial data and forcing. Then, we can obtain an equation for $v = u^{[\nu]} - u^{[0]}$:

$$(1.16) \quad \partial_t v + v \cdot \nabla v + u^{[0]} \cdot \nabla v + v \cdot \nabla u^{[0]} = -\nabla P + \nu \Delta v + f$$

where $f = \nu \Delta u^{[0]}$, $P = p - q$ is the difference of pressure

This is the same equation as for (1.8), with w replaced by $u^{[0]}$, and a different meaning of f . Therefore, the energy estimate (1.15) in this case becomes

(1.17)

$$\sup_{t \in [0, T]} \|u^{[\nu]}(., t) - u^{[0]}(., t)\|_0 \leq \nu \left(\int_0^T \|\Delta u^{[0]}(., t)\|_0 dt \right) \exp \left\{ \int_0^T \|Du^{[0]}(., t)\|_\infty dt \right\} \leq \nu TC(u^{[0]}, T)$$

Notice that (1.14) with w replaced by $u^{[0]}$, and with $f = -\nu \Delta u^{[0]}$ gives rise to

(1.18)

$$\nu \int_0^T \|Dv(., t)\|_0^2 dt \leq \int_0^T \|Du^{[0]}(., t)\|_\infty \|v(., t)\|_0^2 dt + \nu \int_0^T \|v(., t)\|_0 \|\Delta u^{[0]}(., t)\|_0 dt$$

Using estimate (1.17) estimate, we obtain,

(1.19)

$$\int_0^T \|Dv(., t)\|_0^2 dt \leq \nu TC \left\{ CT \int_0^T \|Du^{[0]}(., t)\|_\infty dt + \nu \int_0^T \|\Delta u^{[0]}(., t)\|_0 dt \right\} \leq \nu T^2 c_2(u^{[0]}, T)$$

So,

$$(1.20) \quad \int_0^T \|Dv(., t)\|_0 dt \leq T^{1/2} \left(\int_0^T \|Dv(., t)\|^2 dt \right)^{1/2} \leq \nu^{1/2} T^{3/2} C_2(u^{[0]}, T)$$

This implies the following proposition:

Proposition 1.4. *Comparison of smooth Euler and Navier-Stokes Solution*

Given the same initial data and forcing, then the difference v between smooth $\mathbf{L}^2(\mathbb{R}^N)$ Navier-Stokes and Euler solution over a common interval of existence $[0, T]$ satisfies (1.17) and (1.20). In particular for any fixed T , as $\nu \rightarrow 0$, $u^{[\nu]}(., t) \rightarrow u^{[0]}(., t)$, and $Du^{[\nu]}(., t) \rightarrow Du^{[0]}(., t)$ uniformly for $t \in [0, T]$.

1.3. Kinetic Energy of 2-D flow. The Theorems in the last section hold for solutions to Navier-Stokes/Euler equation that decay sufficiently rapidly as $\mathbf{x} \rightarrow \infty$ so that velocity $u(., t) \in \mathbf{L}^2$. This is a reasonable physical assumption in $\mathbb{R}^3$.

For 2-D flow, this is not necessarily the case, unless the integral of vorticity in the flow is zero, as will be seen shortly. Suppose

$$\text{supp } \omega \subset \{x : x \in \mathbb{R}^2, |x| < R\}$$

Applying 2-D Biot-Savart Law:

$$(1.21) \quad u(x, t) = \int_{|y| \leq R} K(x - y) \omega(y, t) dy, \text{ where } K(x) = \frac{1}{2\pi|x|^2} [-x_2, x_1]$$

We first note that

(1.22)

We note that

$$(1.23) \quad |x - y|^{-2} = |x|^{-2} \left(1 - 2 \frac{y \cdot x}{|x|^2} + \frac{|y|^2}{|x|^2} \right)$$

Now, if $|x| \geq 2R$, then since $|y| \leq R$, it follows that as $|x| \rightarrow \infty$

$$|x - y|^{-2} = |x|^{-2} + O(|x|^{-3})$$

So, from (1.21),

$$(1.24) \quad u(x, t) = K(x) \int_{y \in \mathbb{R}^2} \omega(y, t) + O(|x|^{-3})$$

Since

$$\int_{x \in \mathbb{R}^2} (1 + |x|)^{-l} dx < \infty, \text{ iff } l > N$$

It follows that

Lemma 1.5. *A 2-D incompressible flow with compact vorticity ω has finite energy iff*

$$(1.25) \quad \int_{\mathbb{R}^2} \omega(x) dx = 0$$

Remark 1.6. *Note that the vorticity $\omega(x, t)$ will satisfy (1.25) for $t > 0$, if*

$$(1.26) \quad \int_{\mathbb{R}^2} \omega(x, 0) dx = 0,$$

since integration of 2-D Navier-Stokes equation in the vorticity form gives

$$(1.27) \quad \frac{d}{dt} \int_{x \in \mathbb{R}^2} \omega(x, t) dx = 0$$

Remark 1.7. *The statement that finite energy is implied only iff (1.26) is satisfied is not limited merely to flow with compact support. It is more generally true for $\omega \in \mathbf{L}^1(\mathbb{R}^2)$.*

When (1.26) is violated, it is possible to decompose a solution to Navier-Stokes equation to such that a part of it is in $\mathbf{L}^2(\mathbb{R}^2)$ (hence finite energy), while the other part is generated by a radial distribution of vorticity whose integral is the same as the integral of initial vorticity over $\mathbb{R}^2$.

Consider an initial vorticity distribution $\omega_0(x) \in \mathbf{L}^1(\mathbb{R}^2)$. We chose any compact radial vorticity distribution $\tilde{\omega}_0(|x|)$ such that

$$\int_{\mathbb{R}^2} \tilde{\omega}_0(|x|) dx = \int_{\mathbb{R}^2} \omega_0(x) dx$$

We determine radial vorticity solution $\tilde{\omega}(|x|, t)$ with initial value $\tilde{\omega}(|x|, 0) = \tilde{\omega}_0(|x|)$ to Navier-Stokes equation without forcing. We know from worked out problems two weeks back, that $\tilde{\omega}(|x|, t)$ satisfies 2-D heat equation with corresponding velocity

$$(1.28) \quad \tilde{u}(x, t) = \frac{(-x_2, x_1)}{|x|^2} \int_0^{|x|} s \tilde{\omega}_0(s, t) ds$$

Therefore, we now consider the decomposition

$$(1.29) \quad u(x, t) = \tilde{u}(x, t) + v(x, t)$$

Since $(\nabla \times v)(x, 0) = \omega_0(x) - \tilde{\omega}_0(x)$, it follows that

$$(1.30) \quad \int (\nabla \times v)(x, 0) dx = 0$$

from construction of $\tilde{\omega}_0$. From (1.27) and the fact that heat solution preserves $\int_{\mathbb{R}^2} \tilde{\omega}(x, t) dx$, it follows that

$$(1.31) \quad \frac{d}{dt} \int_{\mathbb{R}^2} (\nabla \times v)(x, t) dx = 0, \text{ implying by above } \int_{x \in \mathbb{R}^2} (\nabla \times v)(x, t) dx = 0$$

This implies that $v(x, t)$ has finite energy.

Thus, we have proved the following Lemma:

Lemma 1.8. *Any smooth solution $u(x, t)$ to 2-D Navier-Stokes equation with an initial $\mathbf{L}^1(\mathbb{R}^2)$ vorticity can be decomposed into*

$$(1.32) \quad u(x, t) = v(x, t) + \tilde{u}(x, t)$$

where $v \in \mathbf{L}^2(\mathbb{R}^2)$ and divergence free, while

$$(1.33) \quad \tilde{u}(x) = (-x_2, x_1)|x|^{-2} \int_0^{|x|} s \tilde{\omega}(s, t) ds$$

for some smooth radial vorticity distribution $\tilde{\omega}(|x|, t)$ with an initial compact support.

1.4. Energy Inequality for 2-D flow. Consider the *radial-Energy* decomposition

$$(1.34) \quad u(x, t) = \tilde{u}(x, t) + v(x, t)$$

of solution to the Navier-Stokes equation where $v \in \mathbf{L}^2(\mathbb{R}^2)$. v satisfies

$$(1.35) \quad \partial_t v + v \cdot \nabla v + \tilde{u} \cdot \nabla v + v \cdot \nabla \tilde{u} = -\nabla p + \nu \Delta v + F$$

Consider two solutions to Navier-Stokes equation u_1, u_2 with radial decompositions:

$$(1.36) \quad u_1 = \tilde{u}_1 + v_1, \quad u_2 = \tilde{u}_2 + v_2$$

Then, if we denote

$$(1.37) \quad w = v_1 - v_2, \quad \tilde{u}_1 - \tilde{u}_2 = \hat{u}, \quad \hat{F} = F_1 - F_2, \quad \hat{p} = p_1 - p_2,$$

then w satisfies

$$(1.38) \quad \partial_t w + v_1 \cdot \nabla w + w \cdot \nabla v_2 + \tilde{u}_1 \cdot \nabla w + \hat{u} \cdot \nabla v_2 + v_2 \cdot \nabla \hat{u} + w \cdot \nabla \tilde{u}_1 = -\nabla \hat{p} + \nu \Delta w + \hat{F}$$

Then using the same integration by parts procedure as in the last section, we have

$$(1.39) \quad \frac{d}{dt} \frac{1}{2} \|w\|_0^2 + \nu \|w\|_0^2 \leq \|w\|_0 \{ \|w\|_0 (\|\nabla v_2\|_\infty + \|\nabla \tilde{u}_1\|_\infty) \\ + \|\nabla(\tilde{u}_1 - \tilde{u}_2)\|_\infty \|v_2\|_0 + \|\hat{F}\|_0 + \|\tilde{u}_1 - \tilde{u}_2\|_\infty \|\nabla v_2\|_0 \}$$

Using Gronwall's inequality, as in previous section, we end up with the following proposition

Proposition 1.9. *2-D Energy Estimate and Gradient Control* Let u_1 and u_2 be two smooth divergence free solutions to the Navier-Stokes equation with radial-energy decomposition $u_j(x, t) = v_j(x, t) + \tilde{u}_j(x, t)$ and with external forces F_1 and F_2 . Then we have the following estimates:

(1.40)

$$\sup_{t \in [0, T]} \|v_1 - v_2\|_0 \leq \|v_1(\cdot, 0) - v_2(\cdot, 0)\|_0 + \exp \left[\int_0^T (\|\nabla v_2\|_\infty + \|\nabla \tilde{u}_1\|_\infty) dt \right] \times \int_0^T [\|(F_1 - F_2)(\cdot, t)\|_0 + \|\tilde{u}_1 - \tilde{u}_2\|_\infty \|\nabla v_2\|_0 + \|\nabla \tilde{u}_1 - \nabla \tilde{u}_2\|_\infty \|v_2\|_0] dt$$

(1.41)

$$\nu \int_0^T \|\nabla(v_1(\cdot, t) - v_2(\cdot, t))\|_0^2 dt \leq C(v_2, \tilde{u}_1, T) \left\{ \|(u_1 - u_2)(\cdot, 0)\|_0^2 + \left[\int_0^T (\|F_1(\cdot, t) - F_2(\cdot, t)\|_0 + \|\tilde{u}_1 - \tilde{u}_2\|_\infty \|\nabla v_2(\cdot, t)\|_0 + \|\nabla \tilde{u}_1 - \nabla \tilde{u}_2\|_\infty \|\nabla v_2(\cdot, t)\|_0 + \|\nabla \tilde{u}_1 - \nabla \tilde{u}_2\|_\infty \|v_1(\cdot, t)\|_0) dt \right]^2 \right\}$$

Exercise: Derive (1.40) and (1.41) and use it to prove the above proposition.

1.5. Calculus Inequalities for Sobolev Spaces and Mollifiers. We have already introduced the Sobolev space $H^m(\mathbb{R}^N)$ for integer $m \geq 0$. We now extend it to $H^s(\mathbb{R}^N)$ for any $s \in \mathbb{R}$. In the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of smooth functions with rapid decay at ∞ , we introduce the norm

$$(1.42) \quad \|u\|_s = \left\{ \int_{\mathbb{R}^N} (1 + |k|)^{2s} \hat{u}(k) dk \right\}^{1/2}$$

where $\hat{u}(k) = \mathcal{F}[u](k)$, i.e. the Fourier-Transform of u . The completion of $\mathcal{S}(\mathbb{R}^N)$ with norm (1.42) will be referred to as $H^s(\mathbb{R}^N)$. You can check that for $s = m$, that this is equivalent to the original definition of H^m .

One of the most important Sobolev space property that we will use is the Sobolev inequality below:

Lemma 1.10. *Sobolev embedding Theorem*

The space $H^{s+k}(\mathbb{R}^N)$, for $s > N/2$, $k \in \mathbb{Z}^+ \cup \{0\}$ is continuously embedded in the space $\mathbf{C}^k(\mathbb{R}^N)$, and there exists a constant $c > 0$ such that

$$(1.43) \quad \|v\|_{\mathbf{C}^k} \leq c \|v\|_{s+k}, \text{ for any } v \in H^{s+k}(\mathbb{R}^N)$$

Some other calculus inequalities in the following Lemma will be useful for our purposes:

Lemma 1.11. i. For all $m \in \mathbb{Z}^+ \cup \{0\}$, there exists $c > 0$ such that for all $u, v \in \mathbf{L}^\infty \cap H^m(\mathbb{R}^N)$,

$$\|uv\|_m \leq c \{ \|u\|_\infty \|D^m v\|_0 + \|D^m u\|_0 \|v\|_\infty \}$$

$$\sum_{0 \leq |\alpha| \leq m} \|D^\alpha(uv) - u D^\alpha v\|_0 \leq c \{ \|\nabla u\|_\infty \|D^{m-1} v\|_0 + \|D^m u\|_0 \|v\|_\infty \}$$

ii. For all $s > N/2$, $H^s(\mathbb{R}^N)$ is a Banach algebra, i.e. there exists a constant c so that for all $u, v \in H^s(\mathbb{R}^N)$,

$$\|uv\|_s \leq c \|u\|_s \|v\|_s$$

We now introduce a mollifier. Suppose $\rho = \rho(|x|) \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, i.e. infinitely smooth function with compact support. Suppose also $\rho \geq 0$, and $\int_{\mathbb{R}^N} \rho dx = 1$. Then we define mollification of v , denoted by $\mathcal{I}_\epsilon v$ to be a function given by:

$$(1.44) \quad (\mathcal{I}_\epsilon v)(x) = \epsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) v(y) dy$$

Lemma 1.12. *Properties of Mollifier*

Let $\mathcal{I}_\epsilon$ be the mollifier defined in (1.44). Then $\mathcal{I}_\epsilon v \in \mathcal{C}^\infty(\mathbb{R}^N)$ and

i. for all $v \in \mathcal{C}^0(\mathbb{R}^N)$, $\mathcal{I}_\epsilon v \rightarrow v$ uniformly on any compact set $\Omega \subset \mathbb{R}^N$ and

$$\|\mathcal{I}_\epsilon v\|_\infty \leq \|v\|_\infty$$

ii. Mollifiers commute with distribution derivatives

$$D^\alpha \mathcal{I}_\epsilon v = \mathcal{I}_\epsilon D^\alpha v \text{ for any } |\alpha| \leq m, v \in H^m$$

iii. For all $u \in \mathbf{L}^p(\mathbb{R}^N)$, $v \in \mathbf{L}^q(\mathbb{R}^N)$, $1/p + 1/q = 1$,

$$\int_{\mathbb{R}^N} (\mathcal{I}_\epsilon u) v dx = \int_{\mathbb{R}^N} u (\mathcal{I}_\epsilon v) dx$$

iv. For all $v \in H^s(\mathbb{R}^N)$, $\mathcal{I}_\epsilon v$ converges to v in H^s and the rate of convergence in the H^{s-1} norm is linear in ϵ , i.e.

$$\lim_{\epsilon \rightarrow 0^+} \|\mathcal{I}_\epsilon v - v\|_s = 0$$

$$\|\mathcal{I}_\epsilon v - v\|_{s-1} \leq C\epsilon \|v\|_s$$

v. For all $v \in H^m(\mathbb{R}^N)$, $k \in \mathbb{Z}^+ \cup \{0\}$, and $\epsilon > 0$,

$$\|\mathcal{I}_\epsilon v\|_{m+k} \leq \frac{c_{mk}}{\epsilon^k} \|v\|_m$$

$$\|\mathcal{I}_\epsilon D^k v\|_\infty \leq \frac{c_k}{\epsilon^{N/2+k}} \|v\|_0$$

2. MORE PROPERTIES OF HODGE PROJECTION

Lemma 2.1. *Any vector field $v \in H^m(\mathbb{R}^N)$ for $m \in \mathbb{Z} \cup \{0\}$ has a unique orthogonal decomposition*

$$v = \nabla \phi + w, \text{ where } \nabla \phi, w \in H^m, \nabla \cdot w = 0$$

We define $w = \mathcal{P}v$ as the Hodge projection of v onto the divergence free vector field.

Further,

i. $(\mathcal{P}v, \nabla \phi)_m = 0$ and $\|\mathcal{P}v\|_m^2 + \|\nabla \phi\|_m^2 = \|v\|_m^2$.

ii. $\mathcal{P}$ commutes with D^α in H^m for $|\alpha| \leq m$: $\mathcal{P}D^\alpha v = D^\alpha \mathcal{P}v$.

iii. $\mathcal{P}\mathcal{I}_\epsilon v = \mathcal{I}_\epsilon \mathcal{P}v$

iv. $\mathcal{P}$ is symmetric: $(\mathcal{P}u, v)_m = (u, \mathcal{P}v)_m$

Proof. We only consider $m = 0$. Other cases follow simply by noting property ii: that $\mathcal{P}$ commutes with D . We further consider only $v \in \mathbf{C}_c^\infty(\mathbb{R}^N)$. This space is dense in $\mathbb{H}^m$ and hence all the results will follow for more general v , except that derivatives have to be understood in the sense of a distribution. Define ϕ by solving

$$\Delta \phi = \nabla \cdot v, \text{ with } \phi \rightarrow 0 \text{ as } x \rightarrow \infty$$

Using Green's function for Laplacian, we know

$$(2.45) \quad \phi(x) = \Delta^{-1} \int_{y \in \mathbb{R}^N} G(x-y) (\nabla \cdot v)(y) dy,$$

where

$$G(x) = \frac{1}{2\pi} \log |x| \text{ for } N = 2, \quad G(x) = -\frac{1}{4\pi|x|} \text{ for } N = 3$$

Then it is clear that

$$(2.46) \quad \nabla \phi = \int_{y \in \mathbb{R}^N} \nabla G(x-y)(\nabla \cdot v)(y) dy \equiv \nabla \Delta^{-1} \nabla \cdot v dy$$

Now, notice that as $x \rightarrow \infty$,

$$\nabla \phi \sim [\nabla G](x) \int_{y \in \mathbb{R}^N} (\nabla \cdot v)(y) dy + O(|x|^{-N})$$

From applying Green's theorem on the first term,

$$\nabla \phi = O(|x|^{-N}) \text{ as } |x| \rightarrow \infty,$$

and hence $\nabla \phi \in L^2(\mathbb{R}^N)$. Define

$$\mathcal{P}v = w = v - \nabla \phi$$

Clearly since $v, \nabla \phi \in L^2(\mathbb{R}^N)$, so is $w = \mathcal{P}v$. It is clear that

$$\nabla \cdot w = \nabla \cdot v - \Delta \phi = 0$$

So, $\mathcal{P}v$ is divergence free, and from the decay rate of $\nabla \phi$ for large ϕ , it follows that

$$w \sim O(|x|^{-N}) \text{ as } |x| \rightarrow \infty$$

Now, property **i.** follows since

$$(w, \nabla \phi)_0 = \int_{\mathbb{R}^N} w_j \partial_{x_j} \phi dx = \int_{\mathbb{R}^N} \partial_{x_j} (w_j \phi) = \lim_{R \rightarrow \infty} \int_{|x|=R} \phi (w \cdot n) dx = 0$$

since for large x , $\phi = O(\log |x|)$ for $N = 2$ and $\phi = O(|x|^{-N+2})$ for $N = 3$, while $w = O(|x|^{-N})$. Also,

$$\|v\|_0^2 = (w + \nabla \phi, w + \nabla \phi)_0 = (w, w) + (\nabla \phi, \nabla \phi)_0 = \|\mathcal{P}v\|_0^2 + \|\nabla \phi\|_0^2$$

because of the orthogonality property.

Property **ii.** follows simply from the observation that

$$D^\alpha \mathcal{P}v = D^\alpha w = D^\alpha v - D^\alpha \nabla \phi = D^\alpha v - \nabla D^\alpha \phi = \mathcal{P}D^\alpha v,$$

since

$$\Delta(D^\alpha \phi) = \nabla \cdot (D^\alpha v)$$

Property **iii.** follows from the commuting property of $\mathcal{I}_\epsilon$ with Δ^{-1} (defined in (2.45)) and with any differential operator, since

$$\mathcal{I}_\epsilon \mathcal{P}v = \mathcal{I}_\epsilon v - \mathcal{I}_\epsilon \nabla \Delta^{-1} \nabla \cdot v = (\mathcal{I}_\epsilon v) - \nabla \Delta^{-1} \nabla \cdot (\mathcal{I}_\epsilon v) = \mathcal{P} \mathcal{I}_\epsilon v$$

Property **iv** follows for $m = 0$ because

$$\begin{aligned} (\mathcal{P}v, u)_0 &= (u, v - \nabla \Delta^{-1} \nabla \cdot v)_0 = (u, v)_0 + (\nabla u, \Delta^{-1} \nabla \cdot v)_0 = (u, v)_0 + (\Delta^{-1} \nabla u, \nabla \cdot v)_0 \\ &= (u, v)_0 - (\nabla \cdot (\Delta^{-1} \nabla u), v)_0 = (u - \nabla \Delta^{-1} \nabla \cdot u, v)_0 = (v, \mathcal{P}u)_0 \end{aligned}$$

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