1. Energy Methods for Euler and Navier-Stokes Equation

We will consider this week basic energy estimates. These are estimates on the $L^2$ spatial norms of the solution $u(x,t)$ and its higher derivatives with respect to $x$. Like other PDE initial value problems, these estimates are most useful in establishing existence and uniqueness of solutions.

For simplicity, we will first take $\Omega = \mathbb{R}^N$, where $N = 2$ or $3$. We will drop the boldfonted notation for vectors since by this time you are sufficiently familiar to recognize which quantities are vectors and scalars. The exposition of this topic follows Bertozzi & Majda (Text), though with some differences in notation.

1.1. Kinetic Energy Dissipation: Consider the incompressible constant density Navier-Stokes equation

$$\partial_t u + u \cdot \nabla u = - \nabla p + \nu \Delta u + b$$

(1.2) $\nabla \cdot u = 0$

Then the $i$-th component of (1.1) is given by

$$\partial_t u_i + u_j \partial_x j u_i = - \partial_x i p + \nu \partial_x^2 x j u_i + b_i$$

(1.3)

Multiplying (1.3) by $u_i$ and summing over $i$ and using (1.2), we have on using Einstein convention of summation over repeated indices:

$$\partial_t \frac{1}{2} u_i^2 + \partial_x i \left( \frac{1}{2} u_j u_i^2 \right) = - \partial_x i (p u_i) + \nu \partial_x^2 x_j u_i - \nu \partial_x x_j u_i \partial_x x_j u_i + b_i u_i$$

(1.4)

Assuming that $|u|(|u|^2 + p) + |u||Du| = o(|x|^{-N+1})$, as $|x| \to \infty$, we obtain from (1.3) after integration over $\mathbb{R}^N$ the following identity:

$$\partial_t \frac{1}{2} \|u(.,t)\|^2_0 = - \nu \|Du\|^2_0 + (b, u)_0$$

(1.5)

The physical kinetic energy in the fluid is

$$E(t) = \int_{\mathbb{R}^N} \frac{1}{2} |u(x,t)|^2 dx = \frac{1}{2} \|u(t)\|^2_0$$

(1.6)

In the absence of any force, i.e. $b = 0$, we obtain from (1.5)

$$\frac{d}{dt} E = - \nu \|Du\|^2_0 = - \nu \int_{\mathbb{R}^N} |Du(x,t)|^2 dx = - \epsilon,$$

(1.7)

where $\epsilon$ is usually referred to as the rate of energy dissipation. For zero viscosity, i.e. Euler flow, kinetic energy $E$ is conserved and $E(t) = E(0)$. In the presence of a force $b$, the term $(b, u)_0$ is the work done by the force $b$ per-unit time. So, (1.5) is a physical statement that rate of change of Kinetic Energy is the rate of work done by the external force minus the Energy dissipated /lost due to viscous friction.

This dissipated energy is actually converted to heat. For compressible fluid flow, we have to couple heat energy and thermodynamics with momentum equation to get a complete set of equations. But this is not the case for incompressible Navier-Stokes equation, which we are studying here, and we do not have to consider thermodynamics to solve for fluid flow field $u$. 

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1.2. Energy Estimate, Uniqueness and $\nu$ dependence of Smooth Solutions. Let $u$, $w$ be two Navier-Stokes solution, corresponding to forcing $b$ and $c$ respectively. We assume $b$ and $c$ to be smooth as well and decaying sufficiently fast in $x$ at $\infty$. We denote the corresponding pressures by $p$ and $q$. Then consider the difference $v = u - w$. It is easy to check that $v$ satisfies:

\[
(1.8) \quad \partial_t v + v \cdot \nabla v + w \cdot \nabla v + v \cdot \nabla w = -\nabla P + \nu \Delta v + f, \quad \text{where } f = b - c, \; P = p - q
\]

The $i$-th component of the above equation may be written as

\[
(1.9) \quad \partial_t v_i + v_j \partial_{x_j} v_i + w_j \partial_{x_j} v_i + v_j \partial_{x_j} w_i = -\partial_{x_i} P + \nu \partial_{x_j}^2 v_i + f_i
\]

Multiplying above by $v_i$ and integrating we obtain that

\[
(1.10) \quad \partial_t \frac{1}{2} v_i^2 + \frac{1}{2} \partial_{x_j} [ (v_j + w_j) \nu v_i^2 ] + v_i v_j \partial_{x_j} w_i = -\partial_{x_i} (u_i P) + \nu \partial_{x_j} (v_i \partial_{x_j} v_i) - \nu (\partial_{x_j} v_i) (\partial_{x_j} v_i) + v_i f_i
\]

So integrating over $\mathbb{R}^N$ with usual assumptions on decay of velocity and pressure fields at $\infty$, we obtain by using

\[
(1.11) \quad \| (v, f) \|_{0} \leq \| v \|_{0} \| f \|_{0}, \quad \text{and } \| (v, v \cdot \nabla w) \|_{0} \leq \| D w (\cdot, t) \|_{\infty} \| v \|_{0}^2,
\]

\[
(1.12) \quad \frac{d}{dt} \frac{1}{2} \| v \|^2_0 + \nu \| D v \|^2_0 \leq \| D w \|_\infty \| v \|_0^2 + \| v \|_0 \| f \|_0
\]

So, in particular,

\[
(1.13) \quad \frac{d}{dt} \| v \|_0 \leq \| \nabla w \|_\infty \| v \|_0 + \| f \|_0
\]

Also, on integrating (1.12) between $t = 0$ to $t = T$, we obtain

\[
(1.14) \quad \frac{1}{2} \| v(\cdot, T) \|_0^2 + \nu \int_0^T \| D v(\cdot, t) \|_0^2 dt \leq \frac{1}{2} \| v(\cdot, 0) \|_0^2 + \int_0^T \| D w(\cdot, t) \|_\infty \| v(\cdot, t) \|_0^2 dt + \int_0^T \| v(\cdot, t) \|_0 \| f(\cdot, t) \|_0 dt
\]

Using well-known Gronwall’s inequality on (1.13) and the definition of $v$, we obtain the following Lemma

**Lemma 1.1.** Let $u$ and $w$ be two smooth $L^2(\mathbb{R}^N)$ solutions to the Navier-Stokes equation for $t \in [0, T]$ for the same viscosity $\nu$, but different forcing $b$ and $c$ respectively. Then,

\[
(1.15) \quad \sup_{t \in [0, T]} \| u(\cdot, t) - w(\cdot, t) \|_0 \leq \left\{ \| u(\cdot, 0) - w(\cdot, 0) \|_0 + \int_0^T \| b(\cdot, t) - c(\cdot, t) \|_0 dt \right\} \exp \left[ \int_0^T \| \nabla w(\cdot, t) \|_\infty dt \right]
\]

**Corollary 1.2.** Uniqueness of smooth solutions

Let $u(\cdot, t)$ and $w(\cdot, t)$ be two smooth $L^2(\mathbb{R}^N)$ solutions to incompressible constant density Navier-Stokes equation for $t \in [0, T]$ with same initial data and forcing. Then, the solution is unique.

**Proof.** This simply follows from Lemma 1.1, since $u(\cdot, 0) - w(\cdot, 0) = 0$ and $b - c = 0$.

**Remark 1.3.** The energy estimate (1.12) does not explicitly depend on $\nu$ and is equally valid for $\nu = 0$, i.e. for the Euler equation.
The energy estimate (1.15) is also useful in estimating the difference between smooth Euler and Navier-Stokes solution with the same initial data and forcing. Let \( u^0 \) be a smooth solution to the Euler equation, i.e. \( \nu = 0 \), while \( u^{[\nu]} \) is a solution to Navier-Stokes equation with the same initial data and forcing. Then, we can obtain an equation for \( v = u^{[\nu]} - u^0 \):

\[
\partial_t v + v \cdot \nabla v + u^0 \cdot \nabla v + v \cdot \nabla u^0 = -\nabla P + \nu \Delta v + f
\]

where \( f = \nu \Delta u^0 \), \( P = p - q \) is the difference of pressure.

This is the same equation as for (1.8), with \( w \) replaced by \( u^0 \), and a different meaning of \( f \). Therefore, the energy estimate (1.15) in this case becomes

\[
\sup_{t \in [0,T]} \| u^{[\nu]}(.,t) - u^0(.,t) \|_0 \leq \nu \left( \int_0^T \| \Delta u^0(.,t) \|_0 dt \right) \exp \left( \int_0^T \| Du^0(.,t) \|_\infty dt \right) \leq \nu TC(u^0, T)
\]

Notice that (1.13) with \( w \) replaced by \( u^0 \), and with \( f = -\nu \Delta u^0 \) gives rise to

\[
\nu \int_0^T \| Dv(.,t) \|_0^2 dt \leq \int_0^T \| Du^0(.,t) \|_\infty \| v(.,t) \|_0^2 dt + \nu \int_0^T v(.,t) \| \Delta u^0(.,t) \|_0 dt
\]

Using estimate (1.14) estimate, we obtain,

\[
\int_0^T \| Dv(.,t) \|_0^2 dt \leq \nu TC \left( CT \int_0^T \| Du^0(.,t) \|_\infty dt + \nu \int_0^T \| \Delta u^0(.,t) \|_0 dt \right) \leq \nu T^2 c_2(u^0, T)
\]

So,

\[
\int_0^T \| Dv(.,t) \|_0 dt \leq T^{1/2} \left( \int_0^T \| Dv(.,t) \|_0^2 dt \right)^{1/2} \leq \nu^{1/2} T^{3/2} c_2(u^0, T)
\]

This implies the following proposition:

**Proposition 1.4. Comparison of smooth Euler and Navier-Stokes Solution**

Given the same initial data and forcing, then the difference \( v \) between smooth \( L^2(\mathbb{R}^N) \) Navier-Stokes and Euler solution over a common interval of existence \([0, T]\) satisfies (1.17) and (1.20). In particular for any fixed \( T \), as \( \nu \to 0 \), \( u^{[\nu]}(.,t) \to u^0(.,t) \), and \( Du^{[\nu]}(.,t) \to Du^0(.,t) \) uniformly for \( t \in [0, T] \).

### 1.3. Kinetic Energy of 2-D flow.

The Theorems in the last section hold for solutions to Navier-Stokes/Euler equation that decay sufficiently rapidly as \( x \to \infty \) so that velocity \( u(.,t) \in L^2 \). This is a reasonable physical assumption in \( \mathbb{R}^3 \).

For 2-D flow, this is not necessarily the case, unless the integral of vorticity in the flow is zero, as will be seen shortly. Suppose

\[
\text{supp} \omega \subset \{ x : x \in \mathbb{R}^2, |x| < R \}
\]

Applying 2-D Biot-Savart Law:

\[
u(x, t) = \int_{|y| \leq R} K(x-y) \omega(y, t) dy, \text{ where } K(x) = \frac{1}{2\pi|x|^2} [-x_2, x_1]
\]

We first note that

(1.22)
We note that
\begin{equation}
|x - y|^2 = |x|^2 \left( 1 - 2 \frac{y \cdot x}{|x|^2} + \frac{|y|^2}{|x|^2} \right)
\end{equation}

Now, if $|x| \geq 2R$, then since $|y| \leq R$, it follows that as $|x| \to \infty$

\begin{equation}
|x - y|^2 = |x|^2 + O(|x|^{-3})
\end{equation}

So, from (1.22),
\begin{equation}
u(x, t) = K(x) \int_{y \in \mathbb{R}^2} \omega(y, t) + O(|x|^{-3})
\end{equation}

Since
\begin{equation}
\int_{x \in \mathbb{R}^2} (1 + |x|)^{-l} dx < \infty, \text{ iff } l > N
\end{equation}

It follows that

**Lemma 1.5.** A 2-D incompressible flow with compact vorticity $\omega$ has finite energy iff
\begin{equation}
\int_{\mathbb{R}^2} \omega(x) dx = 0
\end{equation}

**Remark 1.6.** Note that the vorticity $\omega(x, t)$ will satisfy (1.24) for $t > 0$, if
\begin{equation}
\int_{\mathbb{R}^2} \omega(x, 0) dx = 0,
\end{equation}

since integration of 2-D Navier-Stokes equation in the vorticity form gives
\begin{equation}
\frac{d}{dt} \int_{x \in \mathbb{R}^2} \omega(x, t) dx = 0
\end{equation}

**Remark 1.7.** The statement that finite energy is implied only iff (1.26) is satisfied is not limited merely to flow with compact support. It is more generally true for $\omega \in L^1(\mathbb{R}^2)$.

When (1.26) is violated, it is possible to decompose a solution to Navier-Stokes equation to such that a part of it is in $L^2(\mathbb{R}^2)$ (hence finite energy), while the other part is generated by a radial distribution of vorticity whose integral is the same as the integral of initial vorticity over $\mathbb{R}^2$.

Consider an initial vorticity distribution $\omega_0(x) \in L^1(\mathbb{R}^2)$. We chose any compact radial vorticity distribution $\tilde{\omega}_0(|x|)$ such that
\begin{equation}
\int_{\mathbb{R}^2} \tilde{\omega}_0(|x|) dx = \int_{\mathbb{R}^2} \omega_0(x) dx
\end{equation}

We determine radial vorticity solution $\tilde{\omega}(|x|, t)$ with initial value $\tilde{\omega}(|x|, 0) = \tilde{\omega}_0(|x|)$ to Navier-Stokes equation without forcing. We know from worked out problems two weeks back, that $\tilde{\omega}(|x|, t)$ satisfies 2-D heat equation with corresponding velocity
\begin{equation}
\tilde{u}(x, t) = \frac{(-x_2, x_1)}{|x|^2} \int_0^{|x|} s \tilde{\omega}_0(s, t) ds
\end{equation}

Therefore, we now consider the decomposition
\begin{equation}
u(x, t) = \tilde{u}(x, t) + v(x, t)
\end{equation}
Since \((\nabla \times v)(x, 0) = \omega_0(x) - \tilde{\omega}_0(x)\), it follows that
\[
(1.30) \quad \int (\nabla \times v)(x, 0) dx = 0
\]
from construction of \(\tilde{\omega}_0\). From (1.27) and the fact that heat solution preserves \(\int_{\mathbb{R}^2} \tilde{\omega}(x, t) dx\), it follows that
\[
(1.31) \quad \frac{d}{dt} \int_{\mathbb{R}^2} (\nabla \times v)(x, t) dx = 0 , \text{ implying by above } \int_{x \in \mathbb{R}^2} (\nabla \times v)(x, t) dx = 0
\]
This implies that \(v(x, t)\) has finite energy.

Thus, we have proved the following Lemma:

**Lemma 1.8.** Any smooth solution \(u(x, t)\) to 2-D Navier-Stokes equation with an initial \(L^1(\mathbb{R}^2)\) vorticity can be decomposed into
\[
(1.32) \quad u(x, t) = v(x, t) + \tilde{u}(x, t)
\]
where \(v \in L^2(\mathbb{R}^2)\) and divergence free, while
\[
(1.33) \quad \tilde{u}(x) = (-x_2, x_1)|x|^{-2} \int_0^{|x|} s\tilde{\omega}(s, t) ds
\]
for some smooth radial vorticity distribution \(\tilde{\omega}(|x|, t)\) with an initial compact support.

1.4. **Energy Inequality for 2-D flow.** Consider the radial-Energy decomposition
\[
(1.34) \quad u(x, t) = \tilde{u}(x, t) + v(x, t)
\]
of solution to the Navier-Stokes equation where \(v \in L^2(\mathbb{R}^2)\). \(v\) satisfies
\[
(1.35) \quad \partial_t v + v \cdot \nabla v + \tilde{u} \cdot \nabla v + v \cdot \nabla \tilde{u} = -\nabla p + \nu \Delta v + F
\]
Consider two solutions to Navier-Stokes equation \(u_1, u_2\) with radial decompositions:
\[
(1.36) \quad u_1 = \tilde{u}_1 + v_1 , \quad u_2 = \tilde{u}_2 + v_2
\]
Then, if we denote
\[
(1.37) \quad w = v_1 - v_2 , \quad \tilde{u}_1 - \tilde{u}_2 = \hat{u} , \quad \hat{F} = F_1 - F_2 , \quad \hat{p} = p_1 - p_2 ,
\]
then \(w\) satisfies
\[
(1.38) \quad \partial_t w + v_1 \cdot \nabla w + w \cdot \nabla v + \tilde{u}_1 \cdot \nabla v + v_1 \cdot \nabla \tilde{u} + w \cdot \nabla \tilde{u}_1 = -\nabla \hat{p} + \nu \Delta w + \hat{F}
\]
Then using the same integration by parts procedure as in the last section, we have
\[
(1.39) \quad \frac{d}{dt} \frac{1}{2} \|w\|_0^2 + \nu \|w\|_0^2 \leq \|w\|_0 \left\{ \|\nabla v\|_\infty + \|\nabla \tilde{u}_1\|_\infty \right\}
\]
\[
+ \|\nabla (\tilde{u}_1 - \tilde{u}_2)\|_\infty \|v_2\|_0 + \|\hat{F}\|_0 + |\hat{u}_1 - \hat{u}_2|_\infty \|\nabla v_2\|_0 \right\}
\]
Using Gronwall’s inequality, as in previous section, we end up with the following proposition.
Proposition 1.9. 2-D Energy Estimate and Gradient Control Let $u_1$ and $u_2$ be two smooth divergence free solutions to the Navier-Stokes equation with radial-energy decomposition $u_j(x, t) = v_j(x, t) + \tilde{u}_j(x, t)$ and with external forces $F_1$ and $F_2$. Then we have the following estimates:

\begin{align}
\sup_{t \in [0, T]} \|v_1 - v_2\|_0 & \leq \|v_1(., 0) - v_2(., 0)\|_0 + \exp \left[ \int_0^T (\|\nabla v_2\|_\infty + \|\nabla \tilde{u}_1\|_\infty) \, dt \right] \times \\
\int_0^T \left[ \|(F_1 - F_2)(., t)\|_0 + \|\tilde{u}_1 - \tilde{u}_2\|_\infty \|\nabla v_2\|_0 + \|\nabla \tilde{u}_1 - \nabla \tilde{u}_2\|_\infty \|v_2\|_0 \right] \, dt 
\end{align}

(1.41)

\begin{align}
\nu \int_0^T \|\nabla (v_1(., t) - v_2(., t))\|_0^2 \, dt & \leq C(v_2, \tilde{u}_1, T) \left\{ \||u_1 - u_2(., 0)||_0^2 + \int_0^T (\|F_1(., t) - F_2(., t)||_0 \\
+ \|\tilde{u}_1 - \tilde{u}_2\|_\infty \|\nabla v_2(., t)||_0 + \|\nabla \tilde{u}_1 - \nabla \tilde{u}_2\|_\infty \|v_1(., t)||_0 \right\} dt^2 
\end{align}

Exercise: Derive (1.40) and (1.41) and use it to prove the above proposition.

1.5. Calculus Inequalities for Sobolev Spaces and Mollifiers. We have already introduced the Sobolev space $H^m(\mathbb{R}^N)$ for integer $m \geq 0$. We now extend it to $H^s(\mathbb{R}^N)$ for any $s \in \mathbb{R}$. In the Schwartz space $S(\mathbb{R}^N)$ of smooth functions with rapid decay at $\infty$, we introduce the norm

\begin{align}
\|u\|_s = \left\{ \int_{\mathbb{R}^N} (1 + |k|)^{2s} \hat{u}(k) \, dk \right\}^{1/2}
\end{align}

(1.42)

where $\hat{u}(k) = \mathcal{F}[u](k)$, i.e. the Fourier-Transform of $u$. The completion of $S(\mathbb{R}^N)$ with norm (1.42) will be referred to as $H^s(\mathbb{R}^N)$. You can check that for $s = m$, that this is equivalent to the original definition of $H^m$.

One of the most important Sobolev space property that we will use is the Sobolev inequality below:

Lemma 1.10. Sobolev embedding Theorem

The space $H^{s+k}(\mathbb{R}^N)$, for $s > N/2$, $k \in \mathbb{Z}^+ \cup \{0\}$ is continuously embedded in the space $C^k(\mathbb{R}^N)$, and there exists a constant $c > 0$ such that

\begin{align}
\|v\|_{C^k} \leq c \|v\|_{s+k}, \text{ for any } v \in H^{s+k}(\mathbb{R}^N)
\end{align}

(1.43)

Some other calculus inequalities in the following Lemma will be useful for our purposes:

Lemma 1.11. i. For all $m \in \mathbb{Z}^+ \cup \{0\}$, there exists $c > 0$ such that for all $u, v \in L^\infty \cap H^m(\mathbb{R}^N)$,

\begin{align}
\|uv\|_m & \leq c \{ \|u\|_\infty \|D^m v\|_0 + \|D^m u\|_0 \|v\|_\infty \} \\
\sum_{0 \leq |\alpha| \leq m} \|D^\alpha (uv) - uD^\alpha v\|_0 & \leq c \{ \|\nabla u\|_\infty \|D^{m-1} v\|_0 + \|D^m u\|_0 \|v\|_\infty \}
\end{align}

ii. For all $s > N/2$, $H^s(\mathbb{R}^N)$ is a Banach algebra, i.e. there exits a constant $c$ so that for all $u, v \in H^s(\mathbb{R}^N)$,

\begin{align}
\|uv\|_s & \leq c \|u\|_s \|v\|_s
\end{align}
Using Green’s function for Laplacian, we know derivatives have to be understood in the sense of a distribution. Define
\[(I_\varepsilon v)(x) = \varepsilon^{-N} \int_{\mathbb{R}^N} \rho \left( \frac{x - y}{\varepsilon} \right) v(y) dy\]

**Lemma 1.12. Properties of Mollifier**
Let \(I_\varepsilon\) be the mollifier defined in (1.44). Then \(I_\varepsilon v \in C^\infty(\mathbb{R}^N)\) and

i. For all \(v \in C^0(\mathbb{R}^N)\), \(I_\varepsilon v \to v\) uniformly on any compact set \(\Omega \subset \mathbb{R}^N\) and
\[\|I_\varepsilon v\|_\infty \leq \|v\|_\infty\]

ii. Mollifiers commute with distribution derivatives
\[D^\alpha I_\varepsilon v = I_\varepsilon D^\alpha v\]
for any \(|\alpha| \leq m\), \(v \in H^m\)

iii. For all \(u \in L^p(\mathbb{R}^N)\), \(v \in L^q(\mathbb{R}^N)\), \(1/p + 1/q = 1\),
\[\int_{\mathbb{R}^N} (I_\varepsilon u)v dx = \int_{\mathbb{R}^N} u(I_\varepsilon v) dx\]

iv. For all \(v \in H^s(\mathbb{R}^N)\), \(I_\varepsilon v\) converges to \(v\) in \(H^s\) and the rate of convergence in the \(H^{s-1}\) norm is linear in \(\varepsilon\), i.e.
\[\lim_{\varepsilon \to 0^+} \|I_\varepsilon v - v\|_s = 0\]
\[\|I_\varepsilon v - v\|_{s-1} \leq C\varepsilon \|v\|_s\]

v. For all \(v \in H^m(\mathbb{R}^N)\), \(k \in \mathbb{Z}^+ \cup \{0\}\), and \(\varepsilon > 0\),
\[\|I_\varepsilon v\|_{m+k} \leq \frac{cmk}{\varepsilon} \|v\|_m\]
\[\|I_\varepsilon D^k v\|_\infty \leq \frac{Ck}{\varepsilon^{N/2+k}} \|v\|_0\]

2. More properties of Hodge Projection

**Lemma 2.1.** Any vector field \(v \in H^m(\mathbb{R}^N)\) for \(m \in \mathbb{Z} \cup \{0\}\) has a unique orthogonal decomposition
\[v = \nabla \phi + w\]
where \(\nabla \phi, w \in H^m\), \(\nabla \cdot w = 0\)
We define \(w = \mathcal{P}v\) as the Hodge projection of \(v\) onto the divergence free vector field. Further,

i. \((\mathcal{P}v, \nabla \phi)_m = 0\) and \(\|\mathcal{P}v\|_m^2 + \|\nabla \phi\|_m^2 = \|v\|_m^2\).

ii. \(\mathcal{P}\) commutes with \(D^\alpha\) in \(H^m\) for \(|\alpha| \leq m\):
\[D^\alpha \mathcal{P}v = \mathcal{P} \nabla \phi + \mathcal{P} D^\alpha v\]

iii. \(\mathcal{P}I_\varepsilon v = I_\varepsilon \mathcal{P} v\)

iv. \(\mathcal{P}\) is symmetric: \((\mathcal{P}u, v)_m = (u, \mathcal{P}v)_m\)

**Proof.** We only consider \(m = 0\). Other cases follow simply by noting property ii: that \(\mathcal{P}\) commutes with \(D\). We further consider only \(v \in C^\infty_c(\mathbb{R}^N)\). This space is dense in \(\mathbb{H}^m\) and hence all the results will follow for more general \(v\), except that derivatives have to be understood in the sense of a distribution. Define \(\phi\) by solving
\[\Delta \phi = \nabla \cdot v\]
with \(\phi \to 0\) as \(x \to \infty\)
Using Green’s function for Laplacian, we know
\[\phi(x) = \Delta^{-1} \int_{y \in \mathbb{R}^N} G(x-y)(\nabla \cdot v)(y) dy\]
where
\[ G(x) = \frac{1}{2\pi} \log |x| \text{ for } N = 2, \quad G(x) = -\frac{1}{4\pi|x|} \text{ for } N = 3. \]

Then it is clear that
\[ (2.46) \quad \nabla \phi = \int_{y \in \mathbb{R}^N} \nabla G(x - y)(\nabla \cdot v)(y) \equiv \nabla \Delta^{-1} \nabla \cdot v dy \]

Now, notice that as \( x \to \infty \),
\[ \nabla \phi \sim [\nabla G](x) \int_{y \in \mathbb{R}^N} (\nabla \cdot v)(y) dy + O(|x|^{-N}) \]

From applying Green’s theorem on the first term,
\[ \nabla \phi = O(|x|^{-N}) \text{ as } |x| \to \infty, \]
and hence \( \nabla \phi \in L^2(\mathbb{R}^N) \). Define
\[ \mathcal{P}v = w = v - \nabla \phi \]

Clearly since \( v, \nabla \phi \in L^2(\mathbb{R}^N) \), so is \( w = \mathcal{P}v \). It is clear that
\[ \nabla \cdot w = \nabla \cdot v - \Delta \phi = 0 \]

So, \( \mathcal{P}v \) is divergence free, and from the decay rate of \( \nabla \phi \) for large \( \phi \), it follows that
\[ w \sim O(|x|^{-N}) \text{ as } |x| \to \infty \]

Now, property i. follows since
\[ (w, \nabla \phi)_0 = \int_{\mathbb{R}^N} w_j \nabla_j \phi dx = \int_{\mathbb{R}^N} \nabla_j (w_j \phi) = \lim_{R \to \infty} \int_{|x|=R} \phi(w \cdot n) dx = 0 \]

since for large \( x \), \( \phi = O(\log |x|) \) for \( N = 2 \) and \( \phi = O(|x|^{-N+2}) \) for \( N = 3 \), while \( w = O(|x|^{-N}) \). Also,
\[ \|v\|_0^2 = (w + \nabla \phi, w + \nabla \phi)_0 = (w, w) + (\nabla \phi, \nabla \phi)_0 = \|\mathcal{P}v\|_0^2 + \|\nabla \phi\|_0^2 \]

because of the orthogonality property.

Property ii. follows simply from the observation that
\[ D^\alpha \mathcal{P}v = D^\alpha w = D^\alpha v - D^\alpha \nabla \phi = D^\alpha v - \nabla D^\alpha \phi = \mathcal{P}D^\alpha v, \]

since
\[ \Delta(D^\alpha \phi) = \nabla \cdot (D^\alpha v) \]

Property iii. follows from the commuting property of \( \mathcal{I}_\epsilon \) with \( \Delta^{-1} \) (defined in \( \text{(2.45)} \)) and with any differential operator, since
\[ \mathcal{I}_\epsilon \mathcal{P}v = \mathcal{I}_\epsilon v - \mathcal{I}_\epsilon \Delta^{-1} \nabla \cdot v = (\mathcal{I}_\epsilon v) - \nabla \Delta^{-1} \nabla \cdot (\mathcal{I}_\epsilon v) = \mathcal{P} \mathcal{I}_\epsilon v \]

Property iv follows for \( m = 0 \) because
\[ (\mathcal{P}v, u)_0 = (u, v - \nabla \Delta^{-1} \nabla \cdot v)_0 = (u, v)_0 + (\nabla u, \Delta^{-1} \nabla \cdot v)_0 = (u, v)_0 + (\Delta^{-1} \nabla u, \nabla \cdot v)_0 \]
\[ = (u, v)_0 - (\nabla \cdot (\Delta^{-1} \nabla u), v)_0 = (u - \nabla \Delta^{-1} \nabla \cdot u, v)_0 = (v, \mathcal{P}u)_0 \]

\[ \blacksquare \]