Week 7 Notes, Math 865, Tanveer

1. Global Existence Theorem for regularized Navier-Stokes

Instead of the usual Navier-Stokes equation, we consider the regularized Navier-Stokes equation for \( u^\varepsilon(x,t) \):

\[
\begin{align*}
  u^\varepsilon_t + I_\varepsilon (\{ I_\varepsilon (\nabla I_\varepsilon (u^\varepsilon \cdot \nabla I_\varepsilon (u^\varepsilon))) = -\nabla p^\varepsilon + \nu I_\varepsilon I_\varepsilon \Delta u^\varepsilon, \\
  \nabla \cdot u^\varepsilon = 0, \\
  u^\varepsilon(x,0) = u_0(x)
\end{align*}
\]

Using Hodge Projection operator, we project (1.1) into the space

\[
V^s \equiv \{ v: v \in H^s(\mathbb{R}^N), \nabla \cdot v = 0 \}
\]

It is easily proved that the subspace \( V^s \) of \( H^s \) is itself a Banach space. Since \( P \) commutes with operators \( I_\varepsilon \) and \( D \), it follows from (1.1) that

\[
u I_\varepsilon \Delta u^\varepsilon
\]

This regularized Navier-Stokes equation reduces to an ODE in the Banach space \( V^s \) and can be written symbolically in the form

\[
\frac{d}{dt} u^\varepsilon = F_\varepsilon (u^\varepsilon), \quad u^\varepsilon|_{t=0} = u_0
\]

where

\[
F_\varepsilon (u^\varepsilon) = \nu I_\varepsilon^2 \Delta u^\varepsilon - P \{ I_\varepsilon (\{ I_\varepsilon (u^\varepsilon \cdot \nabla I_\varepsilon (u^\varepsilon))) = F^1_\varepsilon (u^\varepsilon) - F^2_\varepsilon (u^\varepsilon)
\]

Lemma 1.1. Picard Theorem in Banach Space

Let \( O \subset B \) be an open set in a Banach space and \( F: O \to B \) be a mapping that satisfies the following properties:

i. \( F \) maps \( O \) to \( B \).
ii. \( F \) is locally Lipschitz continuous, i.e. for any \( X \in O \), there exists \( L > 0 \) and on open neighborhood \( U \subset O \) containing \( X \) so that

\[
\| F(X_1) - F(X_2) \| \leq L \| X_1 - X_2 \|, \quad \text{for all} \ X_1, X_2 \in U
\]

Then, for any \( X_0 \in O \), there exists time \( T \) such that the ODE

\[
\frac{dX}{dt} = F(X), \quad X|_{t=0} = X_0
\]

has a locally unique solution \( X \in C^1([-T,T]:O] \).

Remark 1.2. In the preceding Lemma, \( \| \cdot \| \) denotes the norm in the Banach space \( B \).

Remark 1.3. The proof of Lemma 1.1 is just like the classical Picard Theorem for ODEs in \( \mathbb{R}^N \); only that \( \mathbb{R}^N \) is replaced by Banach space \( B \). Recall that the classical Picard Theorem is based on contraction mapping theorem applied to the integral equation:

\[
X(t) = X_0 + \int_0^t F(X(\tau))d\tau
\]

Smallness of \( T \) together with Lipschitz property guarantees a unique \( C^0([-T,T],O] \) solution. The differential equation immediately implies that this solution is also in \( C^1([-T,T],O] \).

Remark 1.4. The Lemma above only guarantees local existence in \( t \). To get global existence, the following Lemma is useful.
Lemma 1.5. Continuation of solution to Autonomous ODE in $B$

Let $O \subset B$ be an open set in a Banach space $B$, and let $F : O \to B$ be a locally Lipschitz continuous operator. Then the unique solution $X \in C^1 [(-T, T), O]$ to the autonomous ODE

$$\frac{dX}{dt} = F(X), \quad X|_{t=0} = X_0$$

either exists globally in time, or $T < \infty$ and $X(t)$ leaves the open set $O$ as $t \to T^-$.  

Proof. We define $T$ as the largest possible value for which solution $X(t)$ exists for $t \in (-T, T)$. Suppose contrary to the Lemma statement, $T < \infty$ and the solution cannot be continued past $t = T$ (similar arguments can be repeated if the solution cannot be continued past $t = -T$; yet $\lim_{t \to T^-} X(t) = X(T) \in O$. We can re-start the clock, i.e. change of variable $t \to t - T$. It is clear that since the initial condition satisfies conditions for Lemma 1.13 we have solution $X(t) \in O$ for $t \in (T - \delta, T + \delta)$ for some $\delta > 0$. This contradicts assumption that $T$ is the largest value beyond which solution cannot be continued. $lacksquare$

Remark 1.6. We will now show that each of $F_1^\epsilon$ and $F_2^\epsilon$ satisfies the conditions for applying Lemma 1.5 in the Banach space $V^\epsilon$ for any fixed $\epsilon > 0$. By appropriately choosing an open set $O \subset V^m$, we will use Lemma 1.5 to establish global existence as well.

Lemma 1.7. Local existence for regularized problem

For $O = \{u \in V^m, \|u\|_m < M\}$, the function $F_\epsilon$ defined in (1.2) satisfies the requirement that for any $u_1, u_2 \in O$,

$$\|F_\epsilon(u_1) - F_\epsilon(u_2)\|_m \leq c_M(\epsilon, m, N)\|u_1 - u_2\|_m$$

where constant $c_M$ only depends on $M, m, \epsilon$ and $N$. Thus, $F^\epsilon$ is locally Lipschitz in $O$.

Proof. Consider first $F_1^\epsilon(u_1) - F_1^\epsilon(u_2)$:

$$(1.6) \quad \|F_1^\epsilon(u_1) - F_1^\epsilon(u_2)\|_m = \nu\|\mathcal{I}_\epsilon^2 \Delta (u_1 - u_2)\|_m \leq \nu\|\mathcal{I}_\epsilon^2 (u_1 - u_2)\|_{m+2} \leq \frac{c\nu}{\epsilon^2} \|u_1 - u_2\|_m,$$

where we used Lemma 1.13 of week 5 notes (parts iv and v). Now,

$$(1.7) \quad \|F_\epsilon^2(u_1) - F_\epsilon^2(u_2)\|_m \leq \|\mathcal{P} \{\mathcal{I}_\epsilon (\{\mathcal{I}_\epsilon \{\mathcal{I}_\epsilon \{u_1^\epsilon \cdot \nabla \} \{u_1^\epsilon - u_2^\epsilon\}\} - \{\mathcal{I}_\epsilon \{u_1^\epsilon \cdot \nabla \} \{u_1^\epsilon - u_2^\epsilon\}\}\} \|_m + \|\mathcal{P} \{\mathcal{I}_\epsilon (\{\mathcal{I}_\epsilon \{u_1^\epsilon \cdot \nabla \} \{u_1^\epsilon - u_2^\epsilon\}\} \|_m + \|\mathcal{I}_\epsilon \{u_1^\epsilon - u_2^\epsilon\}\|_m \|\mathcal{I}_\epsilon \{D u_1^\epsilon - D u_2^\epsilon\}\|_m \leq c \left( e^{-N/2-1}\|u_1^\epsilon\|_0\|u_1^\epsilon - u_2^\epsilon\|_m + e^{-N/2-m-1}\|u_1^\epsilon - u_2^\epsilon\|_0\right) \leq \frac{c}{\epsilon^{N/2+1+m}} \|u_1^\epsilon\|_0 \|u_2^\epsilon\|_0 \|u_1^\epsilon - u_2^\epsilon\|_m.$$

$lacksquare$

Remark 1.8. Note that (1.6) and (1.7) implies by using $u_2^\epsilon = 0$ and $u_1^\epsilon = u^\epsilon$:

$$\|F^\epsilon(u^\epsilon)\|_m \leq C(\|u^\epsilon\|_0, \epsilon, N, m)\|u^\epsilon\|_m$$

(1.8)
Proposition 1.9. Consider any initial condition $u_0 \in V^m$, $m \in \mathbb{Z}^+ \cup \{0\}$. Then for any $\varepsilon > 0$, there exists a unique solution $u^\varepsilon \in C^1([0,T^\varepsilon];V^m)$ to (1.4), where $T^\varepsilon = T(\|u_0\|_m,\varepsilon)$. On any time interval $[0,T]$ for which the solution belongs to $C^1([0,T];V^0)$,

$$\sup_{0 \leq t \leq T} \|u^\varepsilon\|_0 \leq \|u_0\|_0$$

Proof. Choose $O \subset V^m$ a ball of radius $M$ that contains $u = u_0$. From Lemma 1.4 it follows that $F_\varepsilon$ is locally Lipschitz in $M$, and therefore from Picard Theorem Lemma 1.1, there exists sufficiently small $T^\varepsilon > 0$, depending on $\|u_0\|_m$ and $\varepsilon$, so that there exists a unique solution $u^\varepsilon \in C^1([0,T^\varepsilon],O)$ to (1.4). This is the only solution in $C^1([0,T^\varepsilon],V^m)$ since for sufficiently small $T^\varepsilon$ continuity implies that $\|u^\varepsilon - u_0\|_m$ is small.

To show the second part of the Theorem, we note that on taking the $L^2$ inner product of (1.4) with $u^\varepsilon$, we obtain on using properties of mollifiers and projections (see Lemmas 1.12 and 2.1 in week 6 notes)

$$\frac{d}{dt} \|u^\varepsilon\|_0^2 = \nu \left( u^\varepsilon, \mathcal{I}_\varepsilon \Delta u^\varepsilon \right)_0 - (u^\varepsilon, \mathcal{P}_\varepsilon [(\mathcal{I}_\varepsilon u^\varepsilon) \cdot \nabla])_0$$

$$= -\nu (\mathcal{I}_\varepsilon u^\varepsilon, \mathcal{I}_\varepsilon \Delta u^\varepsilon) - (\mathcal{I}_\varepsilon u^\varepsilon, (\mathcal{I}_\varepsilon u^\varepsilon) \cdot \nabla)(\mathcal{I}_\varepsilon \nabla u^\varepsilon)_0$$

Now since $\nu \equiv \mathcal{I}^* u^\varepsilon$ is divergence free, it follows that

$$(\nu^\varepsilon, (\nu^\varepsilon \cdot \nabla)\nu^\varepsilon)_0 = 0$$

just as in the usual Navier-Stokes equation. So,

$$\frac{d}{dt} \|u^\varepsilon\|_0^2 + \nu \|\mathcal{I}_\varepsilon Du^\varepsilon\|_0^2 = 0$$

Therefore,

$$\|u^\varepsilon\|_0^2 \leq \|u_0\|_0^2$$

and the second Lemma statement follows.

Theorem 1.1. Global Existence for regularized N-S equation

For any $T > 0$ and initial condition $u_0 \in H^m$, the regularized Navier Stokes equation (1.4) has a solution $u^\varepsilon \in C^1([0,T],H^m)$.

Proof. First, we note from (1.4), (1.8) that

$$\frac{d}{dt} \|u^\varepsilon\|_m^2 = (u^\varepsilon, \partial_t u^\varepsilon)_m = (u^\varepsilon, F^\varepsilon(u^\varepsilon))_m \leq C(\|u_0\|_0, \varepsilon)\|u^\varepsilon\|_m^2$$

Therefore,

$$\|u^\varepsilon(\cdot,t)\|_m \leq \|u_0\|_m e^{CT}$$

Now for any given $T$ and initial condition $u_0$, choose $M$ so large that $\|u_0\|_m e^{CT} < M$. Then, consider the open set

$$O \equiv \{ u^\varepsilon : u^\varepsilon \in V^m, \|u^\varepsilon\|_m < M \}$$

We know local $C^1([0,T],O)$ solution exists from previous proposition (1.8). From Lemma 1.3 this solution can be continued to $[0,T]$ unless $u^\varepsilon$ escapes the ball $O$. However, from the energy estimate above, this can’t happen.
Lemma 1.11. Stokes equation locally in time.

Navier-Stokes equation guaranteed by Theorem 1.1 satisfies the following inequality

\[ \frac{d}{dt} \| u^\varepsilon \|_{m}^2 + \nu \| \mathcal{L}_x \nabla u^\varepsilon \|_{m}^2 \leq c_m \| \nabla \mathcal{L}_x u^\varepsilon \|_{\infty} \| u^\varepsilon \|_{m}^2 \]

Nonetheless, this allows us one to take \( \varepsilon \) to 0 and obtain actual solution of Navier-Stokes equation locally in time.

Remark 1.10. Though the solution to the regularized Navier-Stokes equation \( \{1.4\} \) exists for all time, going to the limit \( \varepsilon \to 0 \) is not possible with the energy bounds obtained so far because they depend badly on \( \varepsilon \). So, now we seek energy bounds independent of \( \varepsilon \); this will be possible only locally in time, as shall be seen shortly.

Nonetheless, this allows us to take \( \varepsilon \to 0 \) and obtain actual solution of Navier-Stokes equation locally in time.

Lemma 1.11. \( \varepsilon \) independent Energy bounds for regularized problem:

Let \( u_0 \in V^m \). Then the unique solution \( u^\varepsilon \in C^1 ([0, \infty); V^m) \) to the regularized Navier-Stokes equation guaranteed by Theorem \([77]\) satisfies the following inequality

\[ \frac{d}{dt} \| u^\varepsilon \|_{m}^2 + \nu \| \mathcal{L}_x \nabla u^\varepsilon \|_{m}^2 \leq c_m \| \nabla \mathcal{L}_x u^\varepsilon \|_{\infty} \| u^\varepsilon \|_{m}^2 \]

Further, for \( m > N/2 + 1 \), we obtain for sufficiently small \( T \),

\[ \sup_{t \in [0, T]} \| u^\varepsilon \|_{m} \leq \frac{\| u_0 \|_{m}}{1 - c_m T \| u_0 \|_{m}} = \| u_0 \|_{m} + \frac{\| u_0 \|_{m} c_m T}{1 - c_m T \| u_0 \|_{m}} \]

Proof. We note that for any \( \alpha \), with \( |\alpha| \leq m \),

\[ (D^\alpha u^\varepsilon, \partial_t D^\alpha u^\varepsilon) = (D^\alpha u^\varepsilon, D^\alpha \mathcal{L}_x \Delta u^\varepsilon) - (D^\alpha u^\varepsilon, D^\alpha \mathcal{P} \{ \mathcal{L}_x (|u^\varepsilon \cdot \nabla| \mathcal{L}_x u^\varepsilon) \}) = (D^\alpha u^\varepsilon, D^\alpha \mathcal{P} \{ \mathcal{L}_x (|u^\varepsilon \cdot \nabla| \mathcal{L}_x u^\varepsilon) \}) \]

However, it is clear from properties of \( \mathcal{L}_x \) that

\[ (D^\alpha u^\varepsilon, D^\alpha \mathcal{L}_x \Delta u^\varepsilon) = - (D^\alpha \nabla \mathcal{L}_x u^\varepsilon, D^\alpha \mathcal{L}_x \Delta u^\varepsilon) \]

Further, on defining \( v^\varepsilon \equiv \mathcal{L}_x u^\varepsilon \), we get

\[ (D^\alpha v^\varepsilon, D^\alpha \mathcal{P} \{ \mathcal{L}_x (|v^\varepsilon \cdot \nabla| \mathcal{L}_x u^\varepsilon) \}) = (D^\alpha v^\varepsilon, D^\alpha \mathcal{P} \{ \mathcal{L}_x (|v^\varepsilon \cdot \nabla| \mathcal{L}_x u^\varepsilon) \}) \]

since for any divergence free vector field \( v^\varepsilon \), \( (w, v^\varepsilon \cdot \nabla w) = 0 \). However, taking \( u^\varepsilon = D^\alpha v^\varepsilon \), we obtain from using Lemma 1.12, week 5 lecture notes:

\[ |(D^\alpha v^\varepsilon, D^\alpha \mathcal{P} \{ \mathcal{L}_x (|v^\varepsilon \cdot \nabla| \mathcal{L}_x u^\varepsilon) \})| \leq \| D v^\varepsilon \|_{\infty} \| D^\alpha v^\varepsilon \| \]

for \( m > N/2 + 1 \). Therefore, it follows from \([1.9] + [1.10]\) summing over \( \alpha \), with \( |\alpha| \leq m \), we obtain

\[ \frac{d}{dt} \| u^\varepsilon \|_{m}^2 + \nu \| \mathcal{L}_x \nabla u^\varepsilon \|_{m}^2 \leq c_m \| \nabla \mathcal{L}_x u^\varepsilon \|_{\infty} \| u^\varepsilon \|_{m}^2 \]

Now, for \( m > N/2 + 1 \),

\[ \| \nabla v^\varepsilon \|_{\infty} \leq c \| v^\varepsilon \|_{m} \leq c \| u^\varepsilon \|_{m} \]

Therefore,

\[ \frac{d}{dt} \| u^\varepsilon \|_{m} \leq c_m \| u^\varepsilon \|_{m}^2 \]

Integration gives rise to the desired energy bounds. \( \square \)
2. LOCAL EXISTENCE FOR NAVIER-STOKES EQUATION

We now use the $\epsilon$-independent energy bounds for solutions to mollified Navier-Stokes equation to prove local existence of solution for the actual Navier-Stokes equation. First, we show that it forms a Cauchy sequence in an appropriate space:

**Lemma 2.1.** For $m > N/2 + 2$, consider the family $\{u^\epsilon\}_\epsilon$ of solution to the regularized N-S equation with same initial condition $u^\epsilon(\cdot, 0) = u_0 \in V^m(\mathbb{R}^N)$ over time interval $[0, T]$, where $T < \frac{1}{\epsilon_m ||u_0||_m}$. Note that we have $\epsilon$-independent energy bounds on this time interval. This forms a Cauchy sequence in $C \{[0, T], L^2(\mathbb{R}^3)\}$. Further, there exists a constant $C$ only depending on $||u_0||_m$ and time $T$ so that for all $\epsilon \geq \epsilon' > 0$,

$$\sup_{t \in [0, T]} ||u^\epsilon - u^{\epsilon'}||_0 \leq C \epsilon$$

**Proof.** Using $\frac{d}{dt} u^\epsilon = F_\epsilon(u^\epsilon)$ for $\epsilon = \epsilon$ and $\epsilon = \epsilon'$, subtracting the equation and taking the inner-product in $L^2$, we obtain

$$\frac{d}{dt} \frac{1}{2} ||u^\epsilon - u^{\epsilon'}||_0^2 = \nu \left( ||I^2_\epsilon \Delta u^\epsilon - I^2_\epsilon \Delta u^{\epsilon'}, u^\epsilon - u^{\epsilon'}\right)$$

$$- \left(\rho I^\epsilon \left[ I^\epsilon u^\epsilon \cdot \nabla I^\epsilon u^{\epsilon'} - I^\epsilon u^{\epsilon'} \cdot \nabla I^\epsilon u^\epsilon\right], u^\epsilon - u^{\epsilon'}\right) \equiv T1 + T2$$

We first estimate $T1$:

$$T1 = \nu \left( (I^2_\epsilon - I^2_\epsilon) \Delta u^\epsilon, u^\epsilon - u^{\epsilon'}\right) + \nu \left( I^2_\epsilon \Delta[u^\epsilon - u^{\epsilon'}], u^\epsilon - u^{\epsilon'}\right)$$

Using part (iv) of Lemma 1.12 of week 6 notes, and taking $w = \Delta u^\epsilon$, we obtain

$$||I^2_\epsilon w - I^2_\epsilon w|| \leq ||I^2_\epsilon w - I^\epsilon w|| + ||I^\epsilon w - w|| + ||I^2_\epsilon w - I^\epsilon w|| + ||I^\epsilon w - w||_0 \leq C \epsilon ||w||_1$$

Therefore, using above and integration by parts on the latter term in $T1$, we obtain

$$|T1| \leq C \nu ||u^\epsilon||_3 ||u^\epsilon - u^{\epsilon'}||_0 + \nu ||I^\epsilon \nabla (u^\epsilon - u^{\epsilon'})||_0^2$$

Now, with respect to $T2$, it is convenient to decompose

$$T2 = \left(\rho I^\epsilon \left[ I^\epsilon u^\epsilon \cdot \nabla I^\epsilon u^{\epsilon'}\right], u^\epsilon - u^{\epsilon'}\right) + \left(\rho I^\epsilon \left[ I^\epsilon (u^\epsilon - u^\epsilon) \cdot \nabla I^\epsilon u^\epsilon\right], u^\epsilon - u^{\epsilon'}\right)$$

$$+ \left(\rho I^\epsilon \left[ I^\epsilon (u^\epsilon - u^\epsilon) \cdot \nabla I^\epsilon u^{\epsilon'}\right], u^\epsilon - u^{\epsilon'}\right) + \left(\rho I^\epsilon \left[ I^\epsilon (u^\epsilon - u^\epsilon) \cdot \nabla (I^\epsilon - I^\epsilon) u^\epsilon\right], u^\epsilon - u^{\epsilon'}\right)$$

$$+ \left(\rho I^\epsilon \left[ I^\epsilon (u^\epsilon - u^\epsilon) \cdot \nabla (I^\epsilon - I^\epsilon) u^{\epsilon'}\right], u^\epsilon - u^{\epsilon'}\right) \equiv T2_1 + T2_2 + T2_3 + T2_4 + T2_5$$

Now, we note that for some $C$, independent of $\epsilon$,

$$|T2_1| \leq C \epsilon ||u^\epsilon||_1 ||u^\epsilon - u^{\epsilon'}||_0 ||I^\epsilon \nabla u^\epsilon||_\infty \leq C \epsilon ||u^\epsilon||_m^2 ||u^\epsilon - u^{\epsilon'}||_0$$

$$|T2_2| \leq C \epsilon ||u^\epsilon||_1 ||u^\epsilon - u^{\epsilon'}||_0 ||I^\epsilon \nabla u^\epsilon||_\infty \leq C \epsilon ||u^\epsilon||_m^2 ||u^\epsilon - u^{\epsilon'}||_0$$

$$|T2_3| \leq C ||u^\epsilon - u^{\epsilon'}||_0 ||I^\epsilon \nabla u^\epsilon||_\infty \leq C ||u^\epsilon||_m ||u^\epsilon - u^{\epsilon'}||_0^2$$

$$|T2_4| \leq C \epsilon ||u^\epsilon||_1 ||u^\epsilon - u^{\epsilon'}||_0 ||I^\epsilon \nabla u^\epsilon||_\infty \leq C \epsilon ||u^\epsilon||_m^2 ||u^\epsilon - u^{\epsilon'}||_0$$

For $T2_5$, it is useful to substitute $v = I^\epsilon u^\epsilon$, $w = I^\epsilon (u^\epsilon - u^{\epsilon'})$. Note that $w$ and $v$ is divergence free. Then we note that

$$T2_5 = (v \cdot \nabla w, w) = \int_{x \in \mathbb{R}^N} w_i v_j \partial_{x_i} w_j = 0$$
Therefore, from (2.11) and previous \( \epsilon \) independent bound on \( \|u^\epsilon\|_m \) over an interval \([0, T]\), (in last week’s notes), it follows that

\[
\frac{d}{dt} \|u^\epsilon - u^\prime\|_0 \leq C_m(T) \left( \epsilon + \|u^\epsilon - u^\prime\|_0 \right)
\]

Using Gronwall’s inequality, it follows that there exists some constant \( C \) depending on \( T \) so that for any \( t \in [0, T] \),

\[
\|u^\epsilon(t) - u^\prime(t)\|_0 \leq \epsilon C
\]

Proposition 2.2. If initial condition \( u_0 \in V^m \) for \( m > N/2 + 2 \), then for \( T < \frac{1}{c_m \|u_0\|_m} \), there exists a solution to Navier-Stokes equation \( u \in C \left( [0, T], V^{m'}(\mathbb{R}^N) \right) \), while \( \partial_t u \in C \left( [0, T], V^{m'-2}(\mathbb{R}^N) \right) \) for any \( N/2 + 2 < m' < m \). More over, this solution is classical in the sense that \( u \in C^0 \left( [0, T], C^2(\mathbb{R}^N) \right) \), \( \partial_t u \in C^0 \left( [0, T], C(\mathbb{R}^N) \right) \).

Proof. Assume without loss of generality that \( c' \leq \epsilon \). We note that for \( t \in [0, T] \), \( \|u^\epsilon\|_m \leq C \), independent of \( \epsilon \). From interpolation inequality for Sobolev norms and Lemma (2.11), for any \( t \in [0, T] \),

\[
\|u^\epsilon(t) - u^\prime(t)\|_m \leq C ||u^\epsilon(t) - u^\prime(t)||_0^{1-m'/m} \leq C \|u^\epsilon(t) - u^\prime(t)\|_m \leq C_m(T)\epsilon^{1-m'/m}
\]

Thus, \( u^\epsilon \) forms a Cauchy sequence in \( C^0 \left( [0, T], V^{m'}(\mathbb{R}^N) \right) \) and hence converges to a function \( u \) in the same space. Since \( m' > N/2 + 2 \), it follows that \( u \in C^0 \left( [0, T], C^2(\mathbb{R}^N) \right) \). Further, by taking the limit of \( \epsilon \to 0 \) it follows that

\[
\lim_{\epsilon \to 0} \nu \left( T^2 \Delta u^\epsilon - \mathcal{P} \mathcal{L}_u [\mathcal{L}_u \cdot \nabla \mathcal{L}_u u^\epsilon] \right) = \nu \Delta u - \mathcal{P} [u \cdot \nabla u]
\]

Therefore,

\[
\lim_{\epsilon \to 0} u^\epsilon = \nu \Delta u - \mathcal{P} [u \cdot \nabla u]
\]

and the limiting function satisfies Navier-Stokes equation. Since \( \lim_{\epsilon \to 0} u^\epsilon = u \) in \( C^0 \left( [0, T], V^{m'}(\mathbb{R}^N) \right) \), it follows that at least in the sense of distribution, we have \( \lim_{\epsilon \to 0} u^\epsilon = u_t \). Therefore, the limiting function \( u \) satisfies the Navier-Stokes equation and satisfies initial condition \( u_0 \). From the equation itself, it follows that we have \( u_t \in C^0 \left( [0, T], V^{m'-2}(\mathbb{R}^N) \right) \). ☐

Remark 2.3. The above proposition is not completely satisfactory since it suggests that if \( u_0 \in V^m \), then it only assures \( u(t) \in V^{m'} \), for \( m' < m \). In reality \( u(t) \in V^m \) as well. However, to show this we need to work a bit harder.

Definition 2.4. A sequence \( \{v_n\}_n \) in a Hilbert Space \( \mathcal{H} \) is said to converge weakly to \( v \) if for any \( w \in \mathcal{H} \), \( \lim_{n \to \infty} \langle w, v_n \rangle = \langle w, v \rangle \).

A property of weakly convergent sequence that will be important for us is that they are also bounded. Also, the following Theorem proved in any standard text in analysis is useful:

Theorem 2.1. (Banach-Alogou Theorem) Any bounded sequence in a Hilbert Space has a subsequence that converges weakly.

(1) More generally a reflexive Banach Space, in which case the definition of weak convergence involves the dual space.
Remark 2.5. In our context, the Hilbert Space $\mathcal{H} = V^m$.

Definition 2.6. A function $v(., t) \in B$, a reflexive Banach Space, is said to be weakly continuous for $t \in [0, T]$ if for any $w \in B^*$, the dual Banach space, we have $< w, v(., t) >$ a continuous function of $t \in [0, T]$.

Theorem 2.2. Local Existence of N-S solutions Let $u$ be the solution described by the previous proposition. Then

$$v \in C([0, T], V^m) \cup C^1_1([0, T], V^{m-2})$$

Proof. We know from prior energy estimates on the regularized Navier-Stokes equation that

$$\sup_{t \in [0, T]} \| u^\epsilon \|_m \leq M$$

and from the regularized N-S equation itself, it follows that

$$\sup_{t \in [0, T]} \| u^\epsilon \|_{m-2} \leq M_1$$

for some constants $M$ and $M_1$. Since $\{ u^\epsilon \}_{\epsilon = 1/n}$ is a bounded sequence in the Hilbert Space $L^2([0, T], V^m)$. Theorem 2.4 implies that there exists a subsequence which converges to $u \in L^2([0, T], V^m)$, as $n \to \infty$ ($\epsilon \to 0$). This must be the same $u$ as in Proposition 2.2 since $V^m \subset V^m$ and $\lim_{n \to 0} u^\epsilon = u$ in $C([0, T], V^m)$. Further, for each $t \in [0, T]$, since $u^\epsilon$ is a bounded sequence in the Hilbert Space $V^m$, there is a subsequence that converges to $u(., t) \in V^m$. Thus, it follows that

$$u \in L^\infty([0, T], V^m)$$

Further, we claim

$$u \in C_W([0, T], V^m)$$

First, we note that for $0 < m' < m$, the space $V^{-m'}$ is dense in $V^{-m}$. Hence we take arbitrary $\phi \in V^{-m'}$ and note that $< \phi, u(., t) >$ is continuous in $t$ for $t \in [0, T]$, because $u \in C([0, T], V^m)$. Therefore, the claim follows.

In view of weak continuity, we note that

$$\lim_{\delta \to 0} (u(., t + \delta) - u(., t), u(., t + \delta) - u(., t))_m = \lim_{\delta \to 0} (\| u(., t + \delta) \|_m^2 - \| u(., t) \|_m^2)$$

Thus to show $u \in C([0, T], V^m)$, it is enough to show $\| u(., t) \|_m$ is continuous.

We first prove the right continuity at $t = 0$. We choose $\phi_0 \in V^{-m}$ so that $\| \phi_0 \| = < \phi_0, u_0 >$. Then from weak continuity,

$$\lim_{t \to 0^+} \langle \phi_0, u(., t) \rangle = \| \phi_0 \|_m^2$$

Therefore,

$$\liminf_{t \to 0^+} \| u(., t) \|_m \geq \| u_0 \|_m$$

Now, from energy bounds,

$$\sup_{t \in [0, T]} \| u(., t) \|_m - \| u_0 \|_m \leq \frac{\| u_0 \|_m^2 c_m T}{1 - c_m T \| u_0 \|_m}$$

Therefore,

$$\limsup_{t \to 0^+} \| u(., t) \| \leq \| u_0 \|_m$$

So, right continuity of $\| u(., t) \|_m$ has been proved at $t = 0$. It is clear that for any $t \in [0, T]$, we can repeat the same argument to show the right continuity.
To show left continuity, we have to deal differently for \( \nu = 0 \) (Euler Equation) and \( \nu > 0 \).

For \( \nu = 0 \), the equations are time reversible, meaning that if we replace \( t \) by \(-t\) and \( u \) by \(-u\), we get back the same (Euler) equation. So, left continuity follows from the same argument as the one above for right continuity.

For \( \nu > 0 \), we recall the energy inequality for \( t \in [0, T] \):

\[
\|u'(., t)\|_{m}^{2} + \nu \int_{0}^{T} \|I_{\epsilon} \nabla u'(., \tau)\|_{m}^{2} d\tau = \|u'(., 0)\|_{m}^{2},
\]

implying that there exists \( C \) independent of \( \epsilon \) so that

\[
\nu \int_{0}^{T} \|I_{\epsilon} \nabla u'(., t)\|_{m}^{2} dt \leq C
\]

Since \( \|I_{\epsilon} v\|_{m+1} \rightarrow \|v\|_{m+1} \) as \( \epsilon \rightarrow 0^{+} \), it follows that \( \{u^{\epsilon}\}_{\epsilon = 1/n} \) is a bounded sequence in the Hilbert space \( L^{2}([0, T], V^{m+1}) \). It follows that there is a subsequence that converges to \( u \) as \( \epsilon \rightarrow 0 \). This implies that for almost any \( t \in [0, T] \), \( u(., t) \in V^{m+1} \). Suppose we want to show left continuity at \( t = T_{1} \in [0, T] \). We choose \( T_{1} > T_{0} > 0 \) so that solution \( u(., T_{0}) \in V^{m+1} \). Then, starting at \( T = T_{0} \), we continue. We can apply Proposition 2.2 with initial condition \( u(., T_{0}) \) This ensures solution in \( C([T_{0}, T], V^{\tilde{m}}) \) for \( \tilde{m} < m+1 \) for some \( T' > T_{0} \). However, from uniqueness of classical solution, it follows that this is the same solution \( u \in C([0, T], V^{m'}) \) for \( m' < m \) guaranteed by Proposition 2.2 Therefore, \( u \in C([T_{0}, T], V^{\tilde{m}}) \) can be continued past \( T' \) if \( \|u(., T')\|_{m+1} < \infty \) Indeed, \( T' \) can be extended to be as large as we like so long as \( \|u(., t)\|_{m+1} \) remains finite for \( t \in [T_{0}, T'] \).

However, in the process of derivation of \( \epsilon \) independent energy bounds, we notice that as long as \( \|I_{\epsilon} \nabla u'\|_{\infty} < C \), where \( C \) is independent of \( \epsilon \), then so is \( \|u'(., t)\|_{m+1} \). However, \( \|I_{\epsilon} \nabla u'(., t)\|_{\infty} \leq c \|u'(., t)\|_{m} < C \), independent of \( \epsilon \) for \( t \in [0, T] \). Therefore, \( \|u(., t)\|_{m+1} < \infty \) for \( t \in [T_{0}, T'] \) for any \( T' \leq T \). Therefore \( u \in C([T_{0}, T], V^{\tilde{m}}) \) for any \( \tilde{m} < m+1 \) and in particular for \( \tilde{m} = m \). Hence the left continuity of \( \|u(., t)\|_{m} \) at \( T = T_{1} \) follows. 

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