Smooth Navier Stokes Solution–a millenium Problem

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Collaborators in joint work in this area: O. Costin, G. Luo

3-D Navier-Stokes (NS) problem

$$v_t + (v \cdot \nabla) v = -\nabla p + \nu \Delta v + f \quad ; \quad \nabla \cdot v = 0,$$

where $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ is the fluid velocity and $p \in \mathbb{R}$ pressure at $x = (x_1, x_2, x_3) \in \Omega$ at time $t \geq 0$. Further, the operator $(v \cdot \nabla) = \sum_{j=1}^{3} v_j \partial_{x_j}$, ν = nondimensional visocity (inverse Reynolds number)

The problem supplemented by initial and boundary conditions:

$$v(x, 0) = v^{(0)}(x) \ \ (IC) \ , \ v = 0 \text{ on } \partial\Omega \text{ for stationary solid boundary}$$

We take $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3[0, 2\pi]$; no-slip boundary condition avoided, but assume in the former case $\|v^{(0)}\|_{L^2(\mathbb{R}^3)} < \infty$.

Millenium problem: Given smooth $v^{(0)}$ and f, prove or disprove that there exists smooth 3-D NS solution v for all $t > 0$. Note: global solution known in 2-D.

NS - a fluid flow model; importance of blow-up

$$v_t + (v \cdot \nabla)v = -\nabla p + \nu \Delta v + f \quad ; \quad \nabla \cdot v = 0,$$

Navier-Stokes equation models incompressible fluid flow.

$v_t + (v \cdot \nabla)v \equiv \frac{Dv}{Dt}$ represents fluid particle acceleration. The right side (force/mass) can be written: $\nabla \cdot \mathrm{T} + f$, where T: a tensor of rank 2, called stress with

$$\mathbf{T}_{jl} = -p\delta_{jl} + \frac{\nu}{2}\left[\frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j}\right]$$

The second term on the right is viscous stress approximated to linear order in ∇v. Invalid for large $||\nabla v||$ or for non-Newtonian fluid (toothpaste, blood)

Incompressibility not valid if v comparable to sound velocity

If NS exhibited blow up, the model itself becomes invalid; terms not included in NS approximation potentially important.

Definition of Spaces of Functions

$H^m(\mathbb{R}^3)$: completion of C_0^∞ functions under the norm

$$\|\phi\|_{H^m} = \left\{ \sum_{0 \le l_1+l_2+l_3 \le m} \| \frac{\partial^{l_1+l_2+l_3}\phi}{\partial x_1^{l_1} \partial x_2^{l_2} \partial x_3^{l_3}} \|_{L_2}^2 \right\}^{1/2}$$

Note $H^0 = L_2$ If ϕ is a vector or tensor, components are also involved in the summation. Note: $\|.\|_{H^m}$ usually called norms.

$H^m(\mathbb{T}^3[0, 2\pi])$: Completion under the above norm of C^∞ periodic functions in $x = (x_1, x_2, x_3)$ with 2π period in each direction.

$L_p\left([0, T], H^m(\mathbb{R}^3)\right)$ will denote the completion of the space of smooth functions of (x, t) under the norm:

$$\|v\|_{L_{p,t}H_{m,x}} \equiv \| \|v(., t)\|_{H^m} \|_{L_p}$$

Basic Steps in a typical evolutionary PDE analysis

Construct an approximate equation for $v^{(\epsilon)}$ that formally reduces to the PDE as $\epsilon \longrightarrow 0$ such that ODE theory guarantees solution $v^{(\epsilon)}$

Find *a priori* **estimate on v that satisfies PDE and also obeyed by $v^{(\epsilon)}$**

Use some compactness argument to pass to the limit $\epsilon \longrightarrow 0$ to obtain local solution of PDE

If *a priori* **bounds on appropriate norms are globally controlled, then global solution follows. One way to get to classical (strong) solutions is to have** *a priori* **bounds on $\|v(.,t)\|_{H^m}$ for any m large enough.**

For weak solutions, starting point is an equation obtained through inner product (in L_2) with a test function.

Some basic observations about Navier Stokes

For $f = 0$, $\Omega = \mathbb{R}^3$, if $v(x,t)$ is a solution, so is
$v_\lambda(x,t) = \frac{1}{\lambda} v\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$.

A space-time norm $\|.\|$ is called sub-critical if for $\lambda > 1$,
$\|v_\lambda\| = \lambda^{-q}\|v\|$ **for some $q > 0$. If the above is for $q = 0$, critical.**
If the above is true for $q < 0$, the norm termed super-critical

Basic Energy Equality for $f = 0$:

$$\frac{1}{2}\|v(.,t)\|_{L_2}^2 + \nu \int_0^t \|\nabla v(.,t'\|_{L_2}^2 dt' = \frac{1}{2}\|v^{(0)}\|_{L_2}^2$$

Therefore, for following *super-critical* norms over time interval $[0,T]$:

$$\|v\|_{L_{\infty,t}L_{2,x}} \le \|v^{(0)}\|_{L_2} \quad , \quad \|v\|_{L_{2,t}H_x^1} \le C$$

These are the only two known globally controlled quantities

More *a priori* bounds for $f = 0$

Taking the gradient of unforced NS-equation j times, doing an L_2 inner-product with $D^j v$ and summing over all indices j upto m we obtain:

$$\frac{d}{dt}\frac{1}{2}\|v(.,t)\|_{H^m}^2 + \nu\|\nabla v\|_{H^m}^2 \le c_m\|\nabla v(.,t)\|_{L_\infty}\|\|v(.,t)\|_{H^m}^2$$

If $m > \frac{5}{2}$, then Sobolev inequality gives $\|\nabla v(.,t)\|_{L_\infty}\| \le C_m\|v(.,t)\|_{H^m}$, meaning that we obtain from above:

$$\frac{d}{dt}\|v(.,t)\|_{H^m} \le C_m\|v(.,t)\|_{H^m}^2, \text{ so } \|v(.,t)\|_{H^m} \le \frac{\|v^{(0)}\|_{H^m}}{1 - tC_m\|v^{(0)}\|_{H^m}}$$

Note the right hand side blows up at $t = T^* = \frac{1}{C_m\|v^{(0)}\|_{H^m}}$

Results by Leray

Leray (1933a,b, 1934) made seminal contributions:

A solution exists, though uniqueness unknown, in
$L_\infty\left((0,T), L_2(\mathbb{R}^3)\right) \cap L_2\left((0,T), H^1(\mathbb{R}^3)\right)$ for any $T > 0$.

For regular f and $v^{(0)}$, unique smooth solution in $(0, T^*)$. For $f = 0$, Leray's weak solution becomes smooth again for $t > T_c$

For $t \in (0, T^*)$, weak and strong solution the same. Only small $v^{(0)}$, f or large viscosity gives $T^* = \infty$

$\int_0^T \|\nabla v(., t)\|_\infty dt < \infty$ guarantees smooth solution on $(0, T]$.

Leray conjectured formation of singular 1-D line vortices where $\nabla \times v$ blows up at some time t_0.

Also conjectured blow up for $f = 0$ via similarity solution

$$v(x,t) = (t_0 - t)^{-1/2} V\left(\frac{x}{(t_0 - t)^{1/2}}\right)$$

Some known important results -II

Cafarelli-Kohn-Nirenberg (1982): 1-D Hausdorff measure of the singular space-time set for Leray's weak solution is 0.

Necas-Ruzicka-Sverak (1996): no Leray similarity solution for $v^{(0)} \in L^3$. Tsai (2003): no Leray-type similarity solution with finite energy and finite dissipation.

Beale-Kato-Majda (1984): $\int_0^T \|\nabla \times v(.,t)\|_\infty dt < \infty$ guarantees smooth v over $[0,T]$

Other controlling norms by Prodi-Serrin-Ladyzhenzkaya and Escauriaza, Seregin & Sverak (2003): $\|.\|_{L_{p_t} L_{s,x}}$ for $\frac{3}{s} + \frac{2}{p} = 1$ for $s \in [3, \infty)$.

Constantin-Fefferman (1994): If $\frac{\nabla \times v}{|\nabla \times v|}$ is uniformly Holder continuous in x in a region where $|\nabla \times v| > c$ for a sufficiently large c for $t \in (0,T]$, then smooth N-S solution exists over $(0,T]$

Difficulty with Navier-Stokes in the usual PDE analysis

Nonlinearity strong unless ν is large enough for given $v^{(0)}$ and f. Rules out perturbation about linear problem.

$\nu = 0$ approximation (3-D Euler equation) formidable, though other techniques available. Rules out perturbative treatment.

The norms that are controlled globally are all super-critical: does not give sufficient control over small scales.

Other techniques include introduction of ϵ regularizations like hyperviscosity, compressibility, etc. and taking limit $\epsilon \rightarrow 0$

Maddingly-Sinai (2003): if $-\Delta$ is replaced by $(-\Delta)^{\alpha}$ in N-S equation, and $\alpha > \frac{5}{4}$ then global smooth solution exists.

Tao (2007) believes that no "soft" estimate can work including introduction of regularization. Believes global control on some critical or subcritical norm a must.

An alternate approach

Sobolev methods give no information about solution at $t = T^*$ when *a priori* Energy estimates breakdown.

A more constructive approach is to use Borel summation ideas for specific $v^{(0)}$, f and ν. We consider $x \in \mathbb{T}^3[0, 2\pi]$

Borel summation, under some conditions, generates an isomorphism between formal series and actual functions they represent. (Ecalle, ..., O. Costin).

Formal expansion of N-S solution possible for small t:
$v(x,t) = v^{(0)}(x) + \sum_{m=1}^{\infty} t^m v^{(m)}(x)$.

Borel Sum of this series, which is sensible for analytic $v^{(0)}$ and f, leads to an actual solution to N-S (O. Costin & S. Tanveer, '06) in the form: $v(x,t) = v^{(0)}(x) + \int_0^{\infty} e^{-p/t} U(x,p) dp$. This form transcends assumptions on analyticity of $v^{(0)}$ and f or of t small

Borel based approach -II

The Fourier-Transform $\mathcal{F}\left[U(.,p)\right](k) \equiv \hat{U}(k,p)$ satisfies an integral equation:

$$\mathrm{U}(\mathrm{k},p) = \int_0^p K(p,p')\hat{\mathrm{R}}(\mathrm{k},p')dp' := \mathcal{N}\left[\hat{U}\right](k,p)$$

$$\hat{\mathrm{R}}(k,p) = -ik_j P_k\left[\hat{v}_{0,j}\hat{*}\hat{\mathrm{U}} + \hat{U}_j\hat{*}\hat{\mathrm{v}}_0 + \hat{U}_j \overset{*}{*} \hat{\mathrm{U}}\right] + \hat{\mathrm{v}}_1\delta(p)$$

where, $P_k = \left(1 - \frac{k(k\cdot)}{|k|^2}\right)$, $\overset{*}{*}$ denote Laplace convolution, followed by Fourier convolution. $K(p,p')$, $\hat{\mathrm{v}}_1(\mathrm{k})$ given by:

$$K(p,p') = \frac{\pi}{z}\left(z'J_1(z)Y_1(z') - z'Y_1(z)J_1(z')\right)\ , z = 2|\mathrm{k}|\sqrt{p},$$

$$z' = 2|\mathrm{k}|\sqrt{p'}\ \ ,\ \ \hat{\mathrm{v}}_1(k) = -|\mathrm{k}|^2\mathrm{v}_0 - ik_jP_k\left[\hat{v}_{0,j}\hat{*}\hat{\mathrm{v}}_0\right] + \hat{\mathrm{f}}(\mathrm{k})$$

Generalized Laplace Representation and Results

It is useful to consider a more general representation:

$$v(x,t) = v^{(0)}(x) + \int_0^\infty U(x,q) e^{-q/t^n} dq$$

Gives rise to an integral equation similar to that for $n = 1$

Have proved (with O. Costin, G. Luo):

1. For regular enough $v^{(0)}$ and f, there exists unique solution $\hat{U}(k,q)$ to the integral equation $\hat{U} = \mathcal{N}\left[\hat{U}\right]$ for functions for which $\int_0^\infty e^{-\alpha q} \|\hat{U}(.,q)\|_{l^1} dq < \infty$ for some $\alpha \geq 0$. Generates smooth NS-solution in $(0, \alpha^{-1/n})$ satisfying I.C.

2. If solution $\hat{U}$ decays for large q, global NS existence follows. On the other hand, if global smooth NS solution exists, then for some large enough n, $\|\hat{U}(.,q)\|_{l^1}$ decreases exponentially in q.

More results using Integral equation approach:

Consider solution based on a finite dimensional Galerkin projection in Fourier-Space and uniform discretization in q of the integral equation:

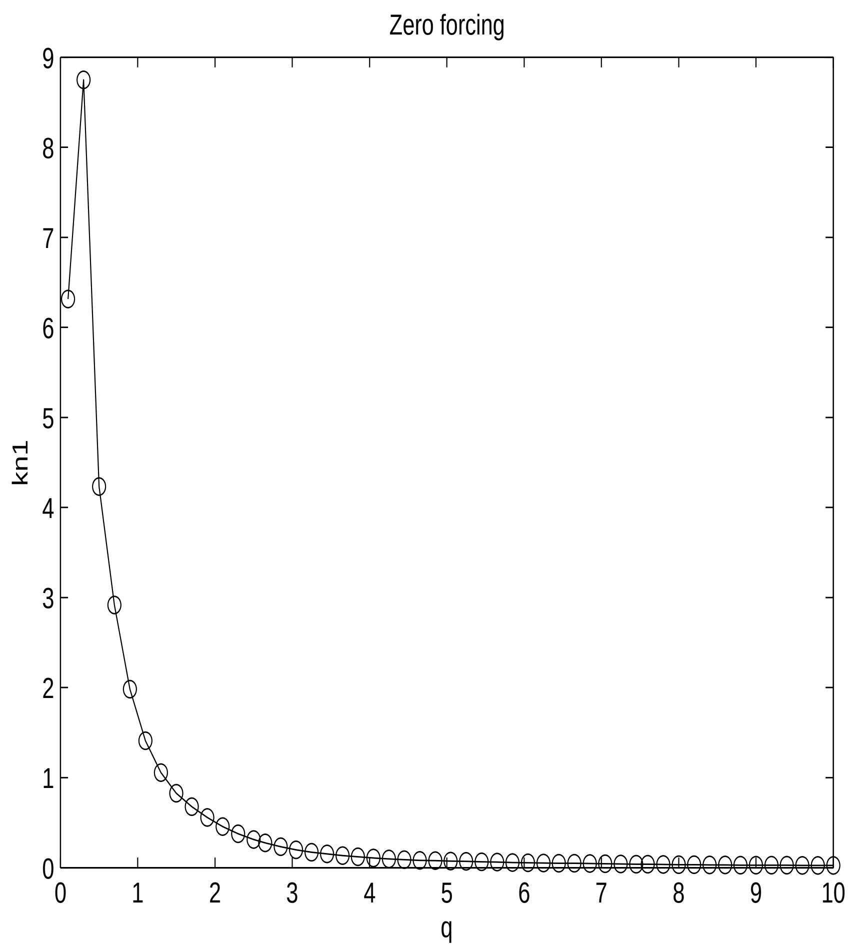
$$\hat{U}_\delta^{(N)} = \mathcal{N}_\delta^{(N)} \left[\hat{U}_\delta^{(N)} \right]$$

3. We proved $\|\hat{U} - \hat{U}_\delta^{(N)}\| \to 0$ as $N \to \infty$, and $\delta \to 0$

4. For given solution in a finite interval $[0, q_0]$, computed numerically or otherwise, a revised asymptotic bound on exponent α is possible based on solution behavior in $[0, q_0]$. This can give rise to long existence time $(0, \alpha^{-1/n})$ for NS.

For given $f = 0$ and $v^{(0)}$ and ν, depending on computed $[0, q_0]$ behavior of v, one an choose in principle δ small enough and large enough N, q_0 so that resulting $\alpha^{-1/n} > T_c$, the critical time beyond which Leray's weak solution becomes smooth.

$\|\hat{U}(.,q)\|_{l^1}$ vs. q, $n = 2$, $\nu = 0.1$



Kida I.C. $v_1^{(0)} = \sin x_1 (\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3)$

Other components from cyclic relation:

$v_1^{(0)}(x_1, x_2, x_3) = v_1^{(0)}(x_3, x_1, x_2) = v_3^{(0)}(x_2, x_3, x_1)$

Conclusions

Global existence problem for smooth 3-D Navier-Stokes solution remains a difficult problem, despite extensive research.

No obvious small or large parameter. Nonlinearity strong except for very large viscosity.

Known globally controlled norms are all super-critical that do not give enough control over small scales.

Alternate Borel based methods casts the global existence problem to an asymptotic problem for a smooth solution to a nonlinear integral equation that is known to exist *a priori***.**

The solution to the integral equation over $[0, q_0]$ can be computed numerically with rigorous error control for specific $v^{(0)}$, ν and f and can be used to obtain better asymptotic bounds at $q = \infty$. Depending on features of computed solution in $[0, q_0]$, this can result in provably large existence time for N-S solution.