GLOBAL SOLUTIONS FOR TWO-PHASE HELE-SHAW BUBBLE FOR A NEAR-CIRCULAR INITIAL SHAPE

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ABSTRACT. Using equal arc-length vortex sheet formulation, we prove global existence of solutions in a two phase Hele-Shaw flow with surface tension for near-circular sufficiently smooth initial interface shape. Further, the circular bubble is shown to be asymptotically stable to all sufficiently smooth initial perturbation.

1. INTRODUCTION.

The displacement of a more viscous fluid by a less viscous one in a Hele-Shaw cell has been a problem of considerable physical as well as mathematical interest. Over the years, many reviews have appeared from a range of perspectives (Saffman [32], Bensimon *et al.* [7], Homsy [17], Pelce [27], Kessler *et al.* [24], Tanveer [34], [35]; Hohlov [16], Howison [21], [22]).

There is a vast literature on the zero surface tension problem though the initial value problem in this case is ill-posed [20], [14] and not always physically relevant [See [35] for detailed discussion of this issue]. With surface tension, there are rigorous local existence results for general initial conditions both for one and two phase problems [10], [12] using different approaches. It is recognized that the global existence problem with surface tension for arbitrary initial shape is a difficult open problem, though there is quite a substantial literature involving formal asymptotic and numerical computations (see cited reviews above). Even the restricted problem of stability of steadily propagating shapes such as a semi-infinite finger [37], [38] or a finite translating bubble [38] for nonzero surface tension remains an open problem for rigorous analysis. Translation causes complications in global analysis due to a less viscous fluid displacing a more viscous fluid – a planar front is known to be unstable [31] in this case.

There are however some global existence and nonlinear stability results [9], [15] for one-phase and two phase Hele-Shaw for near circular initial shapes in the absence of any forcing such as fluid injection or pressure gradient. These have been generalized to non-Newtonian one phase fluids [11]. There are similar results available for the two phase Stefan problem [13], [29], which is mathematically close to but distinct from the two-phase Hele-Shaw (also called Muskat problem) being studied here.

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The unforced two-phase Hele-Shaw problem is described mathematically as follows: let $\Omega(t) \subset \mathbb{R}^2$ be a simply connected bounded domain occupied by fluid with viscosity μ_2 at time t, while a different fluid of viscosity μ_1 occupies the exterior region $\mathbb{R}^2 \setminus \overline{\Omega}$. We define functions ϕ_1 and ϕ_2, outside and inside Ω such that

$$\begin{aligned} \Delta\phi_1 &= 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ \Delta\phi_2 &= 0 \text{ in } \Omega, \end{aligned}$$

with $\phi_1 \to$ constant, as $(x, y) \to \infty$.

On the free boundary $\partial\Omega(t)$ between two fluids, we require two conditions:

$$\mu_1\phi_1 - \mu_2\phi_2 = \sigma\kappa,$$
$$\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n} = v_n,$$

σ is the coefficient of surface tension, $\mathbf{n}$ is the inward unit normal vector on $\partial\Omega(t)$, and v_n is the normal velocity of the interface.

For the problem defined above, global existence for near-circular and analytic initial shape $\partial\Omega(0)$ has been established by Constantin & Pugh [9] for $\mu_2 = 0$, *i.e.* the one-phase problem. They also showed that the circle is asymptotically stable to sufficiently small analytic disturbances. More recently Friedman & Tao [15] proved the similar result for the two-phase Hele-Shaw problem in the exterior of a small circle. While they allow initial shapes to be non-analytic, they are highly constrained.

In the present paper, we extend the Friedman & Tao [15] results to more general non-analytic initial conditions though in the absence of any walls. Our methodology is also different and uses a boundary integral formulation due to Hou *et al* [18]. This formulation has been widely used for numerical calculations for a wide variety of free boundary problems involving Laplace's equation. Ambrose [3] has recently used this formulation to prove local existence for the Hele-Shaw flow of general initial shapes [3] without surface tension. Given the wide use of boundary integral methods in computations, one motivation for the present paper is to further develop the mathematical machinery associated with this method so as to be applicable to more general existence problems. Indeed, in another paper [40], we use some lemmas proved here for global existence results for the much more difficult problem of a translating bubble in a Hele-Shaw channel in the presence of a pressure gradient for any nonzero surface tension.

Adapting the equal arc-length vortex sheet formulation of Hou *et al* [18] to the present geometry, the boundary curve between the two fluids of differing viscosities is described parametrically at any time t by $z = x(\alpha, t) + iy(\alpha, t)$, where α is chosen so that $z(\alpha + 2\pi, t) = z(\alpha, t)$. θ is defined so that $\alpha + \theta$ for the angle formed between the tangent to the curve and the horizontal (x-axis), as the boundary is traversed counter-clockwise with increasing α. Hou, Lowengrub and Shelley in [19] observed that a choice[1] of the tangent velocity T is possible so that s_α becomes independent of α. Here s denotes arc-length. They also observed that this choice simplifies the evolution equation for θ.

It is convenient to introduce the map $\Phi : \mathbb{R}^2 \to \mathbb{C}$ by $\Phi(a, b) = a + ib$. Then the velocity $\mathbf{W}$ (see [18]) generated by a vortex sheet of strength $\gamma(\alpha)$ on the boundary

[1] This choice or any other choice of tangential speed of points on the interface has no effect on the interface shape itself.

is given by the Birkhoff-Rott integral which has the complex representation:

$$[\Phi(\mathbf{W})]^* = \frac{1}{2\pi i} PV \int_0^{2\pi} \frac{\gamma(\alpha')}{z(\alpha) - z(\alpha')} d\alpha'. \tag{1.1}$$

The unit tangent and normal vectors to the curve clearly satisfy

$$\Phi(\mathbf{t}) = \frac{2\pi z_\alpha}{L},\ \Phi(\mathbf{n}) = \frac{2\pi i z_\alpha}{L}.$$

The normal velocity $U(\alpha, t)$ of the curve is given by

$$U(\alpha, t) = \mathbf{W} \cdot \mathbf{n}. \tag{1.2}$$

It is known [18] that the equations for the evolution of a Hele-Shaw interface in the infinite domain with surface tension is equivalent to the following equations:

$$\begin{cases} \theta_t(\alpha, t) = \dfrac{2\pi}{L} U_\alpha(\alpha, t) + \dfrac{2\pi}{L} T(\alpha, t)\big(1 + \theta_\alpha(\alpha, t)\big), \\ L_t(t) = -\displaystyle\int_0^{2\pi} \big(1 + \theta_\alpha(\alpha, t)\big) U(\alpha, t) d\alpha, \end{cases} \tag{A.1}$$

with

(A.2)

$$\begin{cases} \gamma(\alpha, t) = -\dfrac{L}{\pi} A_\mu \mathbf{W} \cdot \mathbf{t} + \dfrac{2\pi}{L} \sigma \theta_{\alpha\alpha}, \\ T(\alpha, t) = \displaystyle\int_0^\alpha \big(1 + \theta_{\alpha'}(\alpha', t)\big) U(\alpha', t) d\alpha' - \frac{\alpha}{2\pi} \int_0^{2\pi} \big(1 + \theta_\alpha(\alpha, t)\big) U(\alpha, t) d\alpha, \end{cases}$$

where

$$A_\mu = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}.$$

The initial condition is given by

$$\theta(\alpha, 0) = \theta_0(\alpha),\ L(0) = 2\pi. \tag{1.3}$$

In order that $z_\alpha = \frac{L}{2\pi} \exp[i\alpha + i\theta]$, the specified $\theta_0(\alpha)$ must satisfy the consistency condition

$$\int_0^{2\pi} \exp[i\alpha + i\theta_0(\alpha)]\, d\alpha = 0. \tag{1.4}$$

Definition 1.1. *Let $s \geq 0$. The Sobolev space $H^s\big(\mathbb{T}[0, 2\pi]\big)$ is the set of all 2π-periodic function $f = \sum_{-\infty}^{\infty} \hat{f}(k) e^{ik\alpha}$ such that*

$$\|f\|_s = \sqrt{\sum_{k=-\infty}^{\infty} |k|^{2s} |\hat{f}(k)|^2 + |\hat{f}(0)|^2} < \infty.$$

Note 1.2. *For $f, g \in H^s(\mathbb{T}[0, 2\pi])$, the Banach Algebra property $\|fg\|_s \leq C_s \|f\|_s \|g\|_s$ for $s \geq 1$ for some constant C_s depending on s is easily proved and will be useful in the sequel.*

Definition 1.3. *The Hilbert transform, $\mathcal{H}$, of a function $f \in H^0(\mathbb{T}[0,2\pi])$ with Fourier Series $f = \sum_{-\infty}^{\infty} \hat{f}(k) e^{ik\alpha}$ is given by*

$$\begin{aligned}\mathcal{H}[f](\alpha) &= \frac{1}{2\pi} PV \int_0^{2\pi} f(\alpha') \cot \frac{1}{2}(\alpha - \alpha') d\alpha' \\ &= \sum_{k \neq 0} -i \,\mathrm{sgn}(k) \hat{f}(k) e^{ik\alpha}.\end{aligned}$$

Note 1.4. *The Hilbert transform commutes with differentiation. We will denote derivative with respect to α, either by D or subscript α. Also, for the sake of brevity of notation, the time t dependence will often be omitted, except where this might cause confusion.*

Definition 1.5. *We define the operator Λ to be one derivative followed by the Hilbert transform: $\Lambda = \mathcal{H}D$.*

Note 1.6. *It is clear that*

$$\left(\int_0^{2\pi} \left(f^2 + f \Lambda f \right) d\alpha \right)^{1/2}$$

is equivalent to $H^{1/2}\big(\mathbb{T}[0,2\pi]\big)$ norm for a real-valued 2π-periodic function f. Further, if $\hat{f}(0) = 0$, then it is easily seen that $\left(\int_0^{2\pi} f \Lambda f d\alpha \right)^{1/2} = \|f\|_{1/2}$. Note that operator Λ is self-adjoint in $H^{1/2}\big(\mathbb{T}[0,2\pi]\big)$.

Definition 1.7. *Following Ambrose* [3]*, we define commutator*

$$[\mathcal{H}, f] g = \mathcal{H}(fg) - f \mathcal{H}(g).$$

The linear integral operator $\mathcal{K}[z]$, depending on z, is also defined by

$$\left(\mathcal{K}[z] f \right)(\alpha) = \frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} f(\alpha') \left[\frac{1}{z(\alpha) - z(\alpha')} - \frac{1}{2 z_\alpha(\alpha')} \cot \frac{1}{2}(\alpha - \alpha') \right] d\alpha'.$$

Remark. For 2π-periodic functions f and z, it is clear that the upper and lower limits of the integral above can be replaced by a and $a+2\pi$ respectively for arbitrary a. Further, in terms of the operators $\left[\mathcal{H}, \frac{1}{z_\alpha}\right]$ and $\mathcal{K}$, we may express $\mathbf{W}$ in the following form (see [1]):

$$[\Phi(\mathbf{W})]^* = \frac{1}{2i} \left[\mathcal{H}, \frac{1}{z_\alpha} \right] \gamma + \frac{1}{2 i z_\alpha} \mathcal{H} \gamma + \mathcal{K}[z] \gamma. \tag{1.5}$$

□

Definition 1.8. *A complex operator $\mathcal{G}[z]$, depending on z, is defined by*

$$\mathcal{G}[z] \gamma = z_\alpha \left[\mathcal{H}, \frac{1}{z_\alpha} \right] \gamma + 2 i z_\alpha \mathcal{K}\left[z \right] \gamma. \tag{1.6}$$

It is also convenient to define a related real operator $\mathcal{F}[z]$, depending on z, so that

$$\mathcal{F}[z] \gamma = \mathrm{Re} \left(\frac{z_\alpha(\alpha)}{\pi i} PV \int_0^{2\pi} \frac{\gamma(\alpha')}{z(\alpha) - z(\alpha')} d\alpha' \right) = \mathrm{Re} \left(\frac{1}{i} \mathcal{G}[z] \gamma \right). \tag{1.7}$$

From the expressions for U and $\mathbf{W}\cdot\mathbf{t}$, it follows that

$$U = \frac{\pi}{L}\mathcal{H}[\gamma] + \frac{\pi}{L}\operatorname{Re}\left(\mathcal{G}[z]\gamma\right), \tag{1.8}$$

$$\mathbf{W}\cdot\mathbf{t} = \frac{\pi}{L}\mathcal{F}[z]\gamma. \tag{1.9}$$

Definition 1.9. *We introduce the projection operator $\mathcal{Q}_1$ so that*

$$\left[\mathcal{Q}_1 f\right](\alpha) = f(\alpha) - \hat{f}(0) - \hat{f}(1)e^{i\alpha} - \hat{f}(-1)e^{-i\alpha},$$

where $f = \sum_{-\infty}^{\infty} \hat{f}(k)e^{ik\alpha}$. In general, $\hat{:}$ symbol will be reserved for Fourier components. Further, we will denote $\tilde{\theta} = \mathcal{Q}_1\theta$.

Definition 1.10. *We define $\dot{H}^s$ as the subspace of $H^s\left(\mathbb{T}[0,2\pi]\right)$ containing real-valued functions so that $\phi \in \dot{H}^s$ implies $\mathcal{Q}_1\phi = \phi$. Note $\|\phi\|_s = \|D^s\phi\|_0$ for $s \geq 1$.*

The significant new aspect of the present paper is a vortex sheet formulation (B.1)-(B.4) equivalent to the evolution system (A.1)-(A.2) with the initial condition (1.3) that projects away the neutral linear modes so that exponential decay of the remaining Fourier modes helps to control small nonlinearities. The equivalent system involves the evolution of $\tilde{\theta}$, $\hat{\theta}(0;t)$ and L, where $\hat{\theta}(1;t)$ and $\hat{\theta}(-1;t)$ are determined as complex functionals of $\tilde{\theta}$.

The main result in this paper is the following theorem:

Theorem 1.11. *There exists $\epsilon > 0$ such that for $r \geq 4$, if $\|\mathcal{Q}_1\theta_0\|_r < \epsilon$, then there exists $(\theta, L) \in C\big([0,\infty); H^r\big(\mathbb{T}[0,2\pi]\big)\times\mathbb{R}\big) \cap C^1\left([0,\infty); H^{r-3}\big(\mathbb{T}[0,2\pi]\big)\times\mathbb{R}\right)$, which satisfies (A.1)-(A.2) with the initial condition (1.3) globally. Furthermore, $\|\tilde{\theta}\|_r$, $\hat{\theta}(1;t)$ and $\hat{\theta}(-1;t)$ each decay exponentially as $t \to \infty$, $|\hat{\theta}(0;t)|$ remains finite, while L approaches $2\sqrt{\pi\mathcal{S}}$, $\mathcal{S}$ being the area of the bubble, which is invariant with time. Thus a near-circular bubble is asymptotically stable for sufficiently small distortions in the $H^r\big(\mathbb{T}[0,2\pi]\big)$ space.*

In §2, we introduce a modified evolution system (B.1)-(B.4) with the initial condition (2.5), which is shown to be equivalent to (A.1)-(A.2) with the initial condition (1.3). We formulate a Galerkin approximation (2.11) and show how Theorem 1.11 follows from Theorem 2.13, Lemma 2.14 and Proposition 2.16.

In §3, we prove several preliminary lemmas. In §4, we prove *a priori* estimates on the growth of solutions to the approximate initial value problem (2.11). In §5, first we use *a priori* estimates to prove global existence and uniqueness of solutions to the Galerkin approximation (2.11), then show the same to be true for (B.1)-(B.4) with the initial condition (2.5). Finally, we also show that $\|\tilde{\theta}\|_r$, for the solution to (B.1)-(B.4) with the initial condition (2.5) decays exponentially in time.

2. Equivalent evolution equations

In this section, we derive an equivalent system of the evolution equations, which will be analyzed in the whole of the paper. Much of the difficulty in this problem is to control the energy appropriately. We find that an equivalent system provides exponentially decaying energy estimates, unlike the original system which contains the neutrally stable modes corresponding to the bubble translation degeneracy.

Definition 2.1. *We introduce the functions*

$$\omega_0(\alpha) = \int_0^\alpha e^{i\alpha'} d\alpha', \ \omega(\alpha) = \int_0^\alpha e^{i\alpha' + i\hat{\theta}(1;t)e^{i\alpha'} + i\hat{\theta}(-1;t)e^{-i\alpha'} + i\tilde{\theta}(\alpha')} d\alpha'.$$

Remark. It is readily checked that

$$\mathrm{Re}\left(\frac{\omega_{0,\alpha}}{\pi} PV \int_0^{2\pi} \frac{f(\alpha')}{\omega_0(\alpha) - \omega_0(\alpha')} d\alpha'\right) = \mathcal{H}(f). \tag{2.1}$$

From the expressions (1.6) and (1.7), it is also easily checked that if $f(\alpha) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\alpha}$, then we have

$$\mathcal{G}[\omega_0] f = i\hat{f}(0), \tag{2.2}$$

which by (1.7) implies

$$\mathcal{F}\left[\omega_0\right] f = \hat{f}(0). \tag{2.3}$$

□

We will show that the evolution system (A.1)-(A.2) is equivalent to the following evolution system for $\left(\tilde{\theta}(\alpha,t), L(t), \hat{\theta}(0;t)\right)$ with $\tilde{\theta}(\alpha,t) = \sum_{k\neq 0,\pm 1} \hat{\theta}(k;t) e^{ik\alpha}$ and $\theta(\alpha,t) = \hat{\theta}(0;t) + \hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha} + \tilde{\theta}(\alpha,t)$:

$$\begin{cases} \dfrac{\partial \tilde{\theta}(\alpha,t)}{\partial t} = \dfrac{2\pi}{L} \mathcal{Q}_1\big(U_\alpha + T(1+\theta_\alpha)\big), \\ \dfrac{dL(t)}{dt} = -\displaystyle\int_0^{2\pi} (1+\theta_\alpha) U d\alpha, \end{cases} \tag{B.1}$$

$$\frac{d\hat{\theta}(0;t)}{dt} = \frac{1}{L} \int_0^{2\pi} T(1+\theta_\alpha) d\alpha, \tag{B.2}$$

with $\gamma(\alpha,t)$, $T(\alpha,t)$, $\hat{\theta}(1;t)$ and $\hat{\theta}(-1;t)$ determined[2] by

$$\begin{cases} \gamma(\alpha,t) = -A_\mu \mathcal{F}[\omega]\gamma + \dfrac{2\pi}{L} \sigma \theta_{\alpha\alpha}, \\ T(\alpha,t) = \displaystyle\int_0^\alpha \big(1+\theta_{\alpha'}(\alpha')\big) U(\alpha') d\alpha' - \dfrac{\alpha}{2\pi} \int_0^{2\pi} \big(1+\theta_\alpha(\alpha)\big) U(\alpha) d\alpha, \end{cases} \tag{B.3}$$

$$\int_0^{2\pi} \exp\left(i\alpha + i\big(\hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha} + \tilde{\theta}(\alpha,t)\big)\right) d\alpha = 0, \tag{B.4}$$

where U is given by

$$U = \frac{\pi}{L} \mathcal{H}[\gamma] + \frac{\pi}{L} \mathrm{Re}\left(\mathcal{G}[\omega]\gamma\right). \tag{2.4}$$

Remark. The formulae for U in (2.4) and $\mathcal{F}[\omega]\gamma$ are equivalent to those in (1.8) and (1.7) since $\hat{\theta}(0;t)$ cancels out. □

The appropriate initial condition is

$$\tilde{\theta}(\alpha,0) = \mathcal{Q}_1 \theta_0, \ L(0) = 2\pi, \ \hat{\theta}(0;0) = \hat{\theta}_0(0). \tag{2.5}$$

Note that the first equation in (B.3) can be rewritten as

$$\big(I + A_\mu \mathcal{F}[\omega]\big)\gamma = \frac{2\pi}{L} \sigma \theta_{\alpha\alpha}. \tag{2.6}$$

[2]Since $\theta(\alpha,t)$ is real valued, note $\hat{\theta}^*(1;t) = \hat{\theta}(-1,t)$.

Note 2.2. *Later we shall see that if $\tilde{\theta} \in H^1\big(\mathbb{T}[0,2\pi]\big)$ and $\|\tilde{\theta}\|_1$ is sufficiently small, then $I + A_\mu \mathcal{F}[\omega]$ is invertible from $\{u \in H^0\big(\mathbb{T}[0,2\pi]\big) | \hat{u}(0) = 0\}$ to itself for any $A_\mu \in [-1,1]$. More general results are available [1], [5] for non self-intersecting interface; however, since we need the sharper estimates for near-circular interface in any case, we construct a direct proof rather than rely on the more general theorems.*

Definition 2.3. *Let $r \geq 3$. We define an open ball $\mathcal{B}$:*

$$\mathcal{B} = \Big\{ u \in \dot{H}^r \big| \|u\|_r < \epsilon \Big\}.$$

We also define the open balls:

$$\begin{aligned}
\mathcal{O} &= \Big\{ (u,v,w) \in \dot{H}^r \times \mathbb{R}^2 \big| u \in \mathcal{B},\ |v - 2\pi| + |w| < 1 \Big\}, \\
\mathcal{V} &= \Big\{ (u,v) \in \dot{H}^r \times \mathbb{R} \big| u \in \mathcal{B},\ |v - 2\pi| < 1 \Big\}, \\
\mathcal{U} &= \big\{ (u,v) \in H^r\big(\mathbb{T}[0,2\pi]\big) \times \mathbb{R} \big| \mathcal{Q}_1 u \in \mathcal{B},\ \|u\|_r + |v - 2\pi| < 1 \big\}.
\end{aligned}$$

Remark. We choose $\epsilon > 0$ is small enough for Lemma 2.14 to apply. □

For (B.4), we also have the following result:

Proposition 2.4. *There exists $\epsilon_1 > 0$ so that (B.4) implicitly defines a unique C^1 function $g : \big\{ u \in \dot{H}^1 \big| \|u\|_1 < \epsilon_1 \big\} \to \mathbb{R}^2$ satisfying $\big(\operatorname{Re}\hat{\theta}(1), \operatorname{Im}\hat{\theta}(1)\big) = g(\tilde{\theta})$ and $g(0) = 0$. Further, g satisfies the following estimates for all $u, u_1, u_2 \in \big\{ u \in \dot{H}^1 \big| \|u\|_1 < \epsilon_1 \big\}$:*

$$|g(u)| \leq \frac{1}{2}\|u\|_1, \tag{2.7}$$

$$|g(u_1) - g(u_2)| \leq \frac{1}{2}\|u_1 - u_2\|_1. \tag{2.8}$$

Remark. Having determined γ, $\hat{\theta}(1;t)$ and $\hat{\theta}(-1;t)$, (2.4) and the second equation in (B.3) also determine U and T needed in (B.1) and (B.2). □

Lemma 2.5. *If $(\theta, L) \in C^1\big([0,S]; \mathcal{U}\big)$ with θ real-valued is the solution of the evolution equations (A.1), where γ, T and U are determined by (A.2) and (1.2) with initial condition (1.3), then $\big(\tilde{\theta} = \mathcal{Q}_1\theta, L, \hat{\theta}(0;t)\big)$ will satisfy the equations (B.1) and (B.2) where γ, T, $\hat{\theta}(\pm 1;t)$ and U are determined by (B.3), (B.4) and (2.4) with the initial condition (2.5) for $t \in [0,S]$.*

Conversely, if $\big(\tilde{\theta}, L, \hat{\theta}(0;t)\big) \in C^1\big([0,S]; \mathcal{O}\big)$ is the solution of the system (B.1) and (B.2) where γ, T, $\hat{\theta}(\pm 1;t)$ and U are determined by (B.3), (B.4) and (2.4) with the initial condition (2.5), then $\theta = \tilde{\theta} + \hat{\theta}(0;t) + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(-1;t)e^{-i\alpha}$ is a real-valued function and (θ, L) satisfies the system (A.1) for $t \in [0,S]$ with the initial condition (1.3), where γ, T and U are determined by (A.2) and (1.2).

Proof. Let $(\theta, L) \in C^1\big([0,S]; \mathcal{U}\big)$ be the solution of the evolution equations (A.1) where γ, T and U are determined by (A.2) and (1.2) with the initial condition (1.3). Then we define $p(t) = \int_0^{2\pi} e^{i\alpha + i\theta(\alpha,t)} d\alpha$. From the consistency condition (1.4), $p(0) = 0$. We consider

$$p'(t) = i \int_0^{2\pi} e^{i\alpha + i\theta(\alpha,t)} \theta_t d\alpha.$$

Substituting for θ_t from (A.1), and using the identity $(e^{i\alpha+i\theta})_\alpha = i(1+\theta_\alpha)e^{i\alpha+i\theta}$, we have

$$p'(t) = \frac{2\pi}{L}\int_0^{2\pi} \left[iU_\alpha e^{i\alpha+i\theta} + T(e^{i\alpha+i\theta})_\alpha \right] d\alpha.$$

We integrate the last term by parts; we use (A.2) to substitute for T_α. There is no boundary term from integrating by parts since T and $e^{i\alpha+i\theta}$ are periodic. We have

$$p'(t) \quad = \quad \frac{2\pi}{L}\int_0^{2\pi} \left(iU_\alpha e^{i\alpha+i\theta} - (1+\theta_\alpha)Ue^{i\alpha+i\theta} - \frac{1}{2\pi}L_t e^{i\alpha+i\theta} \right) d\alpha.$$

Since $iU_\alpha e^{i\alpha+i\theta} - (1+\theta_\alpha)Ue^{i\alpha+i\theta} = (iUe^{i\alpha+i\theta})_\alpha$, we have

$$p' = -\frac{L_t}{L}p.$$

Note that $(\theta, L) \in \mathcal{U}$ implies that $L > 2\pi - 1 > 0$. Furthermore, L_t is continuous in $[0, S]$ from (A.1). So $p(t) = 0$ is the unique solution to the above ordinary differential equation with $p(0) = 0$ for $t \in [0, S]$. Hence

$$e^{i\hat{\theta}(0;t)} \int_0^{2\pi} \exp\Big(i\alpha + i\big(\hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha} + \tilde{\theta}(\alpha, t)\big)\Big) d\alpha = 0,$$

implying

$$\int_0^{2\pi} \exp\Big(i\alpha + i\big(\hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha} + \tilde{\theta}(\alpha, t)\big)\Big) d\alpha = 0 \text{ for } t \in [0, S].$$

Thus $\big(\tilde{\theta} = \mathcal{Q}_1\theta, L, \hat{\theta}(0;t)\big)$ satisfies the equations (B.1) and (B.2) where γ, T, $\hat{\theta}(\pm 1;t)$ and U are determined by (B.3), (B.4) and (2.4) with the initial condition (2.5) for $t \in [0, S]$.

Conversely, suppose that $\big(\tilde{\theta}, L, \hat{\theta}(0;t)\big) \in C^1\big([0, S]; \mathcal{O}\big)$ satisfies (B.1) and (B.2) with the initial condition (2.5), where γ, T, $\hat{\theta}(\pm 1;t)$ and U are determined by (B.3), (B.4) and (2.4). Let $\theta = \tilde{\theta} + \hat{\theta}(0;t) + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(-1;t)e^{-i\alpha}$. We note from Proposition 2.4 that $\hat{\theta}(\pm 1;t)$ scale as ϵ_1 and hence is small. We note from (B.4) that

$$p(t) = e^{i\hat{\theta}(0;t)} \int_0^{2\pi} \exp\Big(i\alpha + i\big(\hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha} + \tilde{\theta}(\alpha, t)\big)\Big) d\alpha = 0.$$

It is convenient to define $\Gamma(\alpha, t) = U_\alpha + T(1+\theta_\alpha)$. From $p'(t) = 0$, using (B.1), we obtain

$$0 = \int_0^{2\pi} e^{i\alpha+i\theta}\Big(\big(\hat{\theta}_t(-1;t) - \frac{2\pi}{L}\widehat{\Gamma}(-1;t)\big)e^{-i\alpha} + \big(\hat{\theta}_t(1;t) - \frac{2\pi}{L}\widehat{\Gamma}(1;t)\big)e^{i\alpha}\Big) d\alpha. \tag{2.9}$$

Let $e^{i\alpha+i\theta} = \sum_{k=-\infty}^{\infty} \hat{c}(k)e^{ik\alpha}$. Hence for sufficiently small ball size ϵ of $\mathcal{B}$, using Proposition 2.4 and Sobolev inequality $|.|_\infty < C\|.\|_1$,

$$|\theta - \hat{\theta}(0;t)|_\infty = |\tilde{\theta}(\alpha, t) + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(-1;t)|_\infty \le C\|\tilde{\theta}\|_1$$

is small, which clearly ensures $|\hat{c}(1)| > |\hat{c}(k)|$ for $k \neq 1$. Note further that (2.9) implies

$$\big(\hat{\theta}_t(-1;t) - \frac{2\pi}{L}\widehat{\Gamma}(-1;t)\big)\hat{c}(1) + \big(\hat{\theta}_t(1;t) - \frac{2\pi}{L}\widehat{\Gamma}(1;t)\big)\hat{c}(-1) = 0.$$

Since $\Gamma(\alpha,t)$ and $\tilde{\theta}$ are real valued, $\hat{\theta}_t(-1;t)-\frac{2\pi}{L}\widehat{\Gamma}(-1;t)$ is the complex conjugate of $\hat{\theta}_t(1;t)-\frac{2\pi}{L}\widehat{\Gamma}(1;t)$. It is clear that if $|a_1|\neq|a_2|$, then the only solution to $a_1\eta+a_2\eta^*=0$ is $\eta=0$. Hence

$$\hat{\theta}_t(-1;t)-\frac{2\pi}{L}\widehat{\Gamma}(-1;t)=0 \text{ and } \hat{\theta}_t(1;t)-\frac{2\pi}{L}\widehat{\Gamma}(1;t)=0.$$

Hence, $(\theta=\tilde{\theta}+\hat{\theta}(0;t)+\hat{\theta}(1;t)e^{i\alpha}+\hat{\theta}(-1;t)e^{-i\alpha}+\hat{\theta}e^{-i\alpha},L)$ will satisfy the system (A.1) where γ, T and U are determined by (A.2) and (1.2) with the initial condition (1.3) for $t\in[0,S]$. □

We will henceforth discuss the global solutions of the evolution equations (B.1) where γ, T, $\hat{\theta}(\pm1;t)$ and U are determined by (B.3), (B.4) and (2.4) with initial condition (2.5).

Definition 2.6. *Define $\hat{\theta}(1;t)=r_1+ir_2$. Then since θ is real valued, $\hat{\theta}(-1;t)=r_1-ir_2$.*

Remark. (B.4) becomes

$$\int_0^{2\pi}\exp\Big(i\alpha+i\big((r_1+ir_2)e^{i\alpha}+(r_1-ir_2)e^{-i\alpha}+\sum_{k=-\infty,\neq0,\pm1}^{\infty}\hat{\theta}(k)e^{ik\alpha}\big)\Big)d\alpha=0. \tag{2.10}$$

□

In order to prove Proposition 2.4, we need the following lemma:

Lemma 2.7. *Implicit function Theorem([30]): Let $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ be Banach spaces and F a mapping from an open subset of $\mathcal{G}_1\times\mathcal{G}_2$ into $\mathcal{G}_3$. Let (u_0,v_0) be a point in $\mathcal{G}_1\times\mathcal{G}_2$ satisfying:*

(i) *$F(u_0,v_0)=0$;*
(ii) *F is continuously differentiable at (u_0,v_0);*
(iii) *the partial Fréchet derivative $D_vF(u_0,v_0)$ is invertible from $\mathcal{G}_2$ to $\mathcal{G}_3$.*

Then, there are neighborhood $\mathcal{V}_1$ of u_0 in $\mathcal{G}_1$ and neighborhood $\mathcal{V}_2$ of v_0 in $\mathcal{G}_2$ and a C^1 map $g:\mathcal{V}_1\to\mathcal{V}_2$ so that $F\big(u,g(u)\big)=0$ for all $u\in\mathcal{V}_1$ and for each $u\in\mathcal{V}_1$, $g(u)$ is the unique point v in $\mathcal{V}_2$ satisfying $F(u,v)=0$.

Definition 2.8. *In the bubble context, we define*

$$F(u,v) \quad = \quad \int_0^{2\pi}\exp\Big(i\alpha+i\big(2(r_1\cos\alpha-r_2\sin\alpha)+u\big)\Big)d\alpha$$

with $v=(r_1,r_2)$.

Remark. Note $F:\dot{H}^1\times\mathbb{R}^2\to\mathbb{C}$. □

Proof of Proposition 2.4: Let us show that the Fréchet derivative of $F(u,v)$ with respect to u exists in $\dot{H}^1\times\mathbb{R}^2$. Since

$$\Big\|\exp\big[ih(\cdot)\big]-1-ih(\cdot)\Big\|_0=\Big\|ih(\cdot)\int_0^1\big(e^{i\tau h(\cdot)}-1\big)d\tau\Big\|_0\le c\|h\|_1^2,$$

we have

$$\begin{aligned}&\Big|F(u+h,v)-F(u,v)-\int_0^{2\pi}ih(\alpha)\exp\Big(i\alpha+i\big[2(r_1\cos\alpha-r_2\sin\alpha)+u(\alpha)\big]\Big)d\alpha\Big|\\ =\ &\Big|\int_0^{2\pi}\exp\Big(i\alpha+i\big[2(r_1\cos\alpha-r_2\sin\alpha)+u(\alpha)\big]\Big)\Big\{\exp\big[ih(\alpha)\big]-1-ih(\alpha)\Big\}d\alpha\Big|\le c\|h\|_1^2.\end{aligned}$$

Hence the Fréchet derivative of F with respect to u is

$$D_uF(u,v)h = \int_0^{2\pi} ih(\alpha)\exp\Big(i\alpha + i\big(2(r_1\cos\alpha - r_2\sin\alpha) + u(\alpha)\big)\Big)d\alpha,$$

for $h \in \dot{H}^1$. It is clear that $D_uF(u,v) : \dot{H}^1 \to \mathbb{C}$ is the bounded linear operator for all $(u,v) \in \dot{H}^1 \times \mathbb{R}^2$.

Similarly,

$$D_vF(u,v)\delta v = 2i\int_0^{2\pi} \big(\delta r_1\cos\alpha - \delta r_2\sin\alpha\big)\exp\Big(i\alpha + i\big(2(r_1\cos\alpha - r_2\sin\alpha) + u(\alpha)\big)\Big)d\alpha,$$

with $\delta v = (\delta r_1, \delta r_2) \in \mathbb{R}^2$ is a bounded linear operator for all $(u,v) \in \dot{H}^1 \times \mathbb{R}^2$, with

$$D_vF(0,0)\delta v = 2i\int_0^{2\pi} \big(\delta r_1\cos\alpha - \delta r_2\sin\alpha\big)e^{i\alpha}d\alpha = 2\pi(\delta r_2 + i\delta r_1).$$

Clearly $D_vF(0,0)$ is invertible. So by the implicit function theorem (Lemma 2.7), for $(u_0,v_0) = (0,0)$, there exist neighborhood $\mathcal{V}_1 = \{u \in \dot{H}^1 : \|u\|_1 < 2\epsilon_1\}$ of 0 in $\dot{H}^1$, and a neighborhood $\mathcal{V}_2$ of $(0,0)$ in $\mathbb{R}^2$, and a C^1 map $g : \mathcal{V}_1 \to \mathcal{V}_2$, so that $F(u,g(u)) = 0$ for all $u \in \mathcal{V}_1$. We also have $\|Dg(u)\| \le \frac{1}{2}$ for $u \in \mathcal{V}_1$ since $Dg(0) = 0$. Hence we have

$$|g(u)| \le \Big|\int_0^1 Dg(tu)udt\Big| \le \frac{1}{2}\|u\|_1,$$
$$|g(u_1) - g(u_2)| \le \Big|\int_0^1 Dg\big(u_1 + t(u_2-u_1)\big)(u_2-u_1)dt\Big| \le \frac{1}{2}\|u_1 - u_2\|_1$$

for all $u, u_1, u_2 \in \{u \in \dot{H}^1 : \|u\|_1 < \epsilon_1\}$.

Corollary 2.9. *There exists sufficiently small $\epsilon_1 > 0$ so that for $\theta \in H^{s+1}(\mathbb{T}[0,2\pi])$ with $s \ge 0$, if $\|\tilde{\theta}\|_1 < \epsilon_1$, then θ satisfying (B.4) implies $\|\theta_\alpha\|_s \le 2\|\tilde{\theta}_\alpha\|_s$.*

Proof. We note from the relation between θ and $\tilde{\theta}$ that

$$\|\theta_\alpha\|_s^2 = \sum_k |k|^{2s+2}|\hat{\theta}(k)|^2 = 2|g(\tilde{\theta})|^2 + \|\tilde{\theta}_\alpha\|_s^2.$$

The rest follows from bounds on $g(\tilde{\theta})$ in Proposition 2.4. □

2.1. **Galerkin approximation.** From the set of equations in (B.1)-(B.4), it is easily seen that $\hat{\theta}(0;t)$ does not effect the evolution of $\tilde{\theta}$ and L, so it is convenient to first determine the solution $(\tilde{\theta}, L)$; determination of $\hat{\theta}(0;t)$ is then simply reduced to an integration of the equation (B.2). It is convenient to introduce a Galerkin approximations as described in this section.

Definition 2.10. *We define a family of Galerkin projections $\{P_n\}_{n=2}^\infty$, as*

$$P_nu(\alpha) = \sum_{k=-n,k\ne0,\pm1}^{n} \hat{u}(k)e^{ik\alpha}, \text{ for all } u = \sum_{-\infty}^{\infty}\hat{u}(k)e^{ik\alpha}.$$

We define the approximate solution $\tilde{\theta}_n(\alpha,t)$ of order n of the problem in the following way:

$$\tilde{\theta}_n(\alpha,t) = \sum_{k=-n,k\ne0,\pm1}^{n} \hat{\theta}_n(k;t)e^{ik\alpha}.$$

The approximate equations are

$$\text{(C.1)}\qquad \begin{cases} \dfrac{\partial\tilde{\theta}_n(\alpha,t)}{\partial t} = \dfrac{2\pi}{L_n}P_n\Big(U_{n,\alpha}+T_n\big(1+\theta_{n,\alpha}\big)\Big), \\ \dfrac{dL_n(t)}{dt} = -\displaystyle\int_0^{2\pi}(1+\theta_{n,\alpha})U_n d\alpha, \end{cases}$$

with γ_n, T_n and $\hat{\theta}_n(\pm 1;t)$ (where $\hat{\theta}_n^*(1;t)=\hat{\theta}_n(-1;t)$ because θ_n is real) determined by

$$\text{(C.2)}\qquad \begin{cases} \big(I+A_\mu\mathcal{F}[\omega_n]\big)\gamma_n(t) = \dfrac{2\pi}{L_n}\sigma\theta_{n,\alpha\alpha}, \\ T_n(\alpha,t) = \displaystyle\int_0^\alpha (1+\theta_{n,\alpha'})U_n(\alpha')d\alpha' - \frac{\alpha}{2\pi}\int_0^{2\pi}(1+\theta_{n,\alpha'})U_n(\alpha')d\alpha', \\ \displaystyle\int_0^{2\pi}\exp\Big(i\alpha+i\big(\hat{\theta}_n(-1;t)e^{-i\alpha}+\hat{\theta}_n(1;t)e^{i\alpha}+\tilde{\theta}_n(\alpha,t)\big)\Big)d\alpha = 0, \end{cases}$$

where

$$\begin{aligned} \theta_n(\alpha,t) &= \tilde{\theta}_n(\alpha,t)+\hat{\theta}_n(-1;t)e^{-i\alpha}+\hat{\theta}_n(1;t)e^{i\alpha}, \\ \omega_n(\alpha) &= \int_0^\alpha e^{i\tau+i\theta_n(\tau)}d\tau, \\ U_n &= \frac{\pi}{L_n}\mathcal{H}[\gamma_n]+\frac{\pi}{L_n}\operatorname{Re}\left(\mathcal{G}[\omega_n]\gamma_n\right). \end{aligned}$$

2.2. **Main results.** Let $X_n = \big(\tilde{\theta}_n,\, L_n\big)$. The Galerkin approximate equations (C.1)-(C.2) reduce to an ODE in the Banach space $\dot{H}^r\times\mathbb{R}$:

$$\frac{dX_n}{dt} = F_n(X_n), \quad X_n(0) = (P_n\theta_0,\, 2\pi), \tag{2.11}$$

where $F_n(X_n) = \big(F_{n,1}(X_n), F_{n,2}(X_n)\big)$ are given by

$$F_{n,1} = \frac{2\pi}{L_n}P_n\left(U_{n,\alpha}+T_n(1+\theta_{n,\alpha})\right), \tag{2.12}$$

$$F_{n,2} = -\int_0^{2\pi}(1+\theta_{n,\alpha})U_n(\alpha)d\alpha. \tag{2.13}$$

For the approximate equation (2.11), we have the following results:

Proposition 2.11. *Assume $P_n\theta_0\in\mathcal{B}$ for $r\geq 3$. For the sufficiently small ball size ϵ of $\mathcal{B}$, there exists the unique solution $X_n\in C^1\left([0,S_n);\mathcal{V}\right)$ to the ODE in Eq. (2.11), where S_n depends on n, r and ϵ.*

Remark. We will prove this proposition in §5 using Picard theorem (See for instance Chapter 3 in [26]). □

Proposition 2.12. *Assume $X_n = \left(\tilde{\theta}_n, L_n\right)\in C^1\left([0,S);\mathcal{V}\right)$ is a solution of the initial value problem (2.11). Then there exists $\epsilon>0$ such that if $\|P_n\theta_0\|_r<\epsilon$ for $r\geq 3$, then*

$$\|\tilde{\theta}_n(\cdot,t)\|_r \leq \|P_n\theta_0(\cdot)\|_r e^{-\frac{1}{36}\sigma t},\ |L_n^3-8\pi^3|\leq C\epsilon\left(1-e^{-\frac{1}{18}\sigma t}\right),$$

with a constant C independent of n for any time $t\geq 0$ where the solution exists.

Remark. We will prove *a priori* estimates in §4. □

Theorem 2.13. *Given the initial condition $X_n(0) \in \mathcal{V}$, for any $n \geq 2$ and $r \geq 3$. For sufficiently small ϵ, there exists for all time a unique solution $X_n(t) \in C^1([0,\infty);\mathcal{V})$ to the approximate equation (2.11).*

Proof. Proposition 2.11 shows the existence and uniqueness of solutions X_n locally in time. Then by continuation of an autonomous ODE on a Banach space (see Chapter 3 in [26]), we know that the unique solution $X_n \in C^1([0,S]);\mathcal{V})$ either exists globally in time or $S < \infty$ and $X_n(t)$ leaves the open set $\mathcal{V}$ as $t \nearrow S$. Suppose $S < \infty$. Combining Propositions 2.12 and 4.3, we know that solution remains in the open set $\mathcal{V}$ as $t \nearrow S$. Hence it shows that the solution to Eq. (2.11) exists globally in time. □

From the solutions to the approximate equation (2.11), we will deduce the existence and uniqueness of solutions to the evolution system (B.1), (B.3) and (B.4) globally in time (Theorem 1.11) using the following lemma and proposition:

Lemma 2.14. *For $r \geq 4$, there exists sufficiently small $\epsilon > 0$ such that for any $S > 0$, solutions $X_n = (\tilde{\theta}_n, L_n) \in C^1([0,\infty);\mathcal{V})$ of the approximate equation (2.11) for different n form a Cauchy sequence in $C\left([0,S];\dot{H}^1 \times \mathbb{R}\right)$. As $n \to \infty$, the limit $X = \left(\tilde{\theta}, L\right) \in C\left([0,S];\dot{H}^r \times \mathbb{R}\right) \cap C^1\left([0,S];\dot{H}^{r-3} \times \mathbb{R}\right)$ and is the unique classical solution to (B.1), (B.3) and (B.4) satisfying the initial condition (2.5).*

Remark. The proof is given in §5. □

Definition 2.15. *The area of bubble is defined by $\mathcal{S}(t)$. That is*

$$\mathcal{S}(t) = \frac{1}{2}\operatorname{Im}\int_0^{2\pi} z_\alpha z^* d\alpha. \tag{2.14}$$

Proposition 2.16. *Let $(\tilde{\theta}, L) \in C\left([0,\infty);\dot{H}^r \times \mathbb{R}\right) \cap C^1\left([0,\infty);\dot{H}^{r-3} \times \mathbb{R}\right)$ be a solution to the system (B.1), (B.3) and (B.4) with the initial condition (2.5) for $r \geq 4$. If $\mathcal{Q}_1\theta_0 \in \mathcal{B}$, then the area $\mathcal{S}$ is invariant with time and for sufficient small ϵ, we have*

$$\|\tilde{\theta}(\cdot,t)\|_r \leq \|\mathcal{Q}_1\theta_0(\cdot)\|_r e^{-\frac{1}{36}\sigma t},$$

$$|\hat{\theta}(1;t)| = |\hat{\theta}(-1;t)| \leq \frac{1}{2}\|\mathcal{Q}_1\theta_0(\cdot)\|_r e^{-\frac{1}{36}\sigma t},$$

$$|L(t) - 2\sqrt{\pi\mathcal{S}}| \leq C\|\mathcal{Q}_1\theta_0\|_r e^{-\frac{1}{36}\sigma t},$$

$$|\hat{\theta}(0;t) - \hat{\theta}_0(0)| \leq C\|\mathcal{Q}_1\theta_0\|_r,$$

where C depends on $\mathcal{S}$.

Remark. We will prove Lemma 2.14 and Proposition 2.16 in §5. Further, the result above together with Proposition 2.4 shows that $\theta(\alpha,t) - \hat{\theta}(0;t)$ goes to 0 exponentially as $t \to \infty$. □

Proof of Theorem 1.11: This immediately follows from Lemma 2.14 and Proposition 2.16 since Lemma 2.5 gives equivalence between (A.1)-(A.2) and (B.1)-(B.4).

3. Preliminary Lemmas

We will need to use a variety of routine estimates for integral operators and other functions in terms of $\tilde{\theta}$ and $\hat{\theta}(0;t)$. Recall tangent angle of the curve is $\alpha+\theta(\alpha)=\alpha+\tilde{\theta}(\alpha)+\hat{\theta}(0;t)+\hat{\theta}(-1;t)e^{-i\alpha}+\hat{\theta}(1;t)e^{i\alpha}$, where $\hat{\theta}(1;t)$ and $\hat{\theta}(-1;t)$ are determined by $g(\tilde{\theta})$.

The next lemma gives a bound for ω_α in terms of $\tilde{\theta}$.

Lemma 3.1. *Assume $\|\tilde{\theta}\|_1<\epsilon_1$ where ϵ_1 is small enough for Corollary 2.9 to apply.*

If ω determined by $\tilde{\theta}\in\dot{H}^s$, then for $s\geq 1$,

(3.1)
$$\|\omega_\alpha\|_s\leq C_1(\|\tilde{\theta}\|_s+1)\exp\left(C_2\|\tilde{\theta}\|_{s-1}\right),\quad \left\|\frac{1}{\omega_\alpha}\right\|_s\leq C_1(\|\tilde{\theta}\|_s+1)\exp\left(C_2\|\tilde{\theta}\|_{s-1}\right),$$

where constants C_1 and C_2, depend only on s, and particularly for $s=1$, $C_2=0$.

Further, if $\omega^{(1)},\omega^{(2)}$ correspond respectively to $\tilde{\theta}^{(1)},\tilde{\theta}^{(2)}\in\dot{H}^s$, where $\|\tilde{\theta}^{(1)}\|_1$, $\|\tilde{\theta}^{(2)}\|_1<\epsilon_1$, then for $s\geq 1$,

$$\|\omega_\alpha^{(1)}-\omega_\alpha^{(2)}\|_s\leq C_1\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_s\exp\left[C_2\left(\|\tilde{\theta}^1\|_s+\|\tilde{\theta}^2\|_s\right)\right], \tag{3.2}$$

$$\left\|\frac{1}{\omega_\alpha^{(1)}}-\frac{1}{\omega_\alpha^{(2)}}\right\|_s\leq C_1\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_s\exp\left[C_2\left(\|\tilde{\theta}^{(1)}\|_s+\|\tilde{\theta}^{(2)}\|_s\right)\right], \tag{3.3}$$

while for $s\geq 2$,

$$\begin{aligned}\|\omega_\alpha^{(1)}-\omega_\alpha^{(2)}\|_s &\leq C_1\left(\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_s+\|\tilde{\theta}^{(2)}\|_s\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_{s-1}\right)\\ &\quad\times\exp\left[C_2\big(\|\tilde{\theta}^{(1)}\|_{s-1}+\|\tilde{\theta}^{(2)}\|_{s-1}\big)\right],\end{aligned} \tag{3.4}$$

$$\begin{aligned}\left\|\frac{1}{\omega_\alpha^{(1)}}-\frac{1}{\omega_\alpha^{(2)}}\right\|_s &\leq C_1\left(\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_s+\|\tilde{\theta}^{(2)}\|_s\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_{s-1}\right)\\ &\quad\times\exp\left[C_2\big(\|\tilde{\theta}^{(1)}\|_{s-1}+\|\tilde{\theta}^{(2)}\|_{s-1}\big)\right],\end{aligned} \tag{3.5}$$

where the constants C_1 and C_2 depend only on s.

Proof. For the formula $\omega_\alpha=e^{i\alpha+i\theta-i\hat{\theta}(0;t)}$, it is easy to obtain

$$\|\omega_\alpha\|_0\leq C.$$

Let us consider for $0<k\leq s$. The chain rule gives

$$D^k\omega_\alpha=\sum_{\beta_1+\cdots+\beta_\mu=k,\beta_i\geq 1}C_\beta D^{\beta_1}(\alpha+\theta)\cdots D^{\beta_\mu}(\alpha+\theta)\omega_\alpha.$$

So by Sobolev embedding Theorem, $|f|_\infty\leq C\|f\|_1$, we have

(3.6)
$$\|D^k\omega_\alpha\|_0\leq C\|1+\theta_\alpha\|_{k-1}(1+\|\theta_\alpha\|_{k-1}+\cdots+\|\theta_\alpha\|_{k-1}^{k-1})\leq C_1\exp\left(C_2\|\theta_\alpha\|_{k-1}\right),$$

where the constants, C_1 and C_2, depend only on s.

For $s=1$, we have

$$\|D\omega_\alpha\|_0=\|1+\theta_\alpha\|_0\leq C(1+\|\tilde{\theta}\|_1).$$

For $s\geq 2$, we note

$$D^s\omega_\alpha=D^{s-1}\big[i(1+\theta_\alpha)\omega_\alpha\big].$$

Hence, by noting Banach algebra property (see Note 1.2), Corollary 2.9 and (3.6), we get

$$\|D^s\omega_\alpha\|_0 \leq \left\|i(1+\theta_\alpha)\omega_\alpha\right\|_{s-1} \leq C_1(\|\tilde{\theta}\|_s + 1)\exp\left(C_2\|\tilde{\theta}\|_{s-1}\right),$$

where the constants, C_1 and C_2, depend only on s. Since $\frac{1}{\omega_\alpha} = e^{-i\alpha - i\theta(\alpha) + i\hat{\theta}(0;t)}$, the preceding arguments are clearly applied to $\frac{1}{\omega_\alpha}$ as well and (3.1) follows from Corollary 2.9 for a modified constant C_2.

To prove (3.2), we note that

$$\omega_\alpha^{(1)} - \omega_\alpha^{(2)} = \left[e^{i\left(\theta^{(1)} - \hat{\theta}^{(1)}(0;t) - \theta^{(2)} + \hat{\theta}^{(2)}(0;t)\right)} - 1\right] e^{i\alpha + i\theta^{(2)} - i\hat{\theta}^{(2)}(0;t)}$$

From the series representation of the exponential and application of Banach algebra property of $\|.\|_s$ norm to each term in the series, we deduce

$$\left\|e^{i\left(\theta^{(1)} - \hat{\theta}^{(1)}(0;t) - \theta^{(2)} + \hat{\theta}^{(2)}(0;t)\right)} - 1\right\|_s$$
$$\leq C_1\|\theta^{(1)} - \hat{\theta}^{(1)}(0;t) - \theta^{(2)} + \hat{\theta}^{(2)}(0;t)\|_s \exp\left(C_2\left\|\theta^{(1)} - \hat{\theta}^{(1)}(0;t) - \theta^{(2)} + \hat{\theta}^{(2)}(0;t)\right\|_s\right),$$

where the constants, C_1 and C_2, depend only on s. Using Banach algebra property and Corollary 2.9, (3.2) follows. Almost identical arguments are applied to prove (3.3).

Further, if $s \geq 2$ we have

$$\left\|D^s(\omega_\alpha^{(1)} - \omega_\alpha^{(2)})\right\|_0 = \left\|D^{s-1}\left[i(1+\theta_\alpha^{(1)})\omega_\alpha^{(1)} - i(1+\theta_\alpha^{(2)})\omega_\alpha^{(2)}\right]\right\|_0$$
$$\leq C_1\left(\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_s + \|\tilde{\theta}^{(2)}\|_s\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_{s-1}\right)\exp\left(C_2\|\tilde{\theta}^{(1)}\|_{s-1} + C_2\|\tilde{\theta}^{(2)}\|_{s-1}\right),$$

where the constants, C_1 and C_2, depend only on s. So (3.4) follows. Almost identical arguments are applied for (3.5). $\square$

In simplifying our integral operators, we find divided differences to be very useful.

Definition 3.2. *The divided differences q_1 and q_2 are defined as follows:*

$$q_1[\omega](\alpha,\alpha') = \frac{\omega(\alpha) - \omega(\alpha')}{\alpha - \alpha'} = \int_0^1 \omega_\alpha(t\alpha + (1-t)\alpha')dt,$$

$$q_2[\omega](\alpha,\alpha') = \frac{\omega(\alpha) - \omega(\alpha') - \omega_\alpha(\alpha)(\alpha - \alpha')}{(\alpha-\alpha')^2} = \int_0^1 (t-1)\omega_{\alpha\alpha}((1-t)\alpha + t\alpha')dt.$$

Proposition 3.3. *There exists $\epsilon_1 > 0$ so that $\|\tilde{\theta}\|_1 \leq \epsilon_1$ implies*

$$|q_1[\omega](\alpha,\alpha')| \geq \frac{1}{8}, \text{ for } 0 < |\alpha - \alpha'| \leq \pi \tag{3.7}$$

Proof. We note that

$$q_1[\omega](\alpha,\alpha') = \frac{\int_{\alpha'}^{\alpha} e^{i\tau + i\tilde{\theta}(\tau) + i\hat{\theta}(1)e^{i\tau} + i\hat{\theta}(-1)e^{-i\tau}}d\tau}{\alpha - \alpha'}.$$

Further,

$$
\begin{aligned}
&\Big|\frac{\int_{\alpha'}^{\alpha} e^{i\tau+i\tilde{\theta}(\tau)+i\hat{\theta}(1)e^{i\tau}+i\hat{\theta}(-1)e^{-i\tau}}d\tau}{\alpha-\alpha'}-\frac{\int_{\alpha'}^{\alpha}e^{i\tau}d\tau}{\alpha-\alpha'}\Big|\\
=\quad&\Big|\frac{\int_{\alpha'}^{\alpha} e^{i\tau}\big(e^{i\tilde{\theta}(\tau)+i\hat{\theta}(1)e^{i\tau}+i\hat{\theta}(-1)e^{-i\tau}}-1\big)d\tau}{\alpha-\alpha'}\Big|\\
\leq\quad& 2\sqrt{2}\max_{\tau\in[0,2\pi]}\big|\tilde{\theta}(\tau)+\hat{\theta}(1)e^{i\tau}+\hat{\theta}(-1)e^{-i\tau}\big|.
\end{aligned}
$$

This bound is a consequence of the inequality

$$|e^{i\zeta}-e^{i\zeta'}|\leq\sqrt{2}|\zeta-\zeta'|,\ \text{for all } \zeta,\,\zeta' \text{ in } \mathbb{R}.$$

We choose $\epsilon_1>0$ small enough so that Proposition 2.4 holds and from Sobolev embedding theorem,

$$2\sqrt{2}\max_{\tau\in[0,2\pi]}\big|\tilde{\theta}(\tau)+\hat{\theta}(1)e^{i\tau}+\hat{\theta}(-1)e^{-i\tau}\big|\leq c\|\tilde{\theta}\|_1\leq\frac{1}{8},$$

where c is some constant.

It is easy to see that

$$\Big|\frac{\int_{\alpha'}^{\alpha}e^{i\tau}d\tau}{\alpha-\alpha'}\Big|\geq\frac{1}{4},\ \text{for } 0<|\alpha-\alpha'|\leq\pi.$$

Thus, if $\|\tilde{\theta}\|_1\leq\epsilon_1$, we have

$$\Big|\frac{\int_{\alpha'}^{\alpha} e^{i\tau+i\tilde{\theta}(\tau)+i\hat{\theta}(1)e^{i\tau}+i\hat{\theta}(-1)e^{-i\tau}}d\tau}{\alpha-\alpha'}\Big|\geq\frac{1}{8}.$$

□

Lemma 3.4. *(See [1] or appendix for proof) Let $\omega_\alpha\in H^k\big(\mathbb{T}[0,2\pi]\big)$ for $k\geq 0$. Then $D_\alpha^k q_1, D_{\alpha'}^k q_1\in H^0[a,a+2\pi]$ in both variables α or α' and satisfy the bounds*

$$\|D_\alpha^k q_1[\omega]\|_0\leq C\|\omega_\alpha\|_k\ ,\ \|D_{\alpha'}^k q_1[\omega]\|_0\leq C\|\omega_\alpha\|_k$$

with C only depending on k (in particular independent of a). Further if $\omega_{\alpha\alpha}\in H^k\big(\mathbb{T}[0,2\pi]\big)$ for $k\geq 0$, then $D_\alpha^k q_2, D_{\alpha'}^k q_2\in H^0[a,a+2\pi]$ in both variables α and α' and satisfy

$$\|D_\alpha^k q_2[\omega]\|_0\leq C\|\omega_{\alpha\alpha}\|_k\ ,\ \|D_{\alpha'}^k q_2[\omega]\|_0\leq C\|\omega_{\alpha\alpha}\|_k$$

with C only depending on k.

Lemma 3.5. *Let $\omega^{(1)},\omega^{(2)}\in H^{k+1}\big(\mathbb{T}[0,2\pi]\big)$ for $k\geq 0$. Suppose*

$$|q_1[\omega^{(1)}](\alpha,\alpha')|\geq\frac{1}{8},\ \textit{for } 0<|\alpha-\alpha'|\leq\pi.$$

Then

$$\Big(\int_{\alpha-\pi}^{\alpha+\pi}\Big|D_\alpha^k\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}\Big|^2d\alpha'\Big)^{\frac{1}{2}}\leq C_1\|\omega_\alpha^{(2)}\|_{k+1}\exp(C_2\|\omega_\alpha^{(1)}\|_k),\tag{3.8}$$

$$\Big(\int_{\alpha-\pi}^{\alpha+\pi}\Big|D_\alpha^k\frac{q_2[\omega^{(2)}](\alpha',\alpha)}{q_1[\omega^{(1)}](\alpha',\alpha)}\Big|^2d\alpha'\Big)^{\frac{1}{2}}\leq C_1\|\omega_\alpha^{(2)}\|_{k+1}\exp(C_2\|\omega_\alpha^{(1)}\|_k),$$

where C_1 and C_2 depend on k alone, but not on α.

Proof. Clearly for $k=0$, (3.8) holds. Consider $k\geq 1$. It is easy to have

$$D_\alpha^k \frac{q_2}{q_1} = \sum_{j=0}^{k} C_{k,j} D_\alpha^{k-j} q_2 D_\alpha^j \frac{1}{q_1}.$$

We have from Lemma 3.4, for $0\leq j\leq k$,

$$\| D_\alpha^{k-j} q_2[\omega^{(2)}](\alpha,\alpha')\|_0 \leq C_1 \|\omega_{\alpha\alpha}^{(2)}\|_{k-j}$$

and

$$\| D_\alpha^{j} q_1[\omega^{(1)}](\alpha,\alpha')\|_0 \leq C_1 \|\omega_{\alpha}^{(1)}\|_{j}.$$

Further, since q_1 is bounded below by $\frac{1}{8}$, it follows that for $0\leq j\leq k$, Hence

$$\|D_\alpha^j \frac{1}{q_1}\|_0 \leq C \exp\left[c_2 \sum_{m=1}^{j} \|D_\alpha^m q_1\|_0 \right] \leq C_1 \exp\left[C_2 \|\omega_\alpha^{(1)}\|_j \right].$$

$$\left\| D_\alpha^k \frac{q_2}{q_1} \right\|_0 \leq \sum_{j=0}^{k} C_{k,j} \left\| D_\alpha^{k-j} q_2 D_\alpha^j \frac{1}{q_1}\right\|_0 \leq C_1 \|\omega_\alpha^{(2)}\|_{k+1} \exp(C_2 \|\omega_\alpha^{(1)}\|_k),$$

since $\|\frac{1}{q_1} D_\alpha^k q_2\|_0 \leq |\frac{1}{q_1}|_\infty \|D_\alpha^k q_2\|_0$ and for $1\leq j\leq k$,

$$\|D_\alpha^{k-j} q_2 \ D_\alpha^j \frac{1}{q_1}\|_0 \leq |D_\alpha^{k-j} q_2|_\infty \|D_\alpha^j \frac{1}{q_1}\|_0 \leq c\|q_2\|_{k-j+1} \|D_\alpha^j \frac{1}{q_1}\|_0.$$

The second part follows in a very similar manner since Lemma 3.4 can be applied by switching variables α' and α in the expression. We note that Lemma 3.4 gives the same $H^0[a, a+2\pi]$ estimates for derivatives of q_1 and q_2 with respect to α or α', independent of a. □

Definition 3.6. *We write the cotangent as a function which is analytic at the origin plus a singular part:*

$$\cot(\beta) = \frac{1}{\beta} + l(\beta).$$

Lemma 3.7. *(See* [1] *or appendix for proof) Let $s\geq 2$ and $\omega\in H^s(\mathbb{T}[0,2\pi])$ with corresponding $\|\tilde\theta\|_1$ sufficiently small to ensure $|q_1[\omega](\alpha,\alpha')|\geq \frac{1}{8}$. Then $\mathcal{K}[\omega]: H^0(\mathbb{T}[0,2\pi]) \to H^{s-2}(\mathbb{T}[0,2\pi])$, and in particular, there are positive constants C_1 and C_2 depending on s such that*

$$\|\mathcal{K}[\omega] f\|_{s-2} \leq C_1 \|f\|_0 \exp\left(C_2 \|\omega_\alpha\|_{s-1}\right). \tag{3.9}$$

Further, $\mathcal{K}[\omega]: H^1(\mathbb{T}[0,2\pi]) \to H^{s-1}(\mathbb{T}[0,2\pi])$, and

$$\|\mathcal{K}[\omega] f\|_{s-1} \leq C_1 \|f\|_1 \exp\left(C_2 \|\omega_\alpha\|_{s-1}\right). \tag{3.10}$$

Lemma 3.8. *If $f\in H^1(\mathbb{T}[0,2\pi])$, $\omega^{(1)}$ and $\omega^{(2)}$ correspond to $\tilde\theta^{(1)}$ and $\tilde\theta^{(2)}$, each in $\dot H^1$, respectively with $\|\tilde\theta^{(1)}\|_1$, $\|\tilde\theta^{(2)}\|_1 < \epsilon_1$, then for sufficient small ϵ_1,*

$$\|\mathcal{K}[\omega^{(1)}] f - \mathcal{K}[\omega^{(2)}] f\|_0 \quad \leq \quad C_1 \|f\|_0 \|\tilde\theta^{(1)} - \tilde\theta^{(2)}\|_1.$$

Suppose $\tilde\theta^1, \tilde\theta^2 \in \dot H^s$. Then for $s\geq 1$,

$$\begin{aligned} &\|\mathcal{K}[\omega^{(1)}] f - \mathcal{K}[\omega^{(2)}] f\|_s \\ \leq \quad & C_1 \exp\Big(C_2 \big(\|\tilde\theta^{(1)}\|_s + \|\tilde\theta^{(2)}\|_s \big) \Big) \|\tilde\theta^{(1)} - \tilde\theta^{(2)}\|_s \|f\|_1, \end{aligned}$$

while for for $s \geq 3$,

$$
\begin{aligned}
&\|\mathcal{K}[\omega^{(1)}]f - \mathcal{K}[\omega^{(2)}]f\|_s \\
\leq\ & C_1 \exp\Big(C_2\big(\|\tilde\theta^{(1)}\|_{s-1} + \|\tilde\theta^{(2)}\|_{s-1}\big)\Big)\Big(\big(\|\tilde\theta^{(1)}\|_s + \|\tilde\theta^{(2)}\|_s\big)\|\tilde\theta^{(1)} - \tilde\theta^{(2)}\|_{s-1} \\
&\qquad\qquad + \|\tilde\theta^{(1)} - \tilde\theta^{(2)}\|_s\Big)\|f\|_1,
\end{aligned}
$$

where the constants C_1 and C_2 depend on s only.

Proof. We note that

$$
\begin{aligned}
\mathcal{K}[\omega^{(1)}]f - \mathcal{K}[\omega^{(2)}]f = &-\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi}\frac{f(\alpha')}{\omega_\alpha^{(1)}(\alpha')}\left(\frac{q_2[\omega^{(1)}](\alpha',\alpha)}{q_1[\omega^{(1)}](\alpha',\alpha)} - \frac{q_2[\omega^{(2)}](\alpha',\alpha)}{q_1[\omega^{(2)}](\alpha',\alpha)}\right)d\alpha' \\
&- \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')\left(\frac{1}{\omega_\alpha^{(1)}(\alpha')} - \frac{1}{\omega_\alpha^{(2)}(\alpha')}\right)\frac{q_2[\omega^{(2)}](\alpha',\alpha)}{q_1[\omega^{(2)}](\alpha',\alpha)}d\alpha' \\
&- \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')\left(\frac{1}{2\omega_\alpha^{(1)}(\alpha')} - \frac{1}{2\omega_\alpha^{(2)}(\alpha')}\right) l\big(\frac{1}{2}(\alpha-\alpha')\big)d\alpha'.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\frac{q_2[\omega^{(1)}](\alpha',\alpha)}{q_1[\omega^{(1)}](\alpha',\alpha)} - \frac{q_2[\omega^{(2)}](\alpha',\alpha)}{q_1[\omega^{(2)}](\alpha',\alpha)} = &\frac{q_2[\omega^{(1)} - \omega^{(2)}](\alpha',\alpha)}{q_1[\omega^{(1)}](\alpha',\alpha)} \\
&- \frac{q_2[\omega^{(2)}(\alpha',\alpha) q_1[\omega^{(1)} - \omega^{(2)}](\alpha',\alpha)}{q_1[\omega^{(2)}](\alpha',\alpha) q_1[\omega^{(1)}](\alpha',\alpha)}.
\end{aligned}
$$

Therefore, using Sobolev inequality $|.|_\infty \leq C\|.\|_1$, we obtain

$$
\begin{aligned}
\|\mathcal{K}[\omega^{(1)}]f - \mathcal{K}[\omega^{(2)}]f\|_0 \leq C_1\|f\|_0 &\left\|\frac{1}{\omega_\alpha^{(1)}}\right\|_1 \Big(\|q_2[\omega^{(1)} - \omega^{(2)}]\|_0 \\
+\|q_1[\omega^{(1)} - \omega^{(2)}]\|_1\|q_2[\omega^{(2)}\|_0\Big) &+ C_2\|f\|_0\left\|\frac{1}{\omega_\alpha^{(1)}} - \frac{1}{\omega_\alpha^{(2)}}\right\|_1\Big(\|q_2[\omega^{(2)}]\|_0 + 1\Big).
\end{aligned}
$$

The first statement follows easily from Lemmas 3.1, 3.4 and 3.5. Further, using one integration by parts, the sth derivative of $\mathcal{K}[\omega]f$ is

$$
\begin{aligned}
D_\alpha^s\mathcal{K}[\omega]f(\alpha) &= D_\alpha^{s-1}\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha(\alpha')}\Big)\Big[\frac{\omega_\alpha(\alpha)}{\omega(\alpha)-\omega(\alpha')} - \frac{1}{2}\cot\frac{1}{2}(\alpha-\alpha')\Big]d\alpha' \\
&= \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha(\alpha')}\Big)D_\alpha^{s-1}\Big[\frac{\omega_\alpha(\alpha)}{\omega(\alpha)-\omega(\alpha')} - \frac{1}{2}\cot\frac{1}{2}(\alpha-\alpha')\Big]d\alpha' \\
&= \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha(\alpha')}\Big)D_\alpha^{s-1}\Big[\frac{\omega_\alpha(\alpha)}{\omega(\alpha)-\omega(\alpha')} - \frac{1}{\alpha-\alpha'}\Big]d\alpha' \\
&\quad -\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} D_{\alpha'}\Big(\frac{f(\alpha')}{2\omega_\alpha(\alpha')}\Big)D_\alpha^{s-1} l\big(\frac{1}{2}(\alpha-\alpha')\big)d\alpha'. \qquad (3.11)
\end{aligned}
$$

Hence, we have

$$\begin{aligned}(3.12)\quad & D_\alpha^s\big(\mathcal{K}[\omega^{(1)}]f(\alpha)-\mathcal{K}[\omega^{(2)}]f(\alpha)\big)\\ =& -\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi}\Big(D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha^{(1)}(\alpha')}\Big)D_\alpha^{s-1}\frac{q_2[\omega^{(1)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}-D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha^{(2)}(\alpha')}\Big)D_\alpha^{s-1}\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(2)}](\alpha,\alpha')}\Big)d\alpha'\\ &-\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi}\Big(D_{\alpha'}\Big(\frac{f(\alpha')}{2\omega_\alpha^{(1)}(\alpha')}\Big)-D_{\alpha'}\Big(\frac{f(\alpha')}{2\omega_\alpha^{(2)}(\alpha')}\Big)\Big)D_\alpha^{s-1}l\big(\frac{1}{2}(\alpha-\alpha')\big)d\alpha'.\end{aligned}$$

Let us see the first part on the right side of (3.12). It can be split as

$$\begin{aligned}(3.13)\quad & \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi}D_{\alpha'}\Big(f(\alpha')\frac{\omega_\alpha^{(2)}(\alpha')-\omega_\alpha^{(1)}(\alpha')}{\omega_\alpha^{(1)}(\alpha')\omega_\alpha^{(2)}(\alpha')}\Big)D_\alpha^{s-1}\frac{q_2[\omega^{(1)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}d\alpha'\\ &+\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi}D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha^{(2)}(\alpha')}\Big)D_\alpha^{s-1}\Big[\frac{q_2[\omega^{(1)}-\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}\Big]d\alpha'\\ &+\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi}D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha^{(2)}(\alpha')}\Big)D_\alpha^{s-1}\Big[\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(2)}](\alpha,\alpha')}\frac{q_1[\omega^{(2)}-\omega^{(1)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}\Big]d\alpha'.\end{aligned}$$

By Proposition 2.4, Lemma 3.1, Lemma 3.5 and Note 1.2, the L^∞-norm of the first part of (3.13) is bounded by

$$\begin{aligned}&C_1\Big\|\frac{f}{\omega_\alpha^{(1)}\omega_\alpha^{(2)}}(\omega_\alpha^{(1)}-\omega_\alpha^{(2)})\Big\|_1\|\omega_\alpha^{(1)}\|_s\exp\big(C_2\|\omega_\alpha^{(1)}\|_{s-1}\big)\\ \le\; & C_1\left(\|\tilde\theta^{(1)}\|_s+\|\tilde\theta^{(2)}\|_s+1\right)\exp\left(C_2\big(\|\tilde\theta^{(1)}\|_{s-1}+\|\tilde\theta^{(2)}\|_{s-1}\big)\right)\|\tilde\theta^{(1)}-\tilde\theta^{(2)}\|_1\|f\|_1,\end{aligned}$$

with C_1 and C_2 depending on s. For the second term in (3.13), we use the Cauchy-Schwartz inequality, Lemmas 3.1, 3.5 to obtain the bound as quoted in the lemma. For the third term, we apply the similar argument. We note that for $0\le l<s-1$,

$$\begin{aligned}&\left\|D_\alpha^l\left[\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(2)}](\alpha,\alpha')}\right]D_\alpha^{s-1-l}\left[\frac{q_1[\omega^{(2)}-\omega^{(1)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}\right]\right\|_0\\ &\qquad\le\left|D_\alpha^l\left[\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(2)}](\alpha,\alpha')}\right]\right|_\infty\left\|D_\alpha^{s-1-l}\left[\frac{q_1[\omega^{(2)}-\omega^{(1)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}\right]\right\|_0.\end{aligned}$$

It is readily checked that

$$D_{\alpha'}\left[\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(2)}](\alpha,\alpha')}\right]=-D_\alpha\left[\frac{q_2[\omega^{(2)}](\alpha',\alpha)}{q_1[\omega^{(2)}](\alpha',\alpha)}\right].$$

Since $|.|_\infty\le C\|,\|_1$, it follows from Lemma 3.5 that for $l<s-1$.

$$\left|D_\alpha^l\left[\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(2)}](\alpha,\alpha')}\right]\right|_\infty\le C_1\|\omega_\alpha^{(2)}\|_s\exp\left(C_2\|\omega_\alpha^{(2)}\|_{s-1}\right)$$

with C_1 and C_2 depending on s. When $l=s-1$,

$$\begin{aligned}&\left\|D_\alpha^{s-1}\left[\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(2)}](\alpha,\alpha')}\right]\frac{q_1[\omega^{(2)}-\omega^{(1)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}\right\|_0\\ &\qquad\le\left|\left[\frac{q_1[\omega^{(2)}-\omega^{(1)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}\right]\right|_\infty\left\|D_\alpha^{s-1}\left[\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(2)}](\alpha,\alpha')}\right]\right\|_0.\end{aligned}$$

Once again using Sobolev inequality $|.|_\infty \le C\|.\|_1$ and using Lemmas 3.1, 3.5 we obtain the stated bounds.

Since the function l is symmetric about α and α', it is easy to see that the stated bounds also hold for the second part on the right side of (3.12).

For $s \ge 3$, we use the more refined estimates in Lemma 3.1 to obtain the third statement.

$\square$

Lemma 3.9. *(See [1] or appendix for proof) For $\psi \in H^s\big(\mathbb{T}[0,2\pi]\big)$ with $s \ge 1$, the operator $[\mathcal{H},\psi]$ is bounded from $H^0\big(\mathbb{T}[0,2\pi]\big)$ to $H^{s-1}\big(\mathbb{T}[0,2\pi]\big)$. And we have*

$$\|[\mathcal{H},\psi]f\|_{s-1} \le C\|f\|_0\|\psi\|_s,$$

where C depends on s.

Lemma 3.10. *For $s > \frac{1}{2}$ and $\psi \in H^s\big(\mathbb{T}[0,2\pi]\big)$, the operator $[\mathcal{H},\psi]$ is bounded from $H^1\big(\mathbb{T}[0,2\pi]\big)$ to $H^s\big(\mathbb{T}[0,2\pi]\big)$, and*

$$\|[\mathcal{H},\psi]f\|_s \le C\|f\|_1\|\psi\|_s,$$

where C depends on s.

Proof. We know that

$$\|[\mathcal{H},\psi]f\|_s^2 \;=\; \sum_{k\neq 0}|k|^{2s}\big|\widehat{\mathcal{H}(\psi f)}(k) - \widehat{\psi\mathcal{H}f}(k)\big|^2 + \big|\widehat{\mathcal{H}(\psi f)}(0) - \widehat{\psi\mathcal{H}f}(0)\big|^2.$$

Since

$$\widehat{\mathcal{H}(\psi f)}(k) = (-i)\,\mathrm{sgn}(k)\widehat{\psi f}(k) = (-i)\,\mathrm{sgn}(k)\sum_{j=-\infty}^{\infty}\hat\psi(j)\hat f(k-j), \text{ for } k\neq 0,$$

and

$$\widehat{\psi\mathcal{H}f}(k) = \sum_{j=-\infty}^{\infty}\hat\psi(j)\widehat{\mathcal{H}f}(k-j) = (-i)\sum_{j\neq k}\hat\psi(j)\,\mathrm{sgn}(k-j)\hat f(k-j),$$

by Cauchy's inequality and the inequality $\|gh\|_0 \le |h|_\infty\|g\|_0 \le C\|h\|_1\|g\|_0$, we have

$$\begin{aligned}
&\big\|[\mathcal{H},\psi]f\big\|_s^2\\
=\;& \sum_{k\neq 0}|k|^{2s}\Big| -i\,\mathrm{sgn}(k)\sum_{j=-\infty}^{\infty}\hat\psi(j)\hat f(k-j) + i\sum_{j\neq k}\hat\psi(j)\,\mathrm{sgn}(k-j)\hat f(k-j)\Big|^2\\
&+\Big| -i\sum_{j\neq 0}\hat\psi(j)\,\mathrm{sgn}(-j)\hat f(-j)\Big|^2\\
=\;& \sum_{k>0}|k|^{2s}\Big|2\sum_{j>k}\hat\psi(j)\hat f(k-j) + \hat\psi(k)\hat f(0)\Big|^2\\
&+\sum_{k<0}|k|^{2s}\Big|2\sum_{j<k}\hat\psi(j)\hat f(k-j) + \hat\psi(k)\hat f(0)\Big|^2 + \Big|\sum_{j\neq 0}\hat\psi(j)\,\mathrm{sgn}(-j)\hat f(-j)\Big|^2\\
\le\;& \sum_{k>0}8|k|^{2s}\Big|\sum_{j>k}\hat\psi(j)\hat f(k-j)\Big|^2 + \sum_{k<0}8|k|^{2s}\Big|\sum_{j<k}\hat\psi(j)\hat f(k-j)\Big|^2
\end{aligned}$$

$$
\begin{aligned}
& +2\|\psi\|_s^2|\hat{f}(0)|^2 + \Big|\sum_{j\neq 0}\hat{\psi}(j)\,\mathrm{sgn}(-j)\hat{f}(-j)\Big|^2 \\
\leq\ & \sum_{k>0} 8\Big|\sum_{j>k}|j|^s\big|\hat{\psi}(j)\hat{f}(k-j)\big|\Big|^2 + \sum_{k<0} 8\Big|\sum_{j<k}|j|^s\big|\hat{\psi}(j)\hat{f}(k-j)\big|\Big|^2 \\
& +2\|\psi\|_s^2|\hat{f}(0)|^2 + \|\psi\|_0^2\|f\|_0^2 \\
\leq\ & 8\sum_{k=-\infty}^{\infty}\Big|\sum_{j=-\infty}^{\infty}\big||j|^s\hat{\psi}(j)\big|\big|\hat{f}(k-j)\big|\Big|^2 + 3\|\psi\|_s^2\|f\|_0^2.
\end{aligned}
$$

We define

$$\left\{|j|^s|\hat{\psi}(s)|\right\}_{j\in\mathbb{Z}} = \Psi \text{ and } \left\{\hat{f}(j)\right\}_{j\in\mathbb{Z}} = \boldsymbol{f}.$$

By Proposition 3.1999 in [23], we know that $\|\boldsymbol{f} * \Psi\|_2 \leq \|\boldsymbol{f}\|_1\|\Psi\|_2$. Hence we obtain the result of the lemma.

□

Lemma 3.11. *If $f \in H^1(\mathbb{T}[0,2\pi])$, $\omega^{(1)}$ and $\omega^{(2)}$ correspond to $\tilde{\theta}^{(1)}$ and $\tilde{\theta}^{(2)}$ respectively, each in $\dot{H}^1$, $\|\tilde{\theta}^{(1)}\|_1$ and $\|\tilde{\theta}^{(2)}\|_1 < \epsilon_1$, then*

$$\|\mathcal{G}[\omega^{(1)}]f - \mathcal{G}[\omega^{(2)}]f\|_0 \quad \leq \quad C_1\|f\|_0\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_1.$$

Suppose $\tilde{\theta}^1, \tilde{\theta}^2 \in \dot{H}^s$. Then for $s \geq 1$,

$$
\begin{aligned}
& \|\mathcal{G}[\omega^{(1)}]f - \mathcal{G}[\omega^{(2)}]f\|_s \\
\leq\ & C_1 \exp\Big(C_2\big(\|\tilde{\theta}^{(1)}\|_s + \|\tilde{\theta}^{(2)}\|_s\big)\Big)\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_s\|f\|_1,
\end{aligned}
$$

while for $s \geq 3$,

$$
\begin{aligned}
& \|\mathcal{G}[\omega^{(1)}]f - \mathcal{G}[\omega^{(2)}]f\|_s \\
\leq\ & C_1 \exp\Big(C_2\big(\|\tilde{\theta}^{(1)}\|_{s-1} + \|\tilde{\theta}^{(2)}\|_{s-1}\big)\Big)\Big(\big(\|\tilde{\theta}^{(1)}\|_s + \|\tilde{\theta}^{(2)}\|_s\big)\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_{s-1} \\
& \qquad + \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_s\Big)\|f\|_1,
\end{aligned}
$$

where the constants C_1 and C_2 depend on s only.

Proof. From (1.6), it follows that

$$
\begin{aligned}
\|\mathcal{G}[\omega^{(1)}]f - \mathcal{G}[\omega^{(2)}]f\|_0 \leq & \left\|(\omega_\alpha^{(1)} - \omega_\alpha^{(2)})\left[\mathcal{H}, \frac{1}{\omega_\alpha^{(1)}}\right] f\right\|_0 + \left\|\omega_\alpha^2\left[\mathcal{H}, \frac{1}{\omega_\alpha^{(1)}} - \frac{1}{\omega_\alpha^{(2)}}\right] f\right\|_0 \\
& + 2\left\|(\omega_\alpha^{(1)} - \omega_\alpha^{(2)})\mathcal{K}[\omega^{(1)}]f\right\|_0 + 2\left\|\omega_\alpha^{(2)}\left(\mathcal{K}[\omega^{(1)}] - \mathcal{K}[\omega^{(2)}]\right)f\right\|_0.
\end{aligned}
$$

Using Lemma 3.1, 3.8, 3.10 and $\|hg\|_0 \leq |h|_\infty\|g\|_0 \leq C\|h\|_1\|g\|_0$, the first statement holds.

Now, consider

$$
\begin{aligned}
\|\mathcal{G}[\omega^{(1)}]f - \mathcal{G}[\omega^{(2)}]f\|_s \leq & \left\|(\omega_\alpha^{(1)} - \omega_\alpha^{(2)})\left[\mathcal{H}, \frac{1}{\omega_\alpha^{(1)}}\right] f\right\|_s + \left\|\omega_\alpha^2\left[\mathcal{H}, \frac{1}{\omega_\alpha^{(1)}} - \frac{1}{\omega_\alpha^{(2)}}\right] f\right\|_s \\
& + 2\left\|(\omega_\alpha^{(1)} - \omega_\alpha^{(2)})\mathcal{K}[\omega^{(1)}]f\right\|_s + 2\left\|\omega_\alpha^{(2)}\left(\mathcal{K}[\omega^{(1)}] - \mathcal{K}[\omega^{(2)}]\right)f\right\|_s.
\end{aligned}
$$

Using Lemmas 3.1, 3.8, 3.10, using $\|hg\|_s \leq C_s\|h\|_s\|g\|_s$ for $s \geq 1$ and $\|hg\|_s \leq C_s(\|h\|_{s-1}\|g\|_s + \|h\|_s\|g\|_{s-1})$ for $s \geq 2$, we see that the last two statements hold.

□

Proposition 3.12. *Assume $\tilde\theta \in \dot H^s$ for $s \geq 3$. If $\|\tilde\theta\|_1 < \epsilon_1$, then for sufficiently small ϵ_1, there exists unique solution $\gamma \in \{u \in H^{s-2}\big(\mathbb{T}[0,2\pi]\big) | \hat u(0) = 0\}$ satisfying (2.6). This solution γ satisfies the estimates*

$$\begin{aligned}
\|\gamma\|_0 &\leq \frac{C_0\sigma}{L}\|\tilde\theta\|_2,\\
\|\gamma\|_{s-2} &\leq \frac{C_1\sigma}{L}\exp(C_2\|\tilde\theta\|_{s-2})\|\tilde\theta\|_s,\\
\left\|\gamma - \frac{2\pi}{L}\sigma\theta_{\alpha\alpha}\right\|_s &\leq \frac{C_3\sigma}{L}\exp(C_4\|\tilde\theta\|_{s-1})\|\tilde\theta\|_s\|\tilde\theta\|_3,
\end{aligned}$$

where C_1, C_2, C_3 and C_4 depend on s, but all are independent of L. And for $s = 3$, $C_2 = 0$.

If $\gamma^{(1)}$ and $\gamma^{(2)}$ correspond respectively to $(\tilde\theta^{(1)}, L^{(1)}) \in \mathcal{V}$ and $(\tilde\theta^{(2)}, L^{(2)}) \in \mathcal{V}$, then for $3 \leq s \leq r$,

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{s-2} \leq C\left(\|\tilde\theta^{(1)} - \tilde\theta^{(2)}\|_s + |L^{(1)} - L^{(2)}|\right),$$

$$\left\|\gamma^{(1)} - \frac{2\pi\sigma}{L^{(1)}}\theta^{(1)}_{\alpha\alpha} - \gamma^{(2)} + \frac{2\pi\sigma}{L^{(2)}}\theta^{(2)}_{\alpha\alpha}\right\|_{s-2} \leq C\left(\|\tilde\theta^{(1)} - \tilde\theta^{(2)}\|_{s-2} + |L^{(1)} - L^{(2)}|\right),$$

where C depends on the diameter of $\mathcal{V}$ and s.

Proof. From (2.3), since $\hat\gamma(0) = 0$, $\mathcal{F}[\omega_0]\gamma = 0$. Therefore, (2.6) implies

$$\left[I + A_\mu\left(\mathcal{F}[\omega] - \mathcal{F}[\omega_0]\right)\right]\gamma = \frac{2\pi\sigma}{L}\theta_{\alpha\alpha}$$

Therefore, if $\tilde\theta \in \dot H^2$, then Lemma 3.11 implies

$$\|\mathcal{F}[\omega]\gamma - \mathcal{F}[\omega_0]\gamma\|_0 \leq C\|\tilde\theta\|_1\|\gamma\|_0.$$

where C depends on ϵ_1. So, for sufficiently small ϵ_1, if $\|\tilde\theta\|_1 \leq \epsilon_1$, then

$$\left[1 + A_\mu\left(\mathcal{F}[\omega] - \mathcal{F}[\omega_0]\right)\right]^{-1}$$

exists and from the bounds above and Corollary 2.9,

$$\|\gamma\|_0 \leq \frac{C_0\sigma}{L}\|\theta_{\alpha\alpha}\|_0 \leq \frac{C_0\sigma}{L}\|\tilde\theta\|_2.$$

Further, we obtain from the second part of Lemma 3.11,

$$\|\mathcal{F}[\omega]\gamma - \mathcal{F}[\omega_0]\gamma\|_{s-2} \leq C_1\exp(C_2\|\tilde\theta\|_{s-2})\|\tilde\theta\|_{s-2}\|\gamma\|_1,$$

where C_1 and C_2 depend on s. Therefore, for $s \geq 3$, it follows from (2.6) that

$$\|\gamma\|_{s-2} \leq \frac{2\pi\sigma}{L}\|\tilde\theta\|_s + C_1\exp(C_2\|\tilde\theta\|_{s-2})\|\tilde\theta\|_{s-2}\|\gamma\|_1$$

which C_1 and C_2 depend on s, which implies for sufficiently small ϵ_1 that the second statement holds.

For the third statement, we note that (2.6) and the third part of Lemma 3.11 implies that

$$\left\|\gamma - \frac{2\pi\sigma}{L}\theta_{\alpha\alpha}\right\|_s \leq \frac{C_3\sigma}{L}\exp(C_4\|\tilde\theta\|_{s-1})\|\tilde\theta\|_s\|\tilde\theta\|_3,$$

where C_3 and C_4 depend on s.

From (2.6), we obtain

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{s-2} \leq \left\|\frac{2\pi\sigma}{L^{(1)}}\theta^{(1)}_{\alpha\alpha} - \frac{2\pi\sigma}{L^{(2)}}\theta^{(2)}_{\alpha\alpha}\right\|_{s-2} + \|\mathcal{F}[\omega^{(1)}]\gamma^{(1)} - \mathcal{F}[\omega^{(2)}]\gamma^{(2)}\|_{s-2}.$$

Using

$$\|\frac{1}{L^{(1)}}\theta_{\alpha\alpha}^{(1)}-\frac{1}{L^{(2)}}\theta_{\alpha\alpha}^{(2)}\|_{s-2}\leq\frac{|L^{(1)}-L^{(2)}|}{L^{(1)}L^{(2)}}\|\theta_{\alpha\alpha}^{(1)}\|_{s-2}+\frac{1}{L^{(2)}}\|\theta_{\alpha\alpha}^{(1)}-\theta_{\alpha\alpha}^{(2)}\|_{s-2},$$

and using Lemmas 3.11 and the first part of the proposition,

(3.14)

$$\begin{aligned}\|\mathcal{F}[\omega^{(1)}]\gamma^{(1)}-\mathcal{F}[\omega^{(2)}]\gamma^{(2)}\|_{s-2}\leq&\left\|\mathcal{F}[\omega^{(1)}](\gamma^{(1)}-\gamma^{(2)})-\mathcal{F}[\omega_0](\gamma^{(1)}-\gamma^{(2)})\right\|_{s-2}\\&+\left\|\mathcal{F}[\omega^{(1)}]\gamma^{(2)}-\mathcal{F}[\omega^{(2)}]\gamma^{(2)}\right\|_{s-2}\\&\leq C\|\tilde{\theta}^{(2)}\|_{s-2}\|\gamma^{(1)}-\gamma^{(2)}\|_1+\frac{C}{L^{(2)}}\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_{s-2}\|\tilde{\theta}^{(2)}\|_3\end{aligned}$$

with C depending on s and the diameter of $\mathcal{V}$. The fourth statement in the proposition follows since $(\tilde{\theta}^{(1)},L^{(1)}),(\tilde{\theta}^{(2)},L^{(2)})\in\mathcal{V}$. The fifth statement follows from (2.6) by the same set of arguments as above.

□

Lemma 3.13. *Assume $\tilde{\theta}\in\dot{H}^s$ for $s\geq 3$. If $\|\tilde{\theta}\|_1<\epsilon_1$, for sufficiently small ϵ_1, the corresponding U and T in (2.4) and (B.3), with $\hat{\theta}(1;t)$ and $\hat{\theta}(-1;t)$ determined from $\tilde{\theta}$ using (B.4), satisfies the following estimates:*

$$\left\|U-\frac{2\pi^2\sigma}{L^2}\mathcal{H}[\theta_{\alpha\alpha}]\right\|_0\leq\frac{C_1\sigma}{L^2}\|\tilde{\theta}\|_1\|\tilde{\theta}\|_2,$$

$$\left\|U-\frac{2\pi^2\sigma}{L^2}\mathcal{H}[\theta_{\alpha\alpha}]\right\|_{s-2}\leq\frac{C_2\sigma}{L^2}\exp(C_3\|\tilde{\theta}\|_{s-2})\|\tilde{\theta}\|_{s-2}\|\tilde{\theta}\|_3,$$

$$\left\|U-\frac{2\pi^2\sigma}{L^2}\mathcal{H}[\theta_{\alpha\alpha}]\right\|_{s}\leq\frac{C_2\sigma}{L^2}\exp(C_3\|\tilde{\theta}\|_{s-1})\|\tilde{\theta}\|_{s}\|\tilde{\theta}\|_3,$$

$$\|U\|_{s-2}\leq\frac{C_2\sigma}{L^2}\exp(C_3\|\tilde{\theta}\|_{s-2})\|\tilde{\theta}\|_s,$$

$$\|T\|_{s-1}\leq\frac{C_2\sigma}{L^2}\exp(C_3\|\tilde{\theta}\|_{s})\|\tilde{\theta}\|_s,$$

where C_1 depends on ϵ_1, C_2 and C_3 depend on s.

If $U^{(1)}$ and $U^{(2)}$ (or $T^{(1)}$, $T^{(2)}$) correspond respectively to $\left(\tilde{\theta}^{(1)},L^{(1)}\right)\in\mathcal{V}$ and $\left(\tilde{\theta}^{(2)},L^{(2)}\right)\in\mathcal{V}$, then for $r\geq 3$,

$$\|U^{(1)}-U^{(2)}\|_{r-2}\leq C_4\left(\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_r+|L^{(1)}-L^{(2)}|\right),$$

$$\begin{aligned}&\left\|U^{(1)}-\frac{2\pi^2}{(L^{(1)})^2}\mathcal{H}[\theta_{\alpha\alpha}^{(1)}]-U^{(2)}-\frac{2\pi^2}{(L^{(2)})^2}\mathcal{H}[\theta_{\alpha\alpha}^{(2)}]\right\|_{r-2}\\&\qquad\leq C_4\left(\|\tilde{\theta}^{(1)}\|_r+\|\tilde{\theta}^{(2)}\|_r\right)\left(\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_{r-2}+|L^{(1)}-L^{(2)}|\right),\end{aligned}$$

$$\|T^{(1)}-T^{(2)}\|_{r-1}\leq C_4\left(\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_r+|L^{(1)}-L^{(2)}|\right),$$

where C_4 depends on the diameter of $\mathcal{V}$ and r.

Proof. From (1.6) and (2.2), Lemma 3.11 it follows that

$$\left\| U - \frac{\pi}{L}\mathcal{H}\gamma \right\|_0 \le \frac{\pi}{L}\|\mathcal{G}[\omega]\gamma - \mathcal{G}[\omega_0]\gamma\|_0 \le \frac{C_1}{L}\|\tilde{\theta}\|_1\|\gamma\|_0,$$

with C_1 depending on ϵ_1. Using Proposition 3.12, we obtain

$$\left\| U - \frac{2\pi^2\sigma}{L^2}\mathcal{H}[\theta_{\alpha\alpha}] \right\|_0 \le \frac{C_1\sigma}{L^2}\|\tilde{\theta}\|_1\|\tilde{\theta}\|_2$$

with C_1 depending on ϵ_1. Again from (1.6) and (2.2), Lemma 3.11 and Proposition 3.12, we obtain

$$\begin{aligned} \left\| U - \frac{2\pi^2\sigma}{L^2}\mathcal{H}[\theta_{\alpha\alpha}] \right\|_s &\le \frac{\pi}{L}\|\mathcal{G}[\omega]\gamma - \mathcal{G}[\omega_0]\gamma\|_s + \frac{\pi}{L}\left\| \mathcal{H}\left[\gamma - \frac{2\pi\sigma}{L}\theta_{\alpha\alpha}\right] \right\|_s \\ &\le \frac{C_2}{L}\exp(C_3\|\tilde{\theta}\|_{s-1})\|\tilde{\theta}\|_s\|\gamma\|_1 \le \frac{C_2\sigma}{L^2}\exp(C_3\|\tilde{\theta}\|_{s-1})\|\tilde{\theta}\|_s\|\tilde{\theta}\|_3, \end{aligned}$$

where C_2 and C_3 depend on s. Similarly, we can get the second and fourth statements. This gives all the desired results for U in terms of $\tilde{\theta}$.

Again, from noting that the second equation from (B.3), and the above estimates on U, we obtain

$$\|T\|_0 \le C\|U(1+\theta_\alpha)\|_1 \le \frac{C_2\sigma}{L^2}\exp(C_3\|\tilde{\theta}\|_3)\|\tilde{\theta}\|_3, \tag{3.15}$$

and

$$\begin{aligned} &\left\| T_\alpha - \frac{2\pi^2\sigma}{L^2}\mathcal{H}[\theta_{\alpha\alpha}] \right\|_{s-2} \\ \le\ &\left\| U - \frac{2\pi^2\sigma}{L^2}\mathcal{H}[\theta_{\alpha\alpha}] \right\|_{s-2} + \|U\theta_\alpha\|_{s-2} \\ \le\ &\frac{C_2\sigma}{L^2}\exp(C_3\|\tilde{\theta}\|_s)\left[\|\tilde{\theta}\|_{s-2}\|\tilde{\theta}\|_3 + \|\tilde{\theta}\|_s\|\tilde{\theta}\|_{s-1}\right], \end{aligned}$$

where C_2 and C_3 depend on s. Hence the fourth statement holds.

Also, we obtain from (1.6), (2.2),

$$\begin{aligned} &\|U^{(1)} - \frac{2\pi^2}{(L^{(1)})^2}\mathcal{H}[\theta^{(1)}_{\alpha\alpha}] - U^{(2)} + \frac{2\pi^2}{(L^{(2)})^2}\mathcal{H}[\theta^{(2)}_{\alpha\alpha}]\|_{r-2} \le \|\frac{\pi}{L_1}\mathcal{G}[\omega^{(1)}]\gamma^{(1)} - \frac{\pi}{L_2}\mathcal{G}[\omega^{(2)}]\gamma^{(2)}\|_{r-2} \\ &\le \frac{|L^{(1)} - L^{(2)}|}{L^{(1)}L^{(2)}}\|\mathcal{G}[\omega^{(1)}]\gamma^{(1)} - \mathcal{G}[\omega_0]\gamma^{(1)}\|_{r-2} + \frac{C}{L^{(2)}}\|\mathcal{G}[\omega^{(1)}]\gamma^{(1)} - \mathcal{G}[\omega^{(2)}]\gamma^{(1)}\|_{r-2} \\ &\qquad + \frac{C}{L^{(2)}}\left\| \mathcal{G}[\omega^{(2)}]\left(\gamma^{(1)} - \gamma^{(2)}\right) \right\|_{r-2}. \end{aligned}$$

The stated results on the differences between $U^{(1)}$ and $U^{(2)}$ follow from Lemma 3.11 and Proposition 3.12 on using the condition that each of $\left(\tilde{\theta}^{(1)}, L^{(1)}\right), \left(\tilde{\theta}^{(2)}, L^{(2)}\right) \in \mathcal{V}$. We note the second equation from (B.3), so the stated result follows for $T^{(1)} - T^{(2)}$ as well. □

4. Energy estimate

We define energy we will use is the $H^r\left(\mathbb{T}[0,2\pi]\right)$ norm of $\tilde{\theta}_n$; it is defined by

$$E_n(t) = \frac{1}{2}\int_0^{2\pi}(D^r\tilde{\theta}_n)^2 d\alpha.$$

We first need to estimate the following terms in the evolution equations.

Lemma 4.1. *Let* $X_n = \left(\tilde{\theta}_n, L_n\right) \in C^1\left([0,S); \mathcal{V}\right)$ *be the solution to the initial value problem (2.11) for* $r \geq 3$. *If the size* ϵ *of the ball* $\mathcal{B}$ *is small enough, then the corresponding energy* E_n*, as defined above, satisfies the inequality*

$$\frac{dE_n}{dt} \leq -\frac{\pi^2\sigma}{L_n^2} E_n.$$

Proof. For $r \geq 3$, taking the derivative of $E_n(t)$ with respect to t, we have

$$\frac{d}{dt} E_n(t) = \int_0^{2\pi} (D^r \tilde{\theta}_n)(D^r \tilde{\theta}_{n,t}) d\alpha.$$

Using (C.1) and (C.2), on integration by parts we find

$$\begin{aligned}
\frac{d}{dt} E_n &= I_1 + I_2 + I_3 + I_4, \text{where} \\
I_1 &= -\int_0^{2\pi} D^{r+1} \tilde{\theta}_n D^r \left(P_n U_n\right) d\alpha, \\
I_2 &= \int_0^{2\pi} D^r \tilde{\theta}_n D^{r-1} \left(P_n U_n\right) d\alpha, \\
I_3 &= \int_0^{2\pi} D^r \tilde{\theta}_n D^{r-1} P_n \left(\theta_{n,\alpha} U_n\right) d\alpha, \\
I_4 &= \int_0^{2\pi} D^r \tilde{\theta}_n D^r P_n \left(T_n \theta_{n,\alpha}\right) d\alpha.
\end{aligned}$$

On using $\tilde{\theta}_n = P_n \theta_n$, we can rewrite

$$I_1 = -\frac{2\pi^2\sigma}{L_n^2} \int_0^{2\pi} D^{r+1} \tilde{\theta}_n D^{r+2} \mathcal{H}[\tilde{\theta}_n] d\alpha - \int_0^{2\pi} D^{r+1} \tilde{\theta}_n P_n D^r \left[U_n - \frac{2\pi^2\sigma}{L_n^2} \mathcal{H}[\theta_{n,\alpha\alpha}]\right] d\alpha.$$

Using Lemma 3.13 to bound the second term I_1, it follows from Cauchy-Scwartz inequality that

$$I_1 \leq -\frac{2\pi^2\sigma}{L_n^2} \|\tilde{\theta}_n\|_{r+3/2}^2 + \frac{C_1\sigma}{L_n^2} \exp(C_2 \|\tilde{\theta}_n\|_{r-1}) \|\tilde{\theta}_n\|_r \|\tilde{\theta}\|_{r+1} \|\tilde{\theta}_n\|_3,$$

where C_1 and C_2 depend on s. Applying Lemma 3.13 once again, we obtain

$$\begin{aligned}
I_2 &= \frac{2\pi^2\sigma}{L_n^2} \int_0^{2\pi} D^r \tilde{\theta}_n D^{r+1} \mathcal{H}[\tilde{\theta}_n] d\alpha + \int_0^{2\pi} D^r \tilde{\theta}_n D^{r-1} P_n \left[U_n - \frac{2\pi\sigma}{L_n^2} \mathcal{H}[\theta_{n,\alpha\alpha}]\right] d\alpha \\
&\leq \frac{2\pi^2\sigma}{L_n^2} \|\tilde{\theta}_n\|_{r+1/2}^2 + \frac{C_1\sigma}{L_n^2} \exp(C_2 \|\tilde{\theta}_n\|_{r-1}) \|\tilde{\theta}_n\|_r^2 \|\tilde{\theta}_n\|_3, \\
I_3 &= \frac{2\pi^2\sigma}{L_n^2} \int_0^{2\pi} D^r \tilde{\theta}_n D^{r-1} P_n \big(\theta_{n,\alpha} \mathcal{H}[\theta_{n,\alpha\alpha}]\big) d\alpha \\
&\quad + \int_0^{2\pi} D^r \tilde{\theta}_n D^{r-1} P_n \left\{\theta_{n,\alpha} \left[U_n - \frac{2\pi\sigma}{L_n^2} \mathcal{H}[\theta_{n,\alpha\alpha}]\right]\right\} d\alpha \\
&\leq \frac{C_1\sigma}{L_n^2} \|\tilde{\theta}_n\|_{r+1} \|\tilde{\theta}_n\|_r \|\tilde{\theta}_n\|_{r-1} + \frac{C_1\sigma}{L_n^2} \exp(C_2 \|\tilde{\theta}_n\|_{r-1}) \|\tilde{\theta}_n\|_r^2 \|\tilde{\theta}_n\|_3,
\end{aligned}$$

$$
\begin{aligned}
I_4 &= \int_0^{2\pi} D^r\tilde{\theta}_n P_n\big(\sum_{j=0}^{r} C_{r,j} D^j T_n D^{r+1-j}\theta_n\big)d\alpha \\
&= \int_0^{2\pi} D^r\tilde{\theta}_n P_n(T_n D^{r+1}\theta_n)d\alpha + C_{r,1}\int_0^{2\pi} D^r\tilde{\theta}_n P_n\left(T_{n,\alpha}D^r\theta_n\right)d\alpha \\
&\quad + \int_0^{2\pi} D^r\tilde{\theta}_n P_n\left(\sum_{j=2}^{r} C_{r,j} D^{j-2}\left(U_n(1+\theta_{n,\alpha})\right)D^{r+1-j}\theta_n\right)d\alpha \\
&\le \frac{C_1\sigma}{L_n^2}\left(\exp(C_3\|\tilde{\theta}_n\|_3)\|\tilde{\theta}_n\|_3\|\tilde{\theta}_n\|_{r+1}^2 + \exp(C_2\|\tilde{\theta}_n\|_{r-1})\|\tilde{\theta}_n\|_r^2\|\tilde{\theta}_n\|_{r-1}\right),
\end{aligned}
$$

where C_1 and C_2 depend on s. Adding up I_1 through I_4, using $\tilde{\theta}_n \in \mathcal{B}$ and the fact that $\|\tilde{\theta}_n\|_{r+1/2}^2 \le \frac{1}{4}\|\tilde{\theta}_n\|_{r+3/2}^2$ since the Fourier 0 and ± 1 modes for $\tilde{\theta}_n$ are zero, we obtain for $r = 3$,

$$
\begin{aligned}
\frac{d}{dt}E_n &\le -\frac{3\pi^2\sigma}{2L_n^2}\|\tilde{\theta}_n\|_{r+3/2}^2 + \frac{C\sigma}{L_n^2}\|\tilde{\theta}_n\|_{r+1}^2\|\tilde{\theta}_n\|_3 \\
&\le -\frac{\sigma}{L_n^2}\|\tilde{\theta}_n\|_{r+3/2}^2\left(\frac{3}{2}\pi^2 - C\|\tilde{\theta}_n\|_3\right) \le -\frac{3\pi^2\sigma}{2L_n^2}E_n\left(1 - \frac{2C}{3\pi^2}(2E_n)^{1/2}\right),
\end{aligned}
\tag{4.1}
$$

and for $r > 3$,

$$
\begin{aligned}
\frac{d}{dt}E_n &\le -\frac{3\pi^2\sigma}{2L_n^2}\|\tilde{\theta}_n\|_{r+3/2}^2 + \frac{C\sigma}{L_n^2}\|\tilde{\theta}_n\|_{r+1}^2\|\tilde{\theta}_n\|_{r-1} \\
&\le -\frac{\sigma}{L_n^2}\|\tilde{\theta}_n\|_{r+3/2}^2\left(\frac{3}{2}\pi^2 - C\|\tilde{\theta}_n\|_{r-1}\right) \le -\frac{3\pi^2\sigma}{2L_n^2}E_n\left(1 - \frac{2C}{3\pi^2}(2E_n)^{1/2}\right),
\end{aligned}
\tag{4.2}
$$

where $C = C_1\exp(C_2\|\tilde{\theta}\|_{r-1})$ with C_1 and C_2 depending on r. It immediately follows that if $1 - C\frac{2}{3\pi^2}(2E_n)^{1/2} > 0$ initially, then $E_n(t)$ decreases in time and $E_n(t) \le E_n(0)$ for all t. This implies that for small enough ϵ, if $\tilde{\theta}_n \in \mathcal{B}$ initially, it remains there for any t for which the solution exists. More, generally, we have

$$
\frac{dE_n}{dt} \le -\frac{\pi^2\sigma}{L_n^2}E_n.
$$

□

Corollary 4.2. *Let* $\left(\tilde{\theta}_n, L_n\right) \in C^1\left([0,S);\mathcal{V}\right)$ *be the solution to the initial value problem (2.11) with* $r \ge 3$. *Then for the sufficiently small ball size* ϵ *of* $\mathcal{B}$, *as long as the solution exists, we have*

$$
\frac{dE_n}{dt} \le -\frac{\pi^2\sigma}{L_n^2}\|\tilde{\theta}_n\|_{r+3/2}^2,
$$

$$
\frac{d\|\tilde{\theta}_n\|_{r+1}^2}{dt} \le -\frac{\pi^2\sigma}{L_n^2}\|\tilde{\theta}_n\|_{r+1}^2.
$$

Proof. The proof of the first statement comes from (4.1) and (4.2).

Replacing r by $r+1$ in (4.2), we obtain

$$\frac{d\|\tilde{\theta}_n\|_{r+1}^2}{dt} \leq -\frac{3\pi^2\sigma}{2L_n^2}\|\tilde{\theta}_n\|_{r+5/2}^2 + \frac{C\sigma}{L_n^2}\|\tilde{\theta}_n\|_{r+2}^2\|\tilde{\theta}_n\|_r \leq -\frac{3\sigma\pi^2}{2L_n^2}\|\tilde{\theta}_n\|_{r+5/2}^2\left(1-\frac{2C}{3\pi^2}\|\tilde{\theta}_n\|_r\right), \tag{4.3}$$

where $C = C_1 \exp\left(C_2\|\tilde{\theta}_n\|_r\right)$ with C_1 and C_2 depending only on r. Hence for small enough ϵ, if $\tilde{\theta}_n \in \mathcal{B}$, then by (4.2), we have

$$\frac{d\|\tilde{\theta}_n\|_{r+1}^2}{dt} \leq -\frac{\sigma\pi^2}{L_n^2}\|\tilde{\theta}_n\|_{r+1}^2.$$

□

Proposition 4.3. *Let $(\tilde{\theta}_n, L_n) \in C^1\big([0,S);\mathcal{V}\big)$ be the solution to the initial value problem (2.11) with $r \geq 3$. Then for the sufficiently small ball size ϵ of $\mathcal{B}$, as long as the solution exists,*

$$E_n(t) \leq E_n(0)\exp\left[-\frac{\sigma t}{18}\right], \tag{4.4}$$

$$\|\tilde{\theta}_n(\cdot,t)\|_{r+1}^2 \leq \|\tilde{\theta}_n(\cdot,0)\|_{r+1}^2 \exp\left[-\frac{\sigma t}{18}\right], \tag{4.5}$$

$$\left|L_n^3(t) - 8\pi^3\right| \quad \leq \quad C\sqrt{E_n(0)}\Big(1-\exp(-\frac{1}{18}\sigma t)\Big), \tag{4.6}$$

where C is independent of n.

Proof. We note from the evolution equation for L_n may be rewritten as

$$L_n^2\frac{dL_n}{dt} = -L_n^2\int_0^{2\pi}\left[U_n - \frac{2\pi^2}{L_n^2}\mathcal{H}[\theta_{n,\alpha\alpha}]\right]d\alpha - L_n^2\int_0^{2\pi}U_n\theta_{n,\alpha}d\alpha.$$

Using Lemma 3.13, on integration, it follows that

$$|L_n^3(t) - (2\pi)^3| \leq C\int_0^t \|\tilde{\theta}(\cdot,t')\|_3^2 dt' \leq C\int_0^t E_n(t')dt', \tag{4.7}$$

where C is independent of n. Since $E_n(t) \leq E_n(0)$, it follows that

$$|L_n^3(t)| \leq 8\pi^3 + CE_n(0)t,$$

where C is independent of n. Using Lemma 4.1, we obtain preliminary estimates:

$$E_n(t) \leq E_n(0)\exp\left\{-\frac{3\sigma\pi^2}{CE_n(0)}\left[\left(8\pi^3 + CE_n(0)t\right)^{1/3} - 2\pi\right]\right\}.$$

Going back to (4.7), it follows that for sufficiently small $E_n(0)$, for any t,

$$|L_n^3(t) - 8\pi^3| < 1 \tag{4.8}$$

which implies that L_n cannot escape the interval $(2\pi-1, 2\pi+1)$. Going back to Lemma 4.1, this implies that

$$\frac{d}{dt}E_n \leq -\frac{\pi^2\sigma}{(2\pi+1)^2}E_n \leq -\frac{\sigma}{18}E_n$$

and therefore (4.4) follows. (4.5) follows from Corollary 4.2 once we use (4.8). Furthermore, plugging estimates (4.4) into (4.7), we have

$$|L_n^3(t) - 8\pi^3| \leq \frac{18CE_n(0)}{\sigma}\left[1 - \exp\left(-\frac{\sigma t}{18}\right)\right].$$

$\square$

Proof of Proposition 2.12: This follows readily from Lemma 4.1 and Proposition 4.3, since Lemma 4.1 assures that as long as the solution X_n to the initial value problem (2.11) exists, the corresponding $\tilde{\theta}_n$ does not exit the ball $\mathcal{B}$ and therefore Proposition 4.3 can be applied to obtain estimates on $E_n(t)$ and $L_n(t)$.

5. Existence of Solutions

In this section, we demonstrate existence of solutions to the initial value problem (2.11). We then show that these solutions converge (as the truncation n tends to ∞) to a solution of (B.1), (B.3) and (B.4) with the initial condition (2.5). We demonstrate that this solution to (B.1), (B.3) and (B.4) with the initial condition (2.5) is unique and has the same regularity as the initial data.

Definition 5.1. *We define*

$$|||X||| = \|u\|_r + |v|$$

for $X = (u, v) \in H^r\big(\mathbb{T}[0, 2\pi]\big) \times \mathbb{R}$.

Proof of Proposition 2.11: First we show that the operator $F_n : \mathcal{V} \to H^r\big(\mathbb{T}[0, 2\pi]\big) \times \mathbb{R}$ is bounded, i.e. $\|F_{n,1}\|_r + |F_{n,2}| < \infty, \forall X_n \in \mathcal{V}$. It follows from Lemma 3.13 that

$$\begin{aligned}
\big\|P_nU_{n,\alpha} + P_nT_n(1+\theta_{n,\alpha})\big\|_r &\leq \|U_{n,\alpha}\|_r + \|T_n\|_r + \|T_n\|_r\|\theta_{n,\alpha}\|_r \\
&\leq C\left(\|\tilde{\theta}_n\|_{r+3} + \|\tilde{\theta}_n\|_{r+1} + \|\tilde{\theta}_n\|^2_{r+1}\right) \leq Cn^3\|\tilde{\theta}_n\|_r, \\
|F_{n,2}| \leq \|1+\theta_{n,\alpha}\|_0\|U_n\|_0 &\leq C\|\tilde{\theta}_n\|_2\left(1 + \|\tilde{\theta}_n\|_1\right),
\end{aligned}$$

where C depends on n, r and the diameter of $\mathcal{V}$.

Consider $X_n^{(1)}, X_n^{(2)} \in \mathcal{V}$. We have

$$\begin{aligned}
\|F_{n,1}(X_n^{(1)}) - F_{n,1}(X_n^{(2)})\|_r &\leq \Big\|\big(\frac{2\pi}{L_n^{(1)}} - \frac{2\pi}{L_n^{(2)}}\big)P_n\big(U_{n,\alpha}^{(1)} + T_n^{(1)}(1+\theta_{n,\alpha}^{(1)})\big)\Big\|_r \\
&+ \frac{2\pi}{L_n^{(2)}}\big\|P_n\big(U_{n,\alpha}^{(1)} - U_{n,\alpha}^{(2)}\big)\big\|_r + \frac{2\pi}{L_n^{(2)}}\Big\|P_n\big(T_n^{(1)}(1+\theta_{n,\alpha}^{(1)}) - T_n^{(2)}(1+\theta_{n,\alpha}^{(2)})\big)\Big\|_r.
\end{aligned} \tag{5.1}$$

It follows from Lemma 3.13 that

$$\begin{aligned}
&\Big\|\big(\frac{2\pi}{L_n^{(1)}} - \frac{2\pi}{L_n^{(2)}}\big)P_n\big(U_{n,\alpha}^{(1)} + T_n^{(1)}(1+\theta_{n,\alpha}^{(1)})\big)\Big\|_r \\
&\leq Cn^3[|L_n^{(1)} - L_n^{(2)}| \leq c|||X_n^{(1)} - X_n^{(2)}|||,
\end{aligned} \tag{5.2}$$

where c depends on n, r and the diameter of $\mathcal{V}$. Further, using Lemma 3.13

$$\begin{aligned}
|F_{n,2}^{(2)} - F_{n,2}^{(2)}| &\leq C\left(\|U_n^{(1)} - U_n^{(2)}\|_0\big(1 + \|\tilde{\theta}_n^{(1)}\|_1\big) + \|U_n^{(2)}\|_0\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_1\right) \\
&\leq C\left(|L_1 - L_2| + \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_1\right) \leq c|||X_n^{(1)} - X_n^{(2)}|||,
\end{aligned}$$

where c depends on n, r and the diameter of $\mathcal{V}$. Therefore, from ODE theory, it follows that there exists local solution $X_n \in C^1\big([0, S_n]; \mathcal{V}\big)$ over some time interval S_n that may depend on n, r and ϵ.

Lemma 5.2. *There exists sufficiently small $\epsilon > 0$ such that solutions $X_n = (\tilde{\theta}_n, L_n) \in C^1([0, S]; \mathcal{V})$ of the initial value problem (2.11) form a Cauchy sequence in $C\left([0, S]; \dot{H}^1 \times \mathbb{R}\right)$ for any $S > 0$.*

Proof. We define difference energy function E_{mn} as

$$E_{mn} = E^1_{mn} + (L_n - L_m)^2$$

where $E^1_{mn} = \frac{1}{2}\int_0^{2\pi}\big(D(\tilde{\theta}_n - \tilde{\theta}_m)\big)^2 d\alpha$. Notice that $E_{mn}(0) = E^1_{mn}(0)$. Without loss of generality, we assume $m > n$ as otherwise we can switch the role of m and n in the ensuing argument.

Using the first equation in (C.1),

$$\frac{dE^1_{mn}}{dt} = \int_0^{2\pi} D(\tilde{\theta}_n - \tilde{\theta}_m) D^2\big(\frac{2\pi}{L_n} P_n U_n - \frac{2\pi}{L_m} P_m U_m\big) d\alpha \tag{5.3}$$

$$+ \int_0^{2\pi} D(\tilde{\theta}_n - \tilde{\theta}_m) D\Big(\frac{2\pi}{L_n} P_n\big(T_n(1+\theta_{n,\alpha})\big) - \frac{2\pi}{L_m} P_m\big(T_m(1+\theta_{m,\alpha})\big)\Big) d\alpha \equiv I_1 + I_2.$$

Defining $\tilde{\theta}_{nm} = \tilde{\theta}_n - \tilde{\theta}_m$, it is clear that

$$\begin{aligned} I_1 &= -2\pi\left(\frac{1}{L_n} - \frac{1}{L_m}\right)\int_0^{2\pi} D^2\tilde{\theta}_{nm} P_n D U_n d\alpha + \frac{2\pi}{L_m}\int_0^{2\pi} D\tilde{\theta}_{mn}(P_n - P_m) D^2 U_n \\ &+ \frac{2\pi}{L_m}\int_0^{2\pi} D\tilde{\theta}_{mn} P_m D^2(U_n - U_m) \equiv I_{1,1} + I_{1,2} + I_{1,3} \end{aligned}$$

From estimates in Lemma 3.13 and restrictions due to $\left(\tilde{\theta}_n, L_n\right), \left(\tilde{\theta}_m, L_n\right) \in \mathcal{V}$, we obtain

$$|I_{1,1}| \le c\epsilon E^{1/2}_{mn} \|\tilde{\theta}_{nm}\|_2,$$

where c depends on the diameter of $\mathcal{V}$. We note that since $P_n\theta_n = \tilde{\theta}_n$ and $P_m\theta_n = \tilde{\theta}_n$, as $m > n$, we can write $I_{1,2}$

$$I_{1,2} = \frac{2\pi}{L_m}\int_0^{2\pi} D\tilde{\theta}_{mn} D^2[P_n - P_m]\left(U_n - \frac{2\pi^2\sigma}{L_n^2}\mathcal{H}[\theta_{n,\alpha\alpha}]\right) d\alpha.$$

Therefore, using Lemma 3.13,

$$|I_{1,2}| \le \frac{c}{n} E^{1/2}_{mn}\left\|U_n - \frac{2\pi^2\sigma}{L_n^2}\mathcal{H}[\theta_{n,\alpha\alpha}]\right\|_3 \le \frac{C\epsilon}{n} E^{1/2}_{mn},$$

where C depends on the diameter of $\mathcal{V}$. Using $P_m\theta_n = \tilde{\theta}_n,\ P_m\theta_m = \tilde{\theta}_m$,

$$\begin{aligned} I_{1,3} &= \frac{2\pi}{L_m}\int_0^{2\pi} D\tilde{\theta}_{nm} P_m D^2\left(U_n - \frac{2\pi^2\sigma}{L_n^2}\mathcal{H}[\theta_{n,\alpha\alpha}] - U_m + \frac{2\pi^2\sigma}{L_m^2}\mathcal{H}[\theta_{m,\alpha\alpha}]\right) \\ &+ \frac{4\pi^3\sigma}{L_m L_n^2}\int_0^{2\pi} D\tilde{\theta}_{nm} D^2\mathcal{H}[\tilde{\theta}_{nm,\alpha\alpha}] - \frac{4\pi^3\sigma}{L_m}\left(\frac{1}{L_n^2} - \frac{1}{L_m^2}\right)\int_0^{2\pi} D^2\tilde{\theta}_{nm} D\mathcal{H}[\tilde{\theta}_{m,\alpha\alpha}] d\alpha. \end{aligned}$$

Integrating by parts the second term in $I_{1,3}$ above and using Lemma 3.13 again, we obtain

$$|I_{1,3}| \le -\frac{4\pi^3\sigma}{L_m L_n^2}\|\tilde{\theta}_{nm}\|_{5/2} + C\epsilon E_{mn} + C\epsilon E^{1/2}_{mn}\|\tilde{\theta}_{nm}\|_2,$$

where C depends on the diameter of $\mathcal{V}$. Now using Lemma 3.13, we obtain

$$\begin{aligned}\frac{d(L_n-L_m)^2}{dt} &= 2(L_m-L_n)\int_0^{2\pi}\left[(U_n-U_m)(1+\theta_{n,\alpha})+U_m(\theta_{n,\alpha}-\theta_{m,\alpha})\right]d\alpha\\ &\leq cE_{nm}^{1/2}\Big(\epsilon\|U_n-U_m\|_0+\left\|U_n-\frac{2\pi^2\sigma}{L_n^2}\mathcal{H}[\theta_{n,\alpha\alpha}]-U_m+\frac{2\pi^2\sigma}{L_m^2}\mathcal{H}[\theta_{m,\alpha\alpha}]\right\|_0\\ &\quad+\|\tilde{\theta}_{nm}\|_1\|U_m\|_0\Big)\leq C\epsilon\left(E_{nm}+E_{nm}^{1/2}\|\tilde{\theta}_{nm}\|_2\right),\end{aligned}$$

C depends on the diameter of $\mathcal{V}$. So for I_2, we use the same method as we did for I_1 and combine all the terms. So we obtain

$$\begin{aligned}\frac{dE_{mn}}{dt} &\leq -\frac{4\pi^3\sigma}{L_mL_n^2}\|\tilde{\theta}_{nm}\|_{5/2}^2+\frac{4\pi^3\sigma}{L_mL_n^2}\|\tilde{\theta}_{nm}\|_{3/2}^2+c\epsilon E_{mn}^{1/2}\|\tilde{\theta}_{nm}\|_2+\frac{c}{n}\epsilon E_{mn}^{1/2}+c\epsilon E_{mn}\\ &\leq -\frac{3\pi^3\sigma}{2(2\pi+1)^3}\|\tilde{\theta}_{nm}\|_{5/2}^2+\frac{c}{2}\epsilon\|\tilde{\theta}_{nm}\|_2^2+\frac{c}{n}E_{mn}^{1/2}+c_1\epsilon E_{mn},\end{aligned}$$

where c and c_1 depends on the diameter of $\mathcal{V}$. Since $\|\tilde{\theta}_{nm}\|_{5/2}\geq\|\tilde{\theta}_{nm}\|_2$, it follows that for ϵ sufficiently small

$$-\frac{3\pi^3\sigma}{(2\pi+1)^3}\|\tilde{\theta}_{nm}\|_{5/2}^2+c\epsilon\|\tilde{\theta}_{nm}\|_2^2\leq 0.$$

So,

$$\frac{dE_{mn}}{dt}\leq cE_{mn}+\frac{c}{n}E_{mn}^{1/2}.$$

This can be restated as

$$\frac{dE_{mn}^{1/2}}{dt}\leq cE_{mn}^{1/2}+\frac{c}{n}.$$

We solve the differential inequality to see that

$$E_{mn}^{1/2}(t)\leq E_{mn}^{1/2}(0)e^{ct}+\frac{1}{n}(e^{ct}-1).$$

Since

$$E_{mn}(0)=E_{mn}^1(0)\leq\frac{c}{n^2}\|\tilde{\theta}_0\|_r^2,$$

we have

$$E_{mn}^{1/2}(t)\leq\frac{c}{n}(\|\tilde{\theta}_0\|_r+1)e^{ct}.$$

Thus, solutions do form a Cauchy sequence in $C\left([0,S];\dot{H}^1\times\mathbb{R}\right)$. $\square$

Remark. We now know that the solutions of the initial value problem (2.11), $(\tilde{\theta}_n,L_n)$, approach a limit as $n\to\infty$ in $C\left([0,S];\dot{H}^1\times\mathbb{R}\right)$. Call this limit $X=(\tilde{\theta},L)$. $\square$

Note 5.3. *By Proposition 2.12, we know that $\|\tilde{\theta}_n(\cdot,t)\|_r\leq\|\mathcal{Q}_1\theta_0\|_r$ for all $t\geq 0$. Since $\dot{H}^r$ is a Hilbert space, its unit ball is weakly compact. Thus, $\tilde{\theta}_n\rightharpoonup\tilde{\theta}$ in $\dot{H}^r$. Furthermore, by Fatou's Lemma, we also have*

$$\|\tilde{\theta}\|_r\leq\liminf_{n\to\infty}\|\tilde{\theta}_n\|_r\leq\|\mathcal{Q}_1\theta_0\|_r.$$

Lemma 5.4. *For $r\geq 3$, there exists the sufficiently small ball size ϵ of $\mathcal{B}$ such that as $n\to\infty$, the limit of the initial value problem (2.11), $X=(\tilde{\theta},L)\in C\left((0,S];\mathcal{V}\right)$ for any $S>0$.*

Proof. Note that estimates in Corollary 4.2 and Proposition 4.3. Since $L_n \in (2\pi - 1, 2\pi + 1)$, we have

$$\frac{dE_n}{dt} \leq -\frac{\sigma}{9}\|\tilde{\theta}_n\|^2_{r+3/2}.$$

It implies

$$\frac{1}{2}E_n(t) + \frac{\sigma}{9}\int_0^t \|\tilde{\theta}_n\|^2_{r+3/2}dt \leq \frac{1}{2}E_n(0) \leq \frac{1}{2}\|\mathcal{Q}_1\theta_0\|_r.$$

Hence $\tilde{\theta}_n$ is a bounded sequence in $L^2\left([0,\infty), \dot{H}^{r+3/2}\right)$. So, there exists a subsequence that converges weakly, and it is easily argued that the limit can only be $\tilde{\theta}$. This means that for any interval $(0, S')$ there exists S_0 in that interval so that $\|\tilde{\theta}(., S_0)\|_{r+3/2} < \infty$. Now consider the solution to (B.1), (B.3) and (B.4) with S_0 as the initial time. In particular, $\tilde{\theta}(., S_0) \in \dot{H}^{r+1} \cap \mathcal{B}$. Taking $\tilde{\theta}(., S_0)$ as initial data in $\dot{H}^{r+1} \cap \mathcal{B}$, repeating the proof of Proposition 2.11 with $r+1$ instead of r, and by Corollary 4.2 and Proposition 4.3, we have global solutions $\tilde{\theta}_n^{S_0} \in C^1\left([S_0,\infty), \dot{H}^{r+1} \cap \mathcal{B}\right)$ for sufficiently small ϵ. Again, by uniqueness of solutions to the approximate equation (2.11) (Proposition 2.11), these solutions are identical to $\tilde{\theta}_n$ in their intervals of existence. Also, by Proposition 4.3, we have
(5.4)

$$\|\tilde{\theta}_n(\cdot,t)\|_{r+1} \leq \|\tilde{\theta}_n(\cdot,S_0)\|_{r+1}e^{-\frac{\sigma}{36}(t-S_0)} \leq \|\tilde{\theta}(\cdot,S_0)\|_{r+1}e^{-\frac{\sigma(t-S_0)}{36}}, \text{ for all } t \geq S_0.$$

From interpolation theorem in Sobolev space, we have

$$\|\tilde{\theta}_m - \tilde{\theta}_n\|_s \leq C\|\tilde{\theta}_m - \tilde{\theta}_n\|_0^{1-\frac{s}{r+1}}\|\tilde{\theta}_m - \tilde{\theta}_n\|_{r+1}^{\frac{s}{r+1}}. \tag{5.5}$$

By Lemma 5.2 and (5.4), we know that the right side of (5.5) goes to zero uniformly on $[S_0, S]$, as $n, m \to \infty$ for any $1 \leq s < r+1$. This implies $X \in C\left([S_0, S]; \dot{H}^s \times \mathbb{R}\right)$. Since the choice of S' is arbitrarily small, it follows that $\tilde{\theta} \in C\left((0, S], \dot{H}^r\right)$. □

Proposition 5.5. *(continuity at $t = 0$ in $\dot{H}^r$) For $r \geq 3$, we have*

$$\lim_{t\to 0+}\|\tilde{\theta}(\cdot,t) - \mathcal{Q}_1\theta_0\|_r = 0. \tag{5.6}$$

Proof. Replacing $r+1$ by r in (5.5), using the uniform bound of $\tilde{\theta}_n$ in $\dot{H}^r$ and $\tilde{\theta}_n \in C^1\left([0,\infty); \dot{H}^r\right)$, we find that $\tilde{\theta}_n \to \tilde{\theta}$ in $C\left([0,S]; \dot{H}^s\right)$ as $n \to \infty$ for any $S > 0$, where $1 \leq s < r$.

Let $\eta > 0$ and $\phi \in H^{-r}\big(\mathbb{T}[0,2\pi]\big)$. For any s satisfying $1 \leq s < r$, choose $\varphi \in H^{-s}\big(\mathbb{T}[0,2\pi]\big)$ so that

$$\|\phi - \varphi\|_{-r} \leq \frac{\eta}{3}. \tag{5.7}$$

We know that such a φ can be found since $H^{-s}\big(\mathbb{T}[0,2\pi]\big)$ is dense in $H^{-r}\big(\mathbb{T}[0,2\pi]\big)$. We have

$$\langle\phi, \tilde{\theta}_n\rangle - \langle\phi, \tilde{\theta}\rangle = \langle\phi - \varphi, \tilde{\theta}_n\rangle + \langle\varphi - \phi, \tilde{\theta}\rangle + \langle\varphi, \tilde{\theta}_n - \tilde{\theta}\rangle, \tag{5.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing with dual spaces. The first two terms can be bounded by $\frac{\eta}{3}$ using (5.7) and uniform bounds on $\tilde{\theta}$ and $\tilde{\theta}_n$ in $\dot{H}^r$. For the third term, we choose n large enough so that $\|\tilde{\theta} - \tilde{\theta}_n\|_s \leq \eta/3$. Thus, (5.8) is bounded by

η. Since η is arbitrary and these bounds are uniform in time, we conclude that $\tilde{\theta} \in C_W\left([0,S];\dot{H}^r\right)$. To prove the lemma, it is enough to show $\lim_{t\to 0+}\|\tilde{\theta}(\cdot,t)\|_r = \|\mathcal{Q}_1\theta_0\|_r = 0$.

By Note 5.3, we know $\|\tilde{\theta}(\cdot,t)\|_r \le \|\mathcal{Q}_1\theta_0\|_r$. This means $\limsup_{t\to 0+}\|\tilde{\theta}(\cdot,t)\|_r \le \|\mathcal{Q}_1\theta_0\|_r$. From the fact that $\tilde{\theta} \in C_W\big([0,S];\dot{H}^r\big)$, we have $\liminf_{t\to 0+}\|\tilde{\theta}(\cdot,t)\|_r \ge \|\mathcal{Q}_1\theta_0\|_r$. Hence, (5.6) holds. This gives us strong right continuity at $t=0$. □

By Lemma 5.4 and Proposition 5.5, we have

Corollary 5.6. *For $r \ge 3$, there exists the sufficiently small ball size ϵ of $\mathcal{B}$ such that $X \in C\left([0,S];\mathcal{V}\right)$ for any $S > 0$.*

Proposition 5.7. *For $r \ge 4$, X is a classical solution to the initial value problem (B.1), (B.3) and (B.4) with the initial condition (2.5) for any $S > 0$, where $\tilde{\theta} \in C\left([0,S];C^3\left(\mathbb{T}[0,2\pi]\right)\right) \cap C^1\left([0,S];C\left(\mathbb{T}[0,2\pi]\right)\right)$ and $L \in C^1[0,S]$.*

Proof. For $r \ge 4$, by Sobolev embedding theorem and Corollary 5.6, we know $X \in C\left([0,S];C^3\left(\mathbb{T}[0,2\pi]\right)\times\mathbb{R}\right)$ and $\tilde{\theta}_n \to \tilde{\theta}$ as $n\to\infty$ in $C\left([0,S];C^3\left(\mathbb{T}[0,2\pi]\right)\right) \cap C\left([0,S];\dot{H}^s\right)$, for $1 \le s < r$.

Since g is C^1 in the open ball $\dot{H}^1$, $g(\tilde{\theta}_n) \to g(\tilde{\theta})$ as $n\to\infty$. So $\hat{\theta}(1;t) = g(\tilde{\theta})$ and $\tilde{\theta}$ satisfy (B.4). By Proposition 3.12 and (3.14), we see that both $\{\gamma_n\}_{n=2}^{\infty}$ and $\{\mathcal{F}[\omega_n]\gamma_n\}_{n=2}^{\infty}$ are Cauchy sequences in $C\left([0,S];H^1\left(\mathbb{T}[0,2\pi]\right)\right)$. Hence, it allows us to pass to the limit as $n\to\infty$ in the equation

$$\left(I + A_\mu\mathcal{F}[\omega_n]\right)\gamma_n = \frac{2\pi}{L_n}\theta_{n,\alpha\alpha},$$

and obtain

$$\big(I + A_\mu\mathcal{F}[\omega]\big)\gamma = \frac{2\pi}{L}\theta_{\alpha\alpha}.$$

By Proposition 3.12 again , we have $\gamma \in C\left([0,S];H^{r-2}\left(\mathbb{T}[0,2\pi]\right)\right)$. We also have

$$\tilde{\theta}_n(\alpha,t) = P_n\theta_0(\alpha) + \int_0^t F_{n,1}\big(X_n(t')\big)dt'. \tag{5.9}$$

From Lemma 3.13, it follows that $\{F_{n,1}\}_{n=2}^{\infty}$ is a Cauchy sequence in $C\left([0,S];\dot{H}^0\right)$. Replacing $r+1$ by $r-3$ and $\tilde{\theta}_n$ by $F_{n,1}$ in (5.5) with the uniform bound of $F_{n,1}$ in $\dot{H}^{r-3}$, we see $\{F_{n,1}\}_{n=2}^{\infty}$ is a Cauchy sequence in $C\left([0,S];\dot{H}^s\right)$ for $0 \le s < r-3$. Hence, we take the limit in (5.9), yielding

$$\tilde{\theta}(\alpha,t) = \mathcal{Q}_1\theta_0(\alpha) + \int_0^t F^1\big(X(t')\big)dt',$$

where F^1 is the right-hand side of the first equation in (B.1). This is differentiable in time, giving $\tilde{\theta}_t = F^1(X) \in C\left([0,S];C\left(\mathbb{T}[0,2\pi]\right)\right)$. Similarly, L satisfies the second equation of (B.1) and $L_t \in C[0,S]$. Thus, X is a classical solution to (B.1), (B.3) and (B.4) with the initial condition (2.5). □

Lemma 5.8. *For $r \ge 3$, there exists the sufficiently small ball size ϵ of $\mathcal{B}$ such that if $X^{(1)}, X^{(2)} \in C\big([0,S];\dot{H}^r\times\mathbb{R}\big) \cap C^1\big([0,S];\dot{H}^{r-3}\times\mathbb{R}\big)$ are solutions to the initial* *value problem (B.1), (B.3) and (B.4) with the initial condition (2.5) for any $S > 0$, and the corresponding initial data $X^{(1)}(\alpha, 0), X^{(2)}(\alpha, 0) \in \mathcal{V}$, then for $0 \le t \le S$,*

$$\left\|\tilde{\theta}^{(1)}(\cdot, t) - \tilde{\theta}^{(2)}(\cdot, t)\right\|_1 + \left|L^{(1)}(t) - L^{(2)}(t)\right|$$
$$\le \left(\left\|\tilde{\theta}^{(1)}(\cdot, 0) - \tilde{\theta}^{(2)}(\cdot, 0)\right\|_1 + \left|L^{(1)}(0) - L^{(2)}(0)\right|\right) \exp\{Bt\}.$$

Proof. This proof is very similar to the proof of Lemma 5.2, and we re-use some notation. Define E_d, the energy function for the difference of two solutions, by $E_d^1 + (L^{(1)} - L^{(2)})^2$. Here,

$$E_d^1 = \frac{1}{2}\int_0^{2\pi} (D_\alpha(\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}))^2 d\alpha.$$

We now wish to estimate how this energy changes over time.

$$\begin{aligned}
\frac{dE_d^1}{dt} &= \int_0^{2\pi} D_\alpha(\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}) D_\alpha(\tilde{\theta}_t^{(1)} - \tilde{\theta}_t^{(2)}) d\alpha \\
&= \int_0^{2\pi} D_\alpha(\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}) D_\alpha^2 \mathcal{Q}_1\big(\frac{2\pi}{L^{(1)}} U^{(1)} - \frac{2\pi}{L^{(2)}} U^{(2)}\big) d\alpha \\
&\quad + \int_0^{2\pi} D_\alpha(\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}) D_\alpha \mathcal{Q}_1\Big(\frac{2\pi}{L^{(1)}}\big(T^{(1)}(1 + \theta_\alpha^{(1)})\big) - \frac{2\pi}{L^{(2)}}\big(T^{(2)}(1 + \theta_\alpha^{(2)})\big)\Big) d\alpha.
\end{aligned}$$

Using the same estimates as that in Lemma 5.2, we have

$$\frac{dE_d^1}{dt} \le -\frac{4\pi^3}{(L^{(2)})^3}\sigma\Big(\big\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\big\|_{5/2}^2 - \big\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\big\|_{3/2}^2\Big) + c\epsilon\big(\big\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\big\|_2^2 + E_d\big),$$

with c depends on the diameter of $\mathcal{V}$. We also have

$$\frac{d(L^{(1)} - L^{(2)})^2}{dt} \le c\epsilon\big(\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_2^2 + E_d\big),$$

with c depends on the diameter of $\mathcal{V}$. As what we did in Lemma 5.2, for sufficiently small ϵ, there exists a positive constant B such that

$$\frac{dE_d}{dt} \le BE_d.$$

We solve the differential inequality to see that

$$E_d(t) \le E_d(0)e^{Bt}.$$

This proves the theorem. □

Hence, uniqueness follows from Lemma 5.8.

Lemma 5.9. *For $r \ge 3$, there exists the sufficiently small ball size ϵ of $\mathcal{B}$ such that solution $X = \big(\tilde{\theta}, L\big) \in C\big([0, S]; \dot{H}^r \times \mathbb{R}\big) \cap C^1\big([0, S]; \dot{H}^{r-3} \times \mathbb{R}\big)$ to (B.1) , (B.3) and (B.4) with initial condition (2.5) is unique in $\dot{H}^1 \times \mathbb{R}$.*

Proof of Lemma 2.14: This follows from Lemmas 5.2, 5.4, 5.9, Corollary 5.6 and Proposition 5.7.

Proof of Proposition 2.16: Taking the derivative with respect to t on both sides of (2.14), we have

$$\begin{aligned}\frac{d\mathcal{S}(t)}{dt} &= \frac{1}{2}\operatorname{Im}\int_0^{2\pi}\big(z_\alpha z_t^* - z_t z_\alpha^*\big)d\alpha\\ &= -\frac{L}{4\pi}\operatorname{Re}\int_0^{2\pi}\big(ie^{i\frac{\pi}{2}+i\alpha+i\theta(\alpha)}z_t^* + z_t(-ie^{-i\frac{\pi}{2}-i\alpha-i\theta(\alpha)})\big)d\alpha\\ &= -\frac{L}{2\pi}\int_0^{2\pi}(x_t,y_t)\cdot\mathbf{n}d\alpha = -\frac{L}{2\pi}\int_0^{2\pi}Ud\alpha\\ &= -\operatorname{Re}\Big(\int_0^{2\pi}\frac{z_\alpha(\alpha)}{2\pi}\mathrm{PV}\int_{\alpha-\pi}^{\alpha+\pi}\frac{\gamma(\alpha')}{z(\alpha)-z(\alpha')}d\alpha' d\alpha\Big).\end{aligned}$$

Since

$$\begin{aligned}&\operatorname{Re}\Big(\mathrm{PV}\int_0^{2\pi}\frac{z_\alpha(\alpha)}{z(\alpha)-z(\alpha')}d\alpha\Big)\\ &\qquad= \operatorname{Re}\Big(\lim_{b\to 0}\int_0^{\alpha'-b}+\int_{\alpha'+b}^{2\pi}\frac{d}{d\alpha}\log(z(\alpha)-z(\alpha'))\Big)\\ &\qquad\qquad= \log\big|z(2\pi)-z(\alpha')\big| - \log\big|z(0)-z(\alpha')\big| = 0,\end{aligned}$$

we have

$$\frac{d\mathcal{S}(t)}{dt} = 0.$$

Hence the area of the bubble is invariant with time.

Since $\big(\tilde{\theta}_n, L_n\big)$ converges to $(\tilde{\theta}, L)$ in $C\left((0,S];\dot{H}^r\times\mathbb{R}\right)$ for any $S>0$, by Proposition 2.12 and $\|P_n\theta_0\|_r \le \|\mathcal{Q}_1\theta_0\|_r$, we have

$$\|\tilde{\theta}(\cdot,t)\|_r = \lim_{n\to\infty}\|\tilde{\theta}_n(\cdot,t)\|_r \le \|\mathcal{Q}_1\theta_0\|_r e^{-\frac{1}{36}\sigma t}. \tag{5.10}$$

By (2.7) and (5.10), the statement for $\hat{\theta}(\pm 1;t)$ hold.

Since the area is invariant with time, we have

$$\frac{L^2}{8\pi^2}\operatorname{Im}\int_0^{2\pi}\omega_\alpha\omega^* d\alpha = \mathcal{S}\frac{1}{2\pi}\operatorname{Im}\int_0^{2\pi}\omega_{0,\alpha}\omega_0^* d\alpha. \tag{5.11}$$

(5.11) gives us

$$\big(L^2-4\pi\mathcal{S}\big)\operatorname{Im}\int_0^{2\pi}\omega_\alpha\omega^* d\alpha + 4\pi\mathcal{S}\Big(\operatorname{Im}\int_0^{2\pi}\omega_\alpha\omega^* d\alpha - \operatorname{Im}\int_0^{2\pi}\omega_{0,\alpha}\omega_0^* d\alpha\Big) = 0.$$

It implies that

$$L-2\sqrt{\pi\mathcal{S}} = -\frac{L^2}{2\pi(L+2\sqrt{\pi\mathcal{S}})}\Big(\operatorname{Im}\int_0^{2\pi}\omega_\alpha\omega^* d\alpha - \operatorname{Im}\int_0^{2\pi}\omega_{0,\alpha}\omega_0^* d\alpha\Big).$$

Hence, using $2\pi-1<L<2\pi+1$, we induce the following estimate:

$$|L-2\sqrt{\pi\mathcal{S}}| \le C\|\tilde{\theta}\|_1$$

with C depending on $\mathcal{S}$. From (5.10), the result for L follows.

From (B.2), using (3.15) and $2\pi-1<L<2\pi+1$, we have

$$\Big|\hat{\theta}(0;t)-\hat{\theta}_0(0)\Big| \le C\int_0^t\|T(\cdot,t')\|_0\left(1+\|\tilde{\theta}(\cdot,t')\|_1\right)dt \le C\int_0^t\|\tilde{\theta}(\cdot,t')\|_3 dt. \tag{5.12}$$

Hence, plugging estimates (5.10) into (5.12), the result for $\hat{\theta}(0;t)$ holds.

6. APPENDIX

Proof of Lemma 3.4 ([1]):

Proof. We note that

$$D^k_\alpha q_1[\omega] = \int_0^1 t^k D^k \omega_\alpha(t\alpha + (1-t)\alpha')dt\ ,\ D^k_{\alpha'} q_1[\omega] = \int_0^1 (1-t)^k D^k \omega_\alpha(t\alpha + (1-t)\alpha')dt.$$

Then, using 2π periodicity of $D^k\omega_\alpha$, we obtain

$$\begin{aligned}
&\int_a^{a+2\pi} \Big| \int_0^1 t^k D^k \omega_\alpha(t\alpha + (1-t)\alpha')dt\Big|^2 d\alpha' \\
\leq\ & \int_a^{a+2\pi} \Big(\int_0^1 |D^k \omega_\alpha(t\alpha + (1-t)\alpha')(1-t)^{1/4}|^2 dt\Big)\Big(\int_0^1 t^{2k}(1-t)^{-1/2} dt\Big) d\alpha' \\
\leq\ & C \int_0^1 \int_a^{a+2\pi} |D^k \omega_\alpha(t\alpha + (1-t)\alpha')(1-t)^{1/4}|^2 d\alpha' dt \\
\leq\ & C \int_0^1 \int_{a(1-t)+t\alpha}^{(a+2\pi)(1-t)+t\alpha} |D^k \omega_\alpha(u)|^2 (1-t)^{-1/2} du dt \\
\leq\ & C \int_0^1 (1-t)^{-1/2} dt \int_0^{2\pi} |D^k \omega_\alpha(u)|^2 du \leq C\|D^k\omega_\alpha\|_0^2.
\end{aligned}$$

So $D^k_\alpha q_1 \in H^k[a, a+2\pi]$ in variable α' and $\|D^k_\alpha q_1[\omega]\|_0 \leq C\|\omega_\alpha\|_k$ with C only dependent on k. Again

$$\begin{aligned}
&\int_a^{a+2\pi} \Big| \int_0^1 (1-t)^k D^k \omega_\alpha(t\alpha + (1-t)\alpha')dt\Big|^2 d\alpha' \\
\leq\ & \int_a^{a+2\pi} \Big(\int_0^1 |D^k \omega_\alpha(t\alpha + (1-t)\alpha')(1-t)^{1/4}|^2 dt\Big)\Big(\int_0^1 (1-t)^{2k-1/2} dt\Big) d\alpha' \\
\leq\ & C \int_0^1 \int_a^{a+2\pi} |D^k \omega_\alpha(t\alpha + (1-t)\alpha')(1-t)^{1/4}|^2 d\alpha' dt \\
\leq\ & C \int_0^1 \int_{a(1-t)+t\alpha}^{(a+2\pi)(1-t)+t\alpha} |D^k \omega_\alpha(u)|^2 (1-t)^{-1/2} du dt \leq C\|D^k\omega_\alpha\|_0^2.
\end{aligned}$$

So $D^k_{\alpha'} q_1 \in H^k[a, a+2\pi]$ in variable α' and $\|D^k_{\alpha'} q_1[\omega]\|_0 \leq C\|\omega_\alpha\|_k$ with C only dependent on k.

We note that for $k \geq 0$

$$D^k_\alpha q_2[\omega] = -\int_0^1 t^k (1-t) D^k \omega_{\alpha\alpha}(t\alpha + (1-t)\alpha')dt,$$

$$D^k_{\alpha'} q_2[\omega] = -\int_0^1 (1-t)^{k+1} D^k \omega_{\alpha\alpha}(t\alpha + (1-t)\alpha')dt.$$

Similar arguments as above leads to the stated bounds for q_2.

From symmetry of q_1, q_2 in α and α', clearly the same results hold with respect to α instead of α' integration. □

Proof of Lemma 3.8 ([1]):

Proof. We begin by taking $r-2$ derivatives of $\mathcal{K}[\omega]f$.

$$\begin{aligned}
D_\alpha^{r-2}\mathcal{K}[\omega]f(\alpha) &= D_\alpha^{r-2}\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')\Big[\frac{1}{\omega(\alpha)-z(\alpha')}-\frac{1}{2\omega_\alpha(\alpha')}\cot\frac{1}{2}(\alpha-\alpha')\Big]d\alpha' \\
&= \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')D_\alpha^{r-2}\Big[\frac{1}{\omega(\alpha)-\omega(\alpha')}-\frac{1}{2\omega_\alpha(\alpha')}\cot\frac{1}{2}(\alpha-\alpha')\Big]d\alpha' \\
&= \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')D_\alpha^{r-2}\Big[\frac{1}{\omega(\alpha)-\omega(\alpha')}-\frac{1}{\omega_\alpha(\alpha')(\alpha-\alpha')}\Big]d\alpha' \\
&\quad -\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi}\frac{f(\alpha')}{2\omega_\alpha(\alpha')}D_\alpha^{r-2}l\big(\frac{1}{2}(\alpha-\alpha')\big)d\alpha' \\
&= P_1+P_2.
\end{aligned}$$

Since the function $l(\beta)$ is analytical for $-\frac{\pi}{2}\le\beta\le\frac{\pi}{2}$, it is easy to have

$$\|P_2\|_0\le\frac{C}{L}\|f\|_0,\ \text{where C depends on } r.$$

Let us see P_1.

$$\begin{aligned}
P_1 &= \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi}\frac{f(\alpha')}{\omega_\alpha(\alpha')}D_\alpha^{r-2}\Big[\frac{\omega_\alpha(\alpha')}{\omega(\alpha)-\omega(\alpha')}-\frac{1}{\alpha-\alpha'}\Big]d\alpha' \\
&= \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi}\frac{f(\alpha')}{\omega_\alpha(\alpha')}D_\alpha^{r-2}\Big(\frac{q_2[\omega](\alpha',\alpha)}{q_1[\omega](\alpha',\alpha)}\Big)d\alpha'.
\end{aligned}$$

(3.7) implies that $\big|q_1[\omega](\alpha,\alpha')\big|\ge\frac{1}{4}$. So by Lemma 3.5, we have

$$\|P_1\|_0\le\frac{C_1}{L}\|f\|_0\exp\big(\frac{C_2}{L}\|\omega_\alpha\|_{r-1}\big).$$

Hence first result follows. Taking α-derivative $r-1$ times $\mathcal{K}[\omega]f$ and integrating by parts once,

$$\begin{aligned}
D_\alpha^{r-1}\mathcal{K}[\omega]f(\alpha) &= D_\alpha^{r-2}\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha(\alpha')}\Big)\Big[\frac{\omega_\alpha(\alpha)}{\omega(\alpha)-\omega(\alpha')}-\frac{1}{2}\cot\frac{1}{2}(\alpha-\alpha')\Big]d\alpha' \\
&= \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha(\alpha')}\Big)D_\alpha^{r-2}\Big[\frac{\omega_\alpha(\alpha)}{\omega(\alpha)-\omega(\alpha')}-\frac{1}{2}\cot\frac{1}{2}(\alpha-\alpha')\Big]d\alpha' \\
&= -\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} D_{\alpha'}\Big(\frac{f(\alpha')}{\omega_\alpha(\alpha')}\Big)D_\alpha^{r-2}\left(\frac{q_2[\omega](\alpha,\alpha')}{q_1[\omega](\alpha,\alpha')}\right)d\alpha' \\
&\quad -\frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} D_{\alpha'}\Big(\frac{f(\alpha')}{2\omega_\alpha(\alpha')}\Big)D_\alpha^{r-2}l\big(\frac{1}{2}(\alpha-\alpha')\big)d\alpha'.
\end{aligned}$$

Using Lemma 3.5, the the second inequality follows from Cauchy-Schwartz inequality after noting that $\|D\left(\frac{f}{\omega_\alpha}\right)\|_0\le C\|f\|_1\|\omega_\alpha\|_1$ □

Proof of Lemma 3.10 ([1]):

Proof. We begin by writing $[\mathcal{H},\psi]$ as an integral operator:

$$[\mathcal{H},\psi]f(\alpha)=\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')\big(\psi(\alpha')-\psi(\alpha)\big)\cot\Big(\frac{1}{2}(\alpha-\alpha')\Big)d\alpha'.$$

We can write the kernel as

$$\Big(\frac{\psi(\alpha')-\psi(\alpha)}{\alpha-\alpha'}\Big)\Big((\alpha-\alpha')\cot\Big(\frac{1}{2}(\alpha-\alpha')\Big).$$

The first part of this product is a divided difference, and the second part is an analytic function on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The lemma now follows from the Generalized Young's Inequality. $\square$

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