GLOBAL EXISTENCE FOR A TRANSLATING NEAR-CIRCULAR HELE-SHAW BUBBLE WITH SURFACE TENSION

J. YE[1] AND S. TANVEER[2]

March 23, 2010

ABSTRACT. This paper concerns global existence for arbitrary nonzero surface tension of bubbles in a Hele-Shaw cell that translate in the presence of a pressure gradient. When the cell width to bubble size is sufficiently large, we show that a unique steady translating near-circular bubble symmetric about the channel centerline exists, where the bubble translation speed in the laboratory frame is found as part of the solution. We prove global existence for symmetric sufficiently smooth initial conditions close to this shape and show that the steady translating bubble solution is an attractor within this class of disturbances. In the absence of side walls, we prove stability of the steady translating circular bubble without restriction on symmetry of initial conditions. These results hold for any nonzero surface tension despite the fact that a local planar approximation near the front of the bubble would suggest Saffman Taylor instability.

We exploit a boundary integral approach that is particularly suitable for analysis of nonzero viscosity ratio between fluid inside and outside the bubble. An important element of the proof was the introduction of a weighted Sobolev norm that accounts for stabilization due to advection of disturbances from the front to the back of the bubble.
Keywords: Free boundary problem, Dissipative equations, Hele-Shaw problem, Translating bubbles, Surface tension
Mathematics Subject Classification: 35K55, 35R35, 76D27

1. INTRODUCTION

The displacement of a more viscous fluid by a less viscous one in a Hele-Shaw cell is a canonical problem in a much wider class of Laplacian growth problems that include dendritic crystal growth, electrochemical growth, diffusion limited aggregation, filtration combusion and tumor growth. It has attracted many physicists and mathematicans. In the recent two decades, there are many reviews about this subject (Saffman [33], Bensimon *et al.* [8], Homsy [18], Pelce [27], Kessler *et al.* [24], Tanveer [41] & [42], Hohlov [17], and Howison [22] & [23]).

There is a vast literature on zero surface tension problem though the initial value problem in this case is ill-posed [21], [15] and not always physically relevant [See [42] for detailed discussion of this issue]. With surface tension, there are rigorous local existence results for general initial conditions both for one and two

[1] Department of Mathematics, Ohio State University, Columbus, OH 43210 (jenny_yeyj@math.ohio-state.edu).
[2] Department of Mathematics, Ohio State University, Columbus, OH 43210 (tanveer@math.ohio-state.edu).

phase problems [11], [13] using different approaches. Also there are some global existence and nonlinear stability results [9], [16] for one and two phase Hele-Shaw for near-circular initial shapes in the absence of any forcing such as fluid injection or pressure gradient. These have been generalized to non-Newtonian one phase fluids [12]. There are similar results available for the two phase Stefan problem [14], [29], which is mathematically close to but distinct from the two-phase Hele-Shaw (also called Muskat problem) being studied here. It is well recognized that global existence problem with surface tension for arbitrary initial shape is a difficult open problem[1] though there is quite a substantial literature involving formal asymptotic and numerical computations (see cited reviews above). Even the restricted problem of stability of steadily propagating shapes such as a semi-infinite finger [45], [46] or a finite translating bubble [46] for nonzero surface tension remains an open problem for rigorous analysis. Translation causes complications in global analysis due to a less viscous fluid displacing a more viscous fluid – a planar front is known to be unstable [32] in this case.

This paper considers the motion of a bubble in a Hele-Shaw cell subject to an external pressure gradient that causes the bubble to translate. We scale the fluid velocity at ∞ in the laboratory frame to be 1; we choose u_0 so that the non-dimensional velocity of the fluid at $+\infty$ in the frame of a steady bubble [2] along the positive x-axis is $-(u_0+1)$. The analysis presented also includes proof of existence and uniqueness of a steady bubble solution together with determination of u_0. We choose the steady bubble perimeter to be 2π; this corresponds to nondimensionalizing all length scales appropriately. The non-dimensional half width of the Hele-Shaw cell will be denoted by $\frac{\pi}{\beta}$.

The two-phase Hele-Shaw problem in the steady bubble frame is described mathematically as follows: $\Omega_2(t) \subset \mathbb{R}^2$ is a simply connected bounded domain occupied by a fluid with viscosity μ_2 at time t, while a different fluid of viscosity $\mu_1 > \mu_2$[3] occupies $\Omega_1(t)$, where $\Omega_1(t) \cup \Omega_2(t)$ constructs the strip which half width is $\frac{\pi}{\beta}$, i. e., $\{(x,y) | x \in \mathbb{R}, -\frac{\pi}{\beta} < y < \frac{\pi}{\beta}\}$ (see Figure 1). We define functions ϕ_1 and ϕ_2, outside and inside Ω_2 such that

$$\text{(O.1)} \qquad \begin{cases} \Delta\phi_1 = 0 \text{ in } \Omega_1, \\ \Delta\phi_2 = 0 \text{ in } \Omega_2, \\ \quad \phi_1 \to -(u_0+1)x + O(1), \text{ as } (x,y) \to \infty, \\ \dfrac{\partial \phi_1}{\partial y}\big(x, \pm\dfrac{\pi}{\beta}\big) = 0, \text{ for } x \in \mathbb{R}. \end{cases}$$

On the free boundary $\partial\Omega_1 \cap \partial\Omega_2$ between two fluids, we require two conditions:

$$\text{(O.2)} \qquad \begin{cases} (2+u_0)x + \phi_1 - \dfrac{\mu_2}{\mu_1}\phi_2 = \sigma\kappa, \\ \qquad \dfrac{\partial \phi_1}{\partial n} = \dfrac{\partial \phi_2}{\partial n} = v_n, \end{cases}$$

[1]Note the "stable" problem where a more viscous fluid displaces a less viscous fluid is relatively simple and will not be considered here; there are many global results available in this case.

[2]This choice implies that the steady bubble translates along the positive x-axis with non-dimensional speed $2+u_0$ in the laboratory frame.

[3]The assumption $\mu_1 > \mu_2$ is not necessary in the analysis.

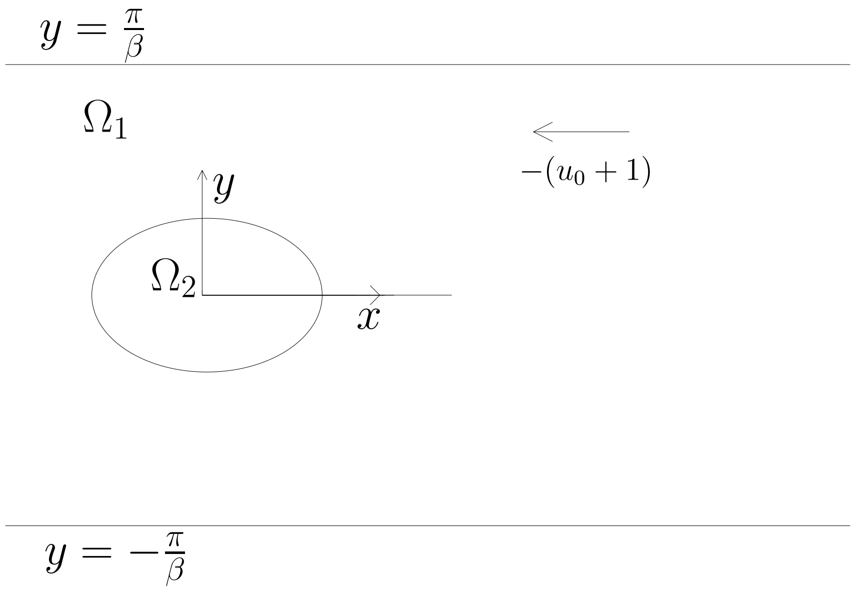

FIGURE 1. The Hele-Shaw flow in the frame of the steady bubble.

where σ is the coefficient of surface tension, $\mathbf{n}$ is the inward unit normal vector on $\partial\Omega_1 \cap \partial\Omega_2$, and v_n is the normal velocity of the interface. The first condition corresponds to jump in pressure balanced by surface tension, while the second is the usual kinematic condition requiring that the normal motion of a point on the interface equals normal fluid velocity on either side of the interface.

The global existence analysis for arbitrary surface tension is complicated by the far-field pressure gradient that causes bubble translation since a planar interface under the same condition is susceptible to well-known Saffman-Taylor instability. This difficulty arises both for finite ($\beta \neq 0$) and infinite cell-width ($\beta = 0$). Locally, near the front of the bubble, at sufficiently small scale a planar approximation would appear reasonable. However, some formal arguments [8], [10]. supported by numerical calculations have suggested that stabilization occurs on a curved interface through advection of disturbances from the front of the interface to the sides. These conclusions are not universally accepted since formal calculations [49] based on a multi-scale hypothesis suggest that the steady state is linearly unstable for sufficiently small surface tension. Here we resolve this controversy rigorously in favor of stability at least in the case of a Hele-Shaw bubble with distant sidewalls for any nonzero surface tension.[4] We have introduced a weighted Sobolev space suitable for controlling terms arising from bubble translation for any nonzero surface tension σ. We are unaware of any previous work for global control of small disturbances superposed on a steadily translating curved interface in Hele-Shaw or any other related problems.

[4]It is to be noted that the problem tackled here is not equivalent to taking $O(1)$ sidewall separation and making bubble size sufficiently small for fixed surface tension, since if we scale down bubble size, we must also scale down surface tension values to make an equivalent problem. In the small bubble limit any fixed surface tension dominates translational effects; in our choice of length scale, this would correspond only to the simpler case of only sufficiently large σ.

In the present paper, we use a boundary integral formulation due to Hou *et al* [19]. This formulation has been widely used for numerical calculations in a wide variety of free boundary problems involving Laplace's equation. Ambrose [4] has recently used this formulation to prove local existence for the Hele-Shaw flow of general initial shapes [4] without surface tension. Given the wide use of boundary integral methods in computations, one motivation for the present paper is to further develop the mathematical machinery associated with this method so as to be applicable to more general existence problems.

Adapting the equal arc-length vortex sheet formulation of Hou *et al* [19] to the present geometry, the boundary curve between the two fluids of differing viscosities is described parametrically at any time t by $z = x(\alpha, t) + iy(\alpha, t)$, where α is chosen so that $z(\alpha + 2\pi, t) = z(\alpha, t)$. We introduce θ so that $\frac{\pi}{2} + \alpha + \theta$ is the angle between the tangent to the curve and the positive x-axis as the boundary is traversed counter-clockwise with increasing α. Hou *et al* [20] observed that a choice[5] of the tangent velocity T is possible so that the rate of change of arc-length $s_\alpha \equiv |z_\alpha|$ is independent of α and corresponds to an equal arc-length interface parametrization. They also observed that this choice simplifies the evolution equation for θ, and used it in their computational scheme. Note in this equal arc-length formulation $z_\alpha = x_\alpha + iy_\alpha = \frac{L}{2\pi} e^{i\pi/2 + i\alpha + i\theta}$, where L is perimeter length of interface. Then the unit tangent vector on the interface $\mathbf{t} = \big(-\sin(\alpha + \theta), \cos(\alpha + \theta)\big)$ and the unit normal vector pointing inward at bubble interface is $\mathbf{n} = \big(-\cos(\alpha+\theta), -\sin(\alpha+\theta)\big)$.

Definition 1.1. *Let $r \geq 0$. The Sobolev space H_p^r is the set of all 2π-periodic function $f = \sum_{-\infty}^{\infty} \hat{f}(k) e^{ik\alpha}$ such that*

$$\|f\|_r = \sqrt{\sum_{k=-\infty}^{\infty} |k|^{2r} |\hat{f}(k)|^2 + |\hat{f}(0)|^2} < \infty.$$

Note 1.2. *For $f, g \in H_p^r$, the Banach Algebra property $\|fg\|_r \leq C_r \|f\|_r \|g\|_r$ for $r \geq 1$ with some constant C_r depending on r is easily proved and will be useful in the sequel. Also, in what follows the ˆ symbol will reserved for Fourier components.*

Definition 1.3. *The Hilbert transform, $\mathcal{H}$, of a function $f \in H_p^0$ (i.e. L_2) with Fourier Series $f = \sum_{-\infty}^{\infty} \hat{f}(k) e^{ik\alpha}$ is given by*

$$\begin{aligned} \mathcal{H}[f](\alpha) &= \frac{1}{2\pi} PV \int_0^{2\pi} f(\alpha') \cot \frac{1}{2}(\alpha - \alpha') d\alpha' \\ &= \sum_{k \neq 0} -i \operatorname{sgn}(k) \hat{f}(k) e^{ik\alpha}. \end{aligned}$$

Note 1.4. *For $f \in H_p^1$, the Hilbert transform commutes with differentiation. We will denote derivative with respect to α, either by D_α or subscript α. Also, for the sake of brevity of notation, the time t dependence will often be omitted, except where it might cause confusion otherwise.*

[5]This choice or any other choice of tangential speed of points on the interface has no effect on the interface shape itself.

Definition 1.5. *We define the operator* Λ *to be a derivative followed by the Hilbert transform:* $\Lambda = \mathcal{H}D_\alpha$. *Following Ambrose* [4], *we also define commutator*

$$[\mathcal{H}, f]g = \mathcal{H}(fg) - f\mathcal{H}(g).$$

Note 1.6. *It is clear that*

$$\Big(\int_0^{2\pi} \big(f^2 + f\Lambda f\big)d\alpha\Big)^{1/2}$$

is equivalent to $H_p^{1/2}$ *norm of a real-valued* 2π*-periodic function* f. *Further, note the operator* Λ *is self-adjoint in* $H_p^{1/2}$ *Hilbert space.*

Definition 1.7. *We define a linear integral operator* $\mathcal{K}[z]$, *depending on* z, *as*

$$\mathcal{K}[z]f = \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')\Big\{\mathfrak{K}(\alpha,\alpha') - \frac{1}{2z_\alpha(\alpha')}\cot\frac{1}{2}(\alpha-\alpha')\Big\}d\alpha', \tag{1.1}$$

where for $\beta = 0$,

$$\mathfrak{K}(\alpha,\alpha') = \frac{1}{z(\alpha) - z(\alpha')}; \tag{1.2}$$

for $\beta \neq 0$,

$$\mathfrak{K}(\alpha,\alpha') = \frac{\beta}{4}\coth\Big[\frac{\beta}{4}\big(z(\alpha) - z(\alpha')\big)\Big] - \frac{\beta}{4}\tanh\Big[\frac{\beta}{4}\big(z(\alpha) - z^*(\alpha')\big)\Big]. \tag{1.3}$$

Remark. For 2π-periodic functions f and z, it is clear that the upper and lower limits of the integral above can be replaced by a and $a+2\pi$ respectively for arbitrary a. □

Definition 1.8. *We define a complex valued operator* $\mathcal{G}[z]$, *depending on* z, *so that*

$$\mathcal{G}[z]\gamma = z_\alpha\Big[\mathcal{H}, \frac{1}{z_\alpha}\Big]\gamma + 2iz_\alpha\mathcal{K}\big[z\big]\gamma. \tag{1.4}$$

It is also convenient to define a related real operator $\mathcal{F}[z]$, *depending on* z, *so that*

$$\mathcal{F}[z]\gamma = \mathrm{Re}\,\Big(\frac{1}{i}\mathcal{G}[z]\gamma\Big). \tag{1.5}$$

From the Hou *et al* [20] equal arc-length formulation, the Hele-Shaw equations (O.1)-(O.2) reduce to the following evolution equations for the boundary $\partial\Omega_1 \cap \partial\Omega_2$:

$$\begin{cases} \theta_t(\alpha,t) = \dfrac{2\pi}{L}U_\alpha(\alpha,t) + \dfrac{2\pi}{L}T(\alpha,t)\big(1+\theta_\alpha(\alpha,t)\big), \\ L_t(t) = -\displaystyle\int_0^{2\pi}\big(1+\theta_\alpha(\alpha,t)\big)U(\alpha,t)d\alpha, \end{cases} \tag{A.1}$$

where U is the normal interface velocity, determined from

$$\begin{aligned} U(\alpha,t) &= \frac{2\pi}{L}\,\mathrm{Re}\,\Big(\frac{z_\alpha}{2\pi}\mathrm{PV}\int_{\alpha-\pi}^{\alpha+\pi}\gamma(\alpha')\mathfrak{K}(\alpha,\alpha')d\alpha'\Big) + (u_0+1)\cos\left(\alpha+\theta(\alpha)\right) \\ &= \frac{\pi}{L}\mathcal{H}[\gamma] + \frac{\pi}{L}\,\mathrm{Re}\left(\mathcal{G}[z]\gamma\right) + (u_0+1)\cos\left(\alpha+\theta(\alpha)\right), \end{aligned} \tag{1.6}$$

vortex sheet γ and the tangent interface velocity are determined, respectively, by

$$\gamma(\alpha,t) = -a_\mu\mathcal{F}[z]\gamma(\alpha,t) + \frac{L}{\pi}\big(1 + \frac{\mu_2}{\mu_1+\mu_2}u_0\big)\sin(\alpha+\theta) + \frac{2\pi}{L}\sigma\theta_{\alpha\alpha}, \tag{A.2}$$

$$T(\alpha,t) = \int_0^{\alpha} \big(1+\theta_{\alpha'}(\alpha',t)\big)U(\alpha',t)d\alpha' - \frac{\alpha}{2\pi}\int_0^{2\pi} \big(1+\theta_\alpha(\alpha,t)\big)U(\alpha,t)d\alpha, \tag{A.3}$$

where $a_\mu = \frac{\mu_1-\mu_2}{\mu_1+\mu_2}$

For (A.1)-(A.3), the initial conditions are

$$\theta(\alpha,0)=\theta_0(\alpha),\ L(0)=L_0. \tag{1.7}$$

Note 1.9. *Since $\big(x_t(\alpha,t),y_t(\alpha,t)\big) = U\mathbf{n}+T\mathbf{t}$, (A.3) implies that the interface evolution at $\alpha=0$ is given by $\big(x_t(0,t),y_t(0,t)\big)=U(0,t)\mathbf{n}(0,t)$. In particular, this implies*

$$y_t(0,t) = -U(0,t)\sin\big(\theta(0,t)\big), \text{with initial condition } y(0,0)=y_0. \tag{1.8}$$

Definition 1.10. *We denote the bubble area by V. From geometric consideration,*

$$V = \frac{1}{2}\operatorname{Im}\int_0^{2\pi} z_\alpha z^* d\alpha. \tag{1.9}$$

Remark. It is well known (indeed easily seen from (O.1)) that the bubble area V will remain invariant in time. That this is also implied by the boundary integral formulation (A.1) is not as obvious and is shown in §2. □

Definition 1.11. *We introduce a family of projections $\{\mathcal{Q}_n\}$ such that*

$$\mathcal{Q}_n f = f - \sum_{k=-n}^{n} \hat{f}(k)e^{ik\alpha}$$

where $f=\sum_{-\infty}^{\infty}\hat{f}(k)e^{ik\alpha}$ and $n\in\mathbb{Z}^+\cup\{0\}$. Henceforth, we will define $\tilde{\theta}=\mathcal{Q}_1\theta$.

Definition 1.12. *We define $\dot{H}^r$ as a subspace of H_p^r containing real valued functions so that $\phi\in\dot{H}^r$ implies $\mathcal{Q}_1\phi=\phi$. Note in this subspace, $\|\phi\|_r=\|D_\alpha^r\phi\|_0$ for $r\geq 1$.*

Without sidewalls, *i.e.* for $\beta=0$, our main result is as follows:

Theorem 1.13. *For any surface tension $\sigma>0$ and $r\geq 3$, there exists $\epsilon>0$ such that if $\|\theta_0\|_r<\epsilon$ and $|L_0-2\pi|<\epsilon<\frac{1}{2}$, then there exists a unique solution $\big(\theta,L\big)\in C\big([0,\infty),H_p^r\times\mathbb{R}\big)$ to the Hele-Shaw problem (A.1)-(A.3) with the initial condition (1.7). Further, $\|\tilde{\theta}\|_r$ and $|\hat{\theta}(\pm1;t)|$ each decay exponentially as $t\to\infty$, $|\hat{\theta}(0;t)|$ remains finite, while L approaches $2\sqrt{\pi V}$ exponentially implying that a steady translating circular bubble is asymptotically stable for sufficiently small initial disturbances in the H_p^r space.*

Remark. The proof is completed at the end of §4 (see Note 4.3). □

We also consider the problem with finite cell-width ($\beta\neq 0$). Here, we first prove the existence of a translating steady bubble; more precisely we have the following theorem:

Theorem 1.14. *For any surface tension $\sigma>0$ and $r\geq 3$, there exist for $\epsilon>0$, $\Upsilon>0$ two balls $O_1=\big\{\beta\in\mathbb{R}:0\leq\beta<\Upsilon\big\}$ and $O_2=\big\{(u,v)\in H_p^r\times\mathbb{R}\big|\|u\|_r<\epsilon,|v|<\epsilon\big\}$, so that for sufficiently small ϵ and Υ, $\big(\theta^{(s)},u_0\big)^T : O_1\to O_2$ is the unique real valued map $\big(\theta^{(s)},u_0\big)$ determining the shape and velocity of a steady translating bubble for $\beta\in O_1$.*

Furthermore, there exists C independent of ϵ and Υ such that

$$\|\theta^{(s)}\|_r + |u_0| + \|\gamma^{(s)} - 2\sin(\cdot)\|_{r-2} \leq C\beta^2,$$

and $\theta^{(s)}$ is an odd function implying that the bubble shape is symmetric about the channel centerline.

Remark. We will prove Theorem 1.14 in §5.3. Note results for steady bubble and finger without restriction on β but small σ is available in [45], [46] and [47]. Here, there is no restriction in $\sigma > 0$, but it is held fixed as β is made sufficiently small. Existence of at least one steady translating finger solution for $\sigma > 0$ has been proved earlier [35] using different methods. □

For $\beta \neq 0$, we also consider the time evolution problem, though only for initial conditions for which the bubble shape is symmetric about the channel centerline. Symmetry implies θ is an odd function of α.

Definition 1.15. *We define unsteady perturbation*

$$\Theta(\alpha, t) = \theta(\alpha, t) - \theta^{(s)}(\alpha). \tag{1.10}$$

We also define $\widetilde{\Theta}(\alpha, t) = \mathcal{Q}_1 \Theta(\alpha, t)$.

The main result for the evolution of a translating bubble with side wall effects ($\beta \neq 0$) is as follows:

Theorem 1.16. *For any surface tension $\sigma > 0$ and $r \geq 3$, there exist ϵ, $\Upsilon > 0$ such that if $\|\Theta(\cdot, 0)\|_r < \epsilon$, $|L_0 - 2\pi| < \epsilon < \frac{1}{2}$ and $0 < \beta < \Upsilon$, with $\Theta(-\alpha, 0) = -\Theta(\alpha, 0)$, then there exists a unique solution $\big(\theta, L\big) \in C\big([0, \infty), H_p^r \times \mathbb{R}\big)$ with $\theta(-\alpha, t) = -\theta(\alpha, t)$ to the Hele-Shaw problem (A.1)-(A.3) with initial condition (1.7). Furthermore, $\|\Theta\|_r$ decays exponentially as $t \to \infty$, while L approaches $2\sqrt{\pi V}$ exponentially. Thus the translating steady bubble determined in Theorem 1.14 is asymptotically stable for sufficiently small symmetric initial disturbances in the H_p^r space.*

Remark. This theorem is proved in §6 (See Note 6.4). □

We organize the paper as follows. In §2, we introduce equations (B.1)-(B.6) equivalent to (A.1)-(A.3). It turns out that linearization of (A.1)-(A.3) about a steady shape gives rise to neutrally stable modes, including $\hat{\theta}(\pm 1; t)$. It is therefore convenient to project away these Fourier modes and introduce instead a constraint to determine $\hat{\theta}(\pm 1; t)$ for given $\tilde{\theta}$. Further, we find it convenient to replace the evolution equation for L in (A.1) by an area constraint relation (B.4) since it is otherwise more difficult to obtain exponential control on L directly. In §3, we prove several preliminary lemmas about some integral operators. In §4, we prove results for near-circular initial shape in the absence of side walls ($\beta = 0$), but without any symmetry assumptions. In §5, we consider the problem of determining a steady translating bubble with side-wall effects ($\beta \neq 0$) and complete the proof of Theorems 1.14. In §6, we consider the global evolution problem for $\beta \neq 0$ for initial shapes symmetric about the channel centerline and complete the proof of Theorem 1.16. Because of technical problems in controlling $\hat{\theta}(0; t)$ for nonzero β, we have restricted our attention to only symmetric initial condition for which $\hat{\theta}(0; t) = 0$.

2. Equivalent evolution equations

Definition 2.1. *We introduce functions*

$$\omega_0(\alpha) = \int_0^\alpha e^{i\alpha'} d\alpha', \ \omega(\alpha) = \int_0^\alpha e^{i\alpha' + i\hat{\theta}(1;t)e^{i\alpha'} + i\hat{\theta}(-1;t)e^{-i\alpha'} + i\tilde{\theta}(\alpha')} d\alpha'. \tag{2.1}$$

Note 2.2. *Given the geometric description of θ in terms of the tangent angle, it is clear that*

$$z(\alpha,t) = \frac{L}{2\pi} e^{i\frac{\pi}{2} + i\hat{\theta}(0;t)} \omega(\alpha,t) + z(0,t). \tag{2.2}$$

Further, from (1.9) and (2.2), it follows that

$$V = \frac{L^2}{8\pi^2} \operatorname{Im} \int_0^{2\pi} (\omega_\alpha \omega^*) \, d\alpha \tag{2.3}$$

The above relation implies equation (B.4) in the sequel.

For $\beta \neq 0$, it is seen that $y(0,t)$ is not decoupled from (A.1)-(A.3); thus (1.8) has to be solved at the same time as (A.1)-(A.3). We will show (A.1)-(A.3) and (1.8) with the initial conditions (1.7), $y(0,0) = y_0$ is equivalent to the following evolution system for $\big(\tilde{\theta}(\alpha,t), \hat{\theta}(0;t), y(0,t)\big) \in \dot{H}^r \times \mathbb{R}^2$:

$$\begin{cases} \tilde{\theta}_t(\alpha,t) = \dfrac{2\pi}{L} \mathcal{Q}_1\big(U_\alpha + T(1+\theta_\alpha)\big), \\ \dfrac{d\hat{\theta}(0;t)}{dt} = \dfrac{1}{L} \displaystyle\int_0^{2\pi} T(\alpha,t)\big(1+\theta_\alpha(\alpha,t)\big) d\alpha \end{cases} \tag{B.1}$$

$$y_t(0,t) = -U(0,t) \sin\big(\theta(0,t)\big), \tag{B.2}$$

where

$$\theta = \hat{\theta}(0;t) + \hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha} + \tilde{\theta}, \tag{2.4}$$

with $\gamma(\alpha,t)$, $L(t)$, $T(\alpha,t)$ and $\hat{\theta}(\pm 1;t)$ determined by

$$(I + a_\mu \mathcal{F}[z])\gamma = \frac{2\pi}{L} \sigma \theta_{\alpha\alpha} + \frac{L}{\pi}\big(1 + \frac{\mu_2}{\mu_1+\mu_2} u_0\big) \sin\big(\alpha + \theta\big), \tag{B.3}$$

(B.4)

$$L = \sqrt{\frac{8\pi^2 V}{\operatorname{Im} \int_0^{2\pi} \omega_\alpha(\alpha,t)\omega^*(\alpha,t) d\alpha}}, \text{where } V = \frac{L_0^2}{8\pi^2} \operatorname{Im} \left\{ \int_0^{2\pi} \omega_\alpha(\alpha,0)\omega^*(\alpha,0) d\alpha \right\},$$

$$T = \int_0^\alpha (1+\theta_{\alpha'}) U(\alpha') d\alpha' - \frac{\alpha}{2\pi} \int_0^{2\pi} (1+\theta_\alpha) U(\alpha) d\alpha, \tag{B.5}$$

$$\int_0^{2\pi} \exp\Big(i\frac{\pi}{2} + i\alpha + i\hat{\theta}(-1;t)e^{-i\alpha} + i\hat{\theta}(1;t)e^{i\alpha} + i\tilde{\theta}(\alpha,t)\Big) d\alpha = 0, \tag{B.6}$$

and U determined by (1.6). The initial condition is

$$\tilde{\theta}(\alpha,0) = \mathcal{Q}_1 \theta_0, \ \hat{\theta}(0;0) = \hat{\theta}_0(0) \text{ and } y(0,0) = y_0. \tag{2.5}$$

Definition 2.3. *Let $r \geq 3$. We define open balls :*

$$\mathcal{B}_\epsilon^r = \left\{ u \in \dot{H}^r \,|\, \|u\|_r < \epsilon \right\};$$

$$S_M = \{ y \in \mathbb{R} \,|\, |y| < M \},$$

for some M independent of β.

Remark. We will eventually choose $\epsilon > 0$ to be small enough for Theorem 1.13 and Theorem 1.16 to apply. □

For the constraint (B.6), we have the following result:

Proposition 2.4. *There exists $\epsilon_1 > 0$ so that (B.6) implicitly defines a unique C^1 function $G : \left\{ u \in \dot{H}^1 \,|\, \|u\|_1 < \epsilon_1 \right\} \to \mathbb{R}^2$ satisfying $\left(\operatorname{Re} \hat{\theta}(1;t), \operatorname{Im} \hat{\theta}(1;t) \right) = G\big(\tilde{\theta}(t)\big)$ with $G(0) = 0$ and $G_{\tilde{\theta}}(0) = 0$. Moreover, G satisfies the following estimates for all $u, u_1, u_2 \in \left\{ u \in \dot{H}^1 \,|\, \|u\|_1 < \epsilon_1 \right\}$:*

$$|G(u)| \leq \frac{1}{2} \|u\|_1, \tag{2.6}$$

$$|G(u_1) - G(u_2)| \leq \frac{1}{2} \|u_1 - u_2\|_1. \tag{2.7}$$

Furthermore, if $\tilde{\theta}$ is odd, then the corresponding $\hat{\theta}(1;t)$ is purely imaginary.

Proof. The proof of the first part appears in [50] (See Proposition 2.4). Furthermore, if $\tilde{\theta}(-\alpha) = -\tilde{\theta}(\alpha)$, then on complex conjugation of (B.6), replacing integration variable $\alpha \to -\alpha$ and local uniqueness of the mapping G, it follows that $\hat{\theta}(1;t) = -\hat{\theta}^*(1;t)$, hence it is imaginary. □

Note 2.5. *Note that calculation of $\hat{\theta}(1;t)$ (and therefore of $\hat{\theta}(-1;t) = \hat{\theta}^*(1;t)$) from $\tilde{\theta}$ in Proposition 2.4 allows compuation of*

$$\mathcal{Q}_0 \theta = \tilde{\theta}(\alpha, t) + \hat{\theta}(1;t) e^{i\alpha} + \hat{\theta}(-1;t) e^{-i\alpha}$$

and this is an odd function of α for odd $\tilde{\theta}$. Also, note that having determined γ, $\hat{\theta}(1;t)$ and $\hat{\theta}(-1;t)$, (1.6) and (B.6) determine U and T needed in (B.1)-(B.2).

Proposition 2.6. *Suppose for $r \geq 3$, $\big(\theta(\alpha, t),\, L(t),\, y(0,t)\big) \in C^1\left([0,S], H_p^r \times \mathbb{R} \times S_M\right)$ with $|L - 2\pi| < \frac{1}{2}$ is a solution to the system (A.1)-(A.3), (1.8) with initial conditions (1.7), $y(0,0) = y_0$. Then the corresponding bubble area V is invariant with time.*

Proof. Taking the derivative with respect to t on both sides of (1.9), it is readily seen that

$$\frac{dV}{dt} = \frac{1}{2} \operatorname{Im} \int_0^{2\pi} \left(z_\alpha z_t^* - z_t z_\alpha^* \right) d\alpha = -\frac{L}{2\pi} \int_0^{2\pi} U d\alpha. \tag{2.8}$$

Using (1.6), we have

$$\frac{dV}{dt} = -\operatorname{Re} \Big(\int_0^{2\pi} \frac{z_\alpha(\alpha)}{2\pi} \mathrm{PV} \int_{\alpha-\pi}^{\alpha+\pi} \gamma(\alpha') \mathfrak{K}(\alpha, \alpha') d\alpha' d\alpha \Big). \tag{2.9}$$

Since

$$\operatorname{Re} \left(\mathrm{PV} \int_0^{2\pi} \frac{z_\alpha(\alpha)}{z(\alpha) - z(\alpha')} d\alpha \right) = \log \left| z(2\pi) - z(\alpha') \right| - \log \left| z(0) - z(\alpha') \right| = 0,$$

the Proposition follows. □

Lemma 2.7. *For $r \geq 3$ and sufficiently small ϵ_1, the following statements* **(i.)** *and* **(ii.)** *are equivalent:*
(i.) $(\theta, L, y(0,t)) \in C^1\Big([0,S], H_p^r \times \mathbb{R} \times S_M\Big)$ *satisfies (A.1) and (1.8) with initial conditions (1.7) and* $y(0,0) = y_0$*, where* θ *is real-valued,* $\|\mathcal{Q}_1\theta\|_1 < \epsilon_1$ *and* $|L - 2\pi| < \epsilon_1 < \frac{1}{2}$*, while* γ*,* T *and* U *are determined by (A.2), (A.3) and (1.6).*
(ii.) $\big(\tilde{\theta}, \hat{\theta}(0;t), y(0,t)\big) \in C^1\Big([0,S], \dot{H}^r \times \mathbb{R} \times S_M\Big)$ *satisfies (B.1)-(B.2), initial conditions (2.5), with* $\|\tilde{\theta}\|_1 < \epsilon_1$*, where* γ*,* T*,* $\hat{\theta}(\pm 1;t)$*,* L *and* U *are determined by (B.3)-(B.6), (1.6) and*

$$\theta = \tilde{\theta} + \hat{\theta}(0;t) + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(-1;t)e^{-i\alpha}.$$

Proof. The first part involves essentially the same arguments as Lemma 2.5 in [50], except that (2.3) is used to derive (B.4) with V determined from initial conditions (see Proposition 2.6).

For the second part, assume $\big(\tilde{\theta}(\alpha,t), \hat{\theta}(0;t), y(0,t)\big) \in C^1\Big([0,S], H_p^r \times \mathbb{R} \times S_M\Big)$ is a solution to (B.1)-(B.2) with $\|\tilde{\theta}\|_1 < \epsilon_1$ and γ, T, $\hat{\theta}(\pm 1;t)$, L and U are determined from (B.3)-(B.6), (1.6). From Lemma 2.5 in [50], $\theta = \tilde{\theta} + \hat{\theta}(0;t) + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(-1;t)e^{-i\alpha}$, is real valued solution to the equation for θ in (A.1), where γ, T and U are determined by (A.2), (A.3) and (1.6) for $t \in [0,S]$.

For the evolution of L, we note that taking time derivative of (B.4), we have

$$\frac{L_t}{2\pi} \operatorname{Im} \int_0^{2\pi} \omega_\alpha \omega^* d\alpha + \frac{L}{2\pi} \operatorname{Im} \int_0^{2\pi} \omega_\alpha \omega_t^* d\alpha = 0. \tag{2.10}$$

Using integration by parts, we also have

$$\begin{aligned} \frac{L}{2\pi}\omega_t &= \frac{Li}{2\pi} \int_0^\alpha e^{i\zeta + i\theta(\zeta)} \theta_t(\zeta) d\zeta \\ &= (iU(\alpha) + T(\alpha))\,\omega_\alpha - iU(0) + \frac{1}{2\pi}\omega \int_0^{2\pi} (1+\theta_\alpha) U d\alpha. \end{aligned}$$

In Proposition 2.6, we noted $\int_0^{2\pi} U(\alpha,t) d\alpha = 0$. Plugging the above formula into (2.10), we obtain

$$\frac{L_t}{2\pi}\left(\operatorname{Im}\int_0^{2\pi} \omega_\alpha \omega^* d\alpha\right) + \frac{1}{2\pi}\Big(\operatorname{Im}\int_0^{2\pi} \omega_\alpha\omega^* d\alpha\Big)\Big(\int_0^{2\pi}(1+\theta_\alpha)U d\alpha\Big) = 0. \tag{2.11}$$

Furthermore, if $\|\tilde{\theta}\|_1 < \epsilon$ is sufficiently small, then using $\operatorname{Im}\int_0^{2\pi} e^{i\alpha}\int_0^\alpha e^{-i\alpha'} d\alpha' d\alpha = 2\pi$, by Sobolev's embedding theorem and Proposition 2.4, we have

(2.12)

$$\begin{aligned} &\Big|\operatorname{Im}\int_0^{2\pi}\omega_\alpha\omega^* d\alpha - \operatorname{Im}\int_0^{2\pi} e^{i\alpha}\int_0^\alpha e^{-i\alpha'} d\alpha' d\alpha\Big| \\ \leq &\Big|\int_0^{2\pi} e^{i\alpha}\big(e^{i\theta}-1\big)\int_0^\alpha e^{-i\alpha' - i\theta(\alpha')} d\alpha' d\alpha + \int_0^{2\pi} e^{i\alpha}\int_0^\alpha e^{-i\alpha'}\big(e^{-i\theta(\alpha')}-1\big) d\alpha' d\alpha\Big| \\ \leq &16\sqrt{2}\pi^2 |\theta|_\infty \leq C\|\tilde{\theta}\|_1. \end{aligned}$$

This implies that $\operatorname{Im}\int_0^{2\pi}\omega_\alpha\omega^* d\alpha \neq 0$ and so (2.11) implies

$$L_t = -\int_0^{2\pi}(1+\theta_\alpha) U d\alpha,$$

which is evolution equation for L in (A.1). □

Remark. Because of the equivalence shown above, it turns out to be more convenient to study solutions to the system (B.1)-(B.6), where U is determined from (1.6). Further, without loss of generality, we take $\hat{\theta}(0;0) = 0$ since it only determines the origin of α. □

3. Preliminary Lemmas

Definition 3.1. *We decompose* coth *and* cot *functions into the singular and regular parts at the origin:*

$$\coth(w) = \frac{1}{w} + l_1(w),$$
$$\cot(w) = \frac{1}{w} + l_2(w).$$

We decompose operator

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2, \tag{3.1}$$

where

$$\mathcal{K}_1[z]f = \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')\Big\{\frac{1}{z(\alpha)-z(\alpha')} - \frac{1}{z_\alpha(\alpha')}\cot\frac{1}{2}(\alpha-\alpha')\Big\}d\alpha',$$
$$\mathcal{K}_2[z]f = \frac{1}{2\pi i}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')\Big\{\frac{\beta}{4}l_1\big(\frac{1}{4}\beta(z(\alpha)-z(\alpha'))\big) - \frac{\beta}{4}\tanh\Big[\frac{\beta}{4}\big(z(\alpha)-z^*(\alpha')\big)\Big]\Big\}d\alpha'.$$

Definition 3.2. *Related to $\mathcal{G}$ and $\mathcal{F}$, we define operators $\mathcal{G}_1$, $\mathcal{F}_1$ so that*

$$\mathcal{G}_1[z]\gamma = z_\alpha\left[\mathcal{H}, \frac{1}{z_\alpha}\right]\gamma + 2iz_\alpha\mathcal{K}_1[z]\gamma, \quad \mathcal{G}_2[z]\gamma = 2iz_\alpha\mathcal{K}_2[z]\gamma, \quad \mathcal{F}_1[z]\gamma = \mathrm{Re}\left(\frac{1}{i}\mathcal{G}_1[z]\gamma\right). \tag{3.2}$$

Note 3.3. *It is readily checked that for any $f \in H_p^0$,*

$$\frac{\omega_{0,\alpha}}{\pi}PV\int_0^{2\pi}\frac{f(\alpha')d\alpha'}{\omega_0(\alpha)-\omega_0(\alpha')} = \mathcal{H}[f](\alpha) + i\hat{f}(0), \tag{3.3}$$

implies that

$$\mathcal{G}_1[\omega_0]f = i\hat{f}(0), \tag{3.4}$$

which is imaginary for real valued f.

Definition 3.4. *We define operators Ξ_e, Ξ_s, Ξ_c so that*

$$\Xi_e[u](\alpha) = e^{iu(\alpha)} - 1 - iu(\alpha),$$
$$\Xi_s[u;a](\alpha) = \sin\left(u(\alpha)+\alpha+a\right) - \sin(\alpha+a) - u(\alpha)\cos(\alpha+a),$$
$$\Xi_c[u;a](\alpha) = \cos\left(u(\alpha)+\alpha+a\right) - \cos(\alpha+a) + u(\alpha)\sin(\alpha+a),$$

for a real function $u \in H_p^r$ with $r \geq 1$.

In the rest of this section, we find some estimates for integral operators and functions in terms of $\tilde{\theta}$ and $\hat{\theta}(0;t)$, which will be useful later. Recall tangent angle of the curve is $\frac{\pi}{2} + \alpha + \theta(\alpha) = \frac{\pi}{2} + \alpha + \tilde{\theta}(\alpha) + \hat{\theta}(0;t) + \hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha}$, where $\hat{\theta}(1;t)$ and $\hat{\theta}(-1;t)$ are determined through $G(\tilde{\theta})$.

Lemma 3.5. *(See Lemma 3.1 in [50]) Assume $\|\tilde{\theta}\|_1 < \epsilon_1$ where ϵ_1 is small enough for Proposition 2.4 to apply. Then ω determined from $\tilde{\theta} \in \dot{H}^r$ through (2.1) satisfies the following estimates for $r \geq 1$,*

(3.5)
$$\|\omega_\alpha\|_r \leq C_1(\|\tilde{\theta}\|_r + 1)\exp\left(C_2\|\tilde{\theta}\|_{r-1}\right), \quad \left\|\frac{1}{\omega_\alpha}\right\|_r \leq C_1(\|\tilde{\theta}\|_r + 1)\exp\left(C_2\|\tilde{\theta}\|_{r-1}\right),$$

where constants C_1 and C_2, depend only on r, and particularly for $r=1$, $C_2 = 0$.

Similarly, if z determined by $\big(\tilde{\theta}, \hat{\theta}(0;t), L\big) \in \dot{H}^r \times \mathbb{R}^2$, then for $r \geq 1$,

$$\|z_\alpha\|_r \leq C_1 L(\|\tilde{\theta}\|_r + 1)\exp\left(C_2\|\tilde{\theta}\|_{r-1}\right), \tag{3.6}$$

where constants C_1 and C_2, depend only on r, and particularly for $r=1$, $C_2 = 0$.

Further, if $\omega^{(1)}, \omega^{(2)}$ correspond respectively to $\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^r$, where $\|\tilde{\theta}^{(1)}\|_1$, $\|\tilde{\theta}^{(2)}\|_1 < \epsilon_1$, then for $r \geq 1$,

$$\|\omega_\alpha^{(1)} - \omega_\alpha^{(2)}\|_r \leq C_1\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r \exp\left(C_2\left(\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r\right)\right), \tag{3.7}$$

$$\left\|\frac{1}{\omega_\alpha^{(1)}} - \frac{1}{\omega_\alpha^{(2)}}\right\|_r \leq C_1\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r \exp\left(C_2\left(\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r\right)\right), \tag{3.8}$$

while for $r \geq 2$,

$$\begin{aligned} \|\omega_\alpha^{(1)} - \omega_\alpha^{(2)}\|_r &\leq C_1\left(\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r + \|\tilde{\theta}^{(2)}\|_r\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_{r-1}\right) \\ &\times \exp\left(C_2\big(\|\tilde{\theta}^{(1)}\|_{r-1} + \|\tilde{\theta}^{(2)}\|_{r-1}\big)\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \left\|\frac{1}{\omega_\alpha^{(1)}} - \frac{1}{\omega_\alpha^{(2)}}\right\|_r &\leq C_1\left(\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r + \|\tilde{\theta}^{(2)}\|_r\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_{r-1}\right) \\ &\times \exp\left(C_2\big(\|\tilde{\theta}^{(1)}\|_{r-1} + \|\tilde{\theta}^{(2)}\|_{r-1}\big)\right), \end{aligned} \tag{3.10}$$

where the constants C_1 and C_2 depend only on r.

Lemma 3.6. *If F is an entire function of order one[6] with $F(u) = \sum_{j=j_0}^\infty a_j u^j$ for $j_0 = 1$ or 2. Then for $u \in H_p^{r+1}$ with $r \geq 1$, $F\big(u(\alpha)\big)$ satisfies*

(i) $j_0 = 1$:

$$\begin{aligned} \big\|F\big(u(\cdot)\big)\big\|_0 &\leq C_1 \exp\left(C_2\|u\|_1\right)\|u\|_1, \\ \big\|F\big(u(\cdot)\big)\big\|_{r+1} &\leq C_1 \exp\left(C_2\|u\|_r\right)\|u\|_{r+1}; \end{aligned}$$

(ii) $j_0 = 2$:

$$\begin{aligned} \big\|F\big(u(\cdot)\big)\big\|_0 &\leq C_1 \exp\left(C_2\|u\|_1\right)\|u\|_1^2, \\ \big\|F\big(u(\cdot)\big)\big\|_{r+1} &\leq C_1 \exp\left(C_2\|u\|_r\right)\|u\|_{r+1}\|u\|_r, \end{aligned}$$

where the constants C_1 and C_2 depend only on r.

Further, if both $u^{(1)}$ and $u^{(2)}$ belong to H_p^{r+1}, then for $r \geq 1$,

---

[6]An entire function f of order m satisfies

$$|f(z)| \leq e^{C|z|^m}, \text{ for } z \in \mathbb{C}.$$

(i) $j_0 = 1$:

$$\begin{aligned}\left\|F\big(u^{(1)}(\cdot)\big) - F\big(u^{(2)}(\cdot)\big)\right\|_0 &\leq C_1\|u^{(1)} - u^{(2)}\|_1 \exp\left[C_2\left(\|u^{(1)}\|_1 + \|u^{(2)}\|_1\right)\right]\\ \left\|F\big(u^{(1)}(\cdot)\big) - F\big(u^{(2)}(\cdot)\big)\right\|_{r+1} &\leq C_1\left(\|u^{(1)} - u^{(2)}\|_{r+1} + \|u^{(1)} - u^{(2)}\|_r\|u^{(2)}\|_{r+1}\right)\\ &\quad\times\exp\left[C_2\left(\|u^{(1)}\|_r + \|u^{(2)}\|_r\right)\right];\end{aligned}$$

(ii) $j_0 = 2$:

$$\begin{aligned}\left\|F\big(u^{(1)}(\cdot)\big) - F\big(u^{(2)}(\cdot)\big)\right\|_0 &\leq C_1\|u^{(1)} - u^{(2)}\|_1\Big\{\exp\left[C_2\left(\|u^{(1)}\|_1 + \|u^{(2)}\|_1\right)\right] - 1\Big\}\\ \left\|F\big(u^{(1)}(\cdot)\big) - F\big(u^{(2)}(\cdot)\big)\right\|_{r+1} &\leq C_1\left(\|u^{(1)} - u^{(2)}\|_{r+1}\|u^{(1)}\|_r + \|u^{(1)} - u^{(2)}\|_r\|u^{(2)}\|_{r+1}\right)\\ &\quad\times\exp\left[C_2\left(\|u^{(1)}\|_r + \|u^{(2)}\|_r\right)\right],\end{aligned}$$

where the constants C_1 and C_2 depend only on r.

Proof. The proof is fairly routine and is relegated to the appendix. □

Note 3.7. *In particular, Ξ_e, Ξ_s and Ξ_c satisfy Lemma 3.6 with $j_0 = 2$. $\sin(\alpha + a + u) - \sin(\alpha + a)$ also satisfies Lemma 3.6 with $j_0 = 1$.*

The following divided differences are useful.

Definition 3.8. *For $z \in H_p^r$, we define operators q_1 and q_2 so that*

$$q_1[z](\alpha, \alpha') = \frac{z(\alpha) - z(\alpha')}{\alpha - \alpha'} = \int_0^1 Dz(t\alpha + (1-t)\alpha')dt,$$

$$q_2[z](\alpha, \alpha') = \frac{z(\alpha) - z(\alpha') - z_\alpha(\alpha)(\alpha - \alpha')}{(\alpha - \alpha')^2} = \int_0^1 (t-1)D^2 z((1-t)\alpha + t\alpha')dt,$$

where D and D^2 denote first and second derivatives with respect to the argument.

Proposition 3.9. *There exists $\epsilon_1 > 0$ so that $\|\tilde{\theta}\|_1 \leq \epsilon_1$ implies*

$$\big|q_1[\omega](\alpha, \alpha')\big| \geq \frac{1}{8}, \tag{3.11}$$

and

$$\big|q_1[z](\alpha, \alpha')\big| \geq \frac{1}{2\pi}\sqrt{\frac{\pi V}{24}}\ ,\ \text{for } 0 < |\alpha - \alpha'| \leq \pi, \tag{3.12}$$

which implies that the curve $z(\alpha)$ is non-self-intersecting.

Proof. The first part follows from Proposition 3.3 in [50]. Since $\mathrm{Im}\int_0^{2\pi}\omega_{0,\alpha}\omega_0^* d\alpha = 2\pi$, using (2.12), we obtain for $C\epsilon_1 \leq \pi$,

$$\pi \leq \mathrm{Im}\int_0^{2\pi}\omega_\alpha\omega^* d\alpha \leq 3\pi. \tag{3.13}$$

From (B.4), we obtain

$$\sqrt{\frac{8\pi V}{3}} \leq L \leq \sqrt{8\pi V}. \tag{3.14}$$

Combining (3.11) and (3.14), if $\|\tilde{\theta}\|_1 < \epsilon_1$, then

$$\big|q_1[z](\alpha,\alpha')\big| = \frac{L}{2\pi}\big|q_1[\omega](\alpha,\alpha')\big| \geq \frac{1}{2\pi}\sqrt{\frac{\pi V}{24}}, \text{ for all } 0<|\alpha-\alpha'|\leq\pi.$$

□

Lemma 3.10. *(See Lemma 5 in [2]) Assume z and ω are related through (2.1) and (2.2). Let $z_\alpha \in H_p^j$ for $j \geq 0$. Then for any real a, $D_\alpha^j q_1, D_{\alpha'}^j q_1 \in H^0[a, a+2\pi]$ in both variables α or α' and satisfy the bounds*

$$\|D_\alpha^j q_1[z]\|_0 \leq CL\|\omega_\alpha\|_j \ , \ \|D_{\alpha'}^j q_1[z]\|_0 \leq CL\|\omega_\alpha\|_j$$

with C only depending on j (in particular independent of a). Further if $z_{\alpha\alpha} \in H_p^j$ for $j \geq 0$, then $D_\alpha^j q_2, D_{\alpha'}^j q_2 \in H^0[a, a+2\pi]$ in both variables α and α' and satisfy

$$\|D_\alpha^j q_2[z]\|_0 \leq CL\|\omega_{\alpha\alpha}\|_j \ , \ \|D_{\alpha'}^j q_2[z]\|_0 \leq CL\|\omega_{\alpha\alpha}\|_j$$

with C only depending on j.

Lemma 3.11. *Let $\omega^{(1)}, \omega^{(2)} \in H_p^{j+1}$ for $j \geq 0$. Suppose*

$$|q_1[\omega^{(1)}](\alpha,\alpha')| \geq \frac{1}{8}, \text{ for } 0<|\alpha-\alpha'|\leq\pi.$$

Then for $j=0$, there exists constant C_1 independent of α such that

$$\Big(\int_{\alpha-\pi}^{\alpha+\pi}\Big|\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}\Big|^2 d\alpha'\Big)^{\frac{1}{2}} \leq C_1\|\omega_\alpha^{(2)}\|_1.$$

Further, for $j \geq 3$,

$$\Big(\int_{\alpha-\pi}^{\alpha+\pi}\Big|D_\alpha^j\frac{q_2[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}\Big|^2 d\alpha'\Big)^{\frac{1}{2}} \leq C_2\left(\|\omega_\alpha^{(2)}\|_{j+1} + \|\omega_\alpha^{(2)}\|_{j-1}\|\omega_\alpha^{(1)}\|_j\right)(\|\omega_\alpha^{(1)}\|_{j-1}^{j-1}+1), \tag{3.15}$$

where C_2 depends on j alone, but not on α.

Proof. We note that

$$D_\alpha^j\frac{q_2}{q_1} = \sum_{l=0}^{j} C_{j,l} D_\alpha^{j-l} q_2 D_\alpha^l \frac{1}{q_1}.$$

Using Lemma 3.10 with $L=2\pi$ it follows that for $l \geq 1$

$$\left\|D_\alpha^l\frac{1}{q_1}\right\|_0 \leq C_1\|q_1\|_l\left(1+\|q_1\|_{l-1}^{l-1}\right) \leq C_1\|\omega_\alpha^{(1)}\|_l\left(\|\omega_\alpha^{(1)}\|_{l-1}^{l-1}+1\right),$$

and

$$\left\|D_\alpha^j\frac{q_2}{q_1}\right\|_0 \leq C\sum_{l=1}^{j-1}\left\|D_\alpha^{j-l}q_2\right\|_0\left\|D_\alpha^l\frac{1}{q_1}\right\|_\infty + C\left\|D_\alpha^j\frac{1}{q_1}\right\|_0\|q_2\|_\infty + C\left\|D_\alpha^j q_2\right\|_0\left\|\frac{1}{q_1}\right\|_\infty.$$

The lemma immediately follows from Lemma 3.10 on using $\|\frac{1}{q_1}\|_\infty \leq C$ and $\left\|D_\alpha^l\frac{1}{q_1}\right\|_\infty \leq C\left\|\frac{1}{q_1}\right\|_{l+1}$. □

Lemma 3.12. *Assume $\omega^{(1)}, \omega^{(2)} \in H_p^{j+1}$ for $j \geq 0$. Assume further that*

$$\left| q_1[\omega^{(1)}](\alpha,\alpha') \right| \geq \frac{1}{8} \text{ for } 0 < |\alpha - \alpha'| \leq \pi.$$

Then for $j = 0$, there exists constant C_1 independent of α such that

$$\Big(\int_0^{2\pi} \Big| \frac{q_1[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')} \Big|^2 d\alpha' \Big)^{\frac{1}{2}} \leq C_1 \|\omega_\alpha^{(2)}\|_0.$$

Further, for $j \geq 3$,

$$\Big(\int_0^{2\pi} \Big| D_\alpha^j \frac{q_1[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')} \Big|^2 d\alpha' \Big)^{\frac{1}{2}} \leq C_2 \left(\|\omega_\alpha^{(2)}\|_j + \|\omega_\alpha^{(2)}\|_{j-2} \|\omega_\alpha^{(1)}\|_j \right) (1 + \|\omega_\alpha^{(1)}\|_{j-1}^{j-1}), \tag{3.16}$$

where C_2 depends on j only.

Proof. The proof is almost identical to that of Lemma 3.11. It uses Lemma 3.10 and the lower bound on $q_1[\omega^{(1)}]$. We note that integrand on the left of (3.16) is 2π-periodic in α', noting that factors of $(\alpha - \alpha')$ in $q_1[\omega^{(1)}]$ and $q_1[\omega^{(2)}]$ cancel each other. We are therefore free to replace the upper and lower bound in the integral in α' by $\alpha + \pi$ and $\alpha - \pi$ respectively for which $|q_1|$ is bounded below as needed. □

Lemma 3.13. *Assume $f, g \in H_p^j$, for $j \geq 0$, with Fourier components $\hat{f}(0), \hat{g}(0) = 0$ and $h \in H_p^0$. Suppose*

$$\Big| \int_0^1 g(t\alpha + (1-t)\alpha') dt \Big| \geq \frac{1}{8}, \text{ for } 0 \leq |\alpha' - \alpha| \leq \pi. \tag{3.17}$$

Then for $j = 0$, there exists constant C_1 independent of α such that

$$\int_0^{2\pi} \Big| h(\alpha') \frac{\int_{\alpha'}^{\alpha} f(\tau) d\tau}{\int_{\alpha'}^{\alpha} g(\tau) d\tau} \Big| d\alpha' \leq C_1 \|h\|_0 \|f\|_0.$$

Further, for $j \geq 3$,

$$\int_0^{2\pi} \Big| h(\alpha') D_\alpha^j \frac{\int_{\alpha'}^{\alpha} f(\tau) d\tau}{\int_{\alpha'}^{\alpha} g(\tau) d\tau} \Big| d\alpha' \leq C_2 \|h\|_0 \left(\|f\|_j + \|f\|_{j-2} \|g\|_j \right) (1 + \|g\|_{j-1}^{j-1}),$$

where C_2 depends on j only.

Proof. We define

$$\omega^{(1)}(\alpha) = \int_0^\alpha g(s) ds, \quad \omega^{(2)}(\alpha) = \int_0^\alpha f(s) ds.$$

Clearly, $\omega^{(1)}, \omega^{(2)} \in H_p^{k+1}$ since $\hat{g}(0) = 0 = \hat{f}(0)$. We note

$$\frac{\int_{\alpha'}^{\alpha} f(\tau) d\tau}{\int_{\alpha'}^{\alpha} g(\tau) d\tau} = \frac{q_1[\omega^{(2)}](\alpha,\alpha')}{q_1[\omega^{(1)}](\alpha,\alpha')}.$$

Further, we note that the given condition on lower bound involving g becomes

$$\left| q_1[\omega^{(1)}](\alpha,\alpha') \right| \geq \frac{1}{8}.$$

Using Lemma 3.12, the proof follows using Cauchy Schwartz inequality. □

Lemma 3.14. *Assume $\omega^{(1)}, \omega^{(2)} \in H_p^{j+2}$ with $j \geq 0$. Suppose*

$$\left|q_1[\omega^{(1)}](\alpha,\alpha')\right| \geq \frac{1}{8}\ ,\ \left|q_1[\omega^{(2)}](\alpha,\alpha')\right| \geq \frac{1}{8} \text{ for } 0 < |\alpha-\alpha'| \leq \pi.$$

Then $j = 0$, for any $a \in \mathbb{R}$, there exists constant C_1 independent of α and a such that

$$\left\{\Big(\int_a^{a+2\pi}\Big|\frac{q_2[\omega^{(2)}](\alpha',\alpha)}{q_1[\omega^{(2)}](\alpha',\alpha)} - \frac{q_2[\omega^{(1)}](\alpha',\alpha)}{q_1[\omega^{(1)}](\alpha',\alpha)}\Big|^2 d\alpha'\right\}^{1/2} \leq C_1\|\omega_\alpha^{(2)} - \omega_\alpha^{(1)}\|_1\left(1+\|\omega_\alpha^{(1)}\|_1\right\}.$$

Further, for $j \geq 3$,

$$\left\{\Big(\int_a^{a+2\pi}\Big|D_\alpha^j\left(\frac{q_2[\omega^{(2)}](\alpha',\alpha)}{q_1[\omega^{(2)}](\alpha',\alpha)} - \frac{q_2[\omega^{(1)}](\alpha',\alpha)}{q_1[\omega^{(1)}](\alpha',\alpha)}\right)\Big|^2 d\alpha'\right\}^{1/2}$$
$$\leq C\left(\|\omega_\alpha^{(2)} - \omega_\alpha^{(1)}\|_{j+1} + \|\omega_\alpha^{(2)} - \omega_\alpha^{(1)}\|_j\|\omega_\alpha^{(1)}\|_{j+1}\right)\left(1+\|\omega_\alpha^{(1)}\|_j^j + \|\omega_\alpha^{(2)}\|_j^j\right\},$$

where C depends on j alone, but not on a and α.

Proof. We note from the definitions of q_1 and q_2 that the nonperiodic term $\frac{1}{\alpha-\alpha'}$ that appears in each $\frac{q_2}{q_1}$ in the integrand cancels each other out and we are left with integrating a 2π-periodic function in α'; hence there is no dependence on a, and we may choose $a = \alpha - \pi$ in the proof. The rest of the proof is similar to that of Lemma 3.11. We note that

$$\frac{q_2[\omega^{(2)}]}{q_1[\omega^{(2)}]} - \frac{q_2[\omega^{(1)}]}{q_1[\omega^{(1)}]} = \frac{q_2[\omega^{(2)} - \omega^{(1)}]}{q_1[\omega^{(2)}]} - \frac{q_2[\omega^{(1)}]q_1[\omega^{(2)} - \omega^{(1)}]}{q_1[\omega^{(1)}]q_1[\omega^{(2)}]}$$

and that the denominators are bounded away from zero. We use Lemmas 3.10, 3.11 and the Banach algebra property for $\|\cdot\|_j$ norms in α' for $j \geq 1$. For $j = 0$, the result follows from

$$\left\|\frac{q_2[\omega^{(1)}]}{q_1[\omega^{(1)}]}\right\|_{L^\infty} \leq C\left\|\frac{q_2[\omega^{(1)}]}{q_1[\omega^{(1)}]}\right\|_1,$$

where the norms are taken in α'. □

Lemma 3.15. *For $\|\tilde{\theta}\|_1 < \epsilon$ sufficiently small, ω determined from $\tilde{\theta}$ through (2.1), then for $\tilde{\theta}, J \in H_p^r$ for $r \geq 3$ and any a, there exists constant C_r only depending on r such that*

$$\left\|\frac{1}{\pi}PV\int_a^{a+2\pi}\frac{\omega_\alpha(\alpha)J(\alpha')d\alpha'}{\omega(\alpha)-\omega(\alpha')}\right\|_r \leq C_r\left[\|J\|_r + \|J\|_0\|\tilde{\theta}\|_{r+1}\exp\left(C_r\|\tilde{\theta}\|_r\right)\right].$$

Proof. We note from (3.4) that $J \in H_p^0$,

$$\frac{\omega_{0_\alpha}}{\pi}\text{PV}\int_a^{a+2\pi}\frac{J(\alpha')d\alpha'}{\omega_0(\alpha)-\omega_0(\alpha')} = \mathcal{H}[J](\alpha) + i\hat{J}(0)$$

and we know that $\|\mathcal{H}J\|_r = \|J\|_r$. Therefore, the integrand may be written as

$$i\hat{J}(0) + \mathcal{H}[J](\alpha) + \frac{1}{\pi}\int_{\alpha-\pi}^{\alpha+\pi}d\alpha' J(\alpha')\left\{\frac{q_2[\omega](\alpha,\alpha')}{q_1[\omega](\alpha,\alpha')} - \frac{q_2[\omega_0](\alpha,\alpha')}{q_1[\omega_0](\alpha,\alpha')}\right\}.$$

The proof follows from applying Lemmas 3.14, 3.5, Proposition 3.9 and using Cauchy Schwartz inequality and noting $\omega = \omega_0$, when $\tilde{\theta} = 0$. □

Lemma 3.16. *Assume $\tilde{\theta} \in \dot{H}^{r+1}$, $J \in H_p^r$ and $\omega^{[3]} \in H_p^{r+1}$ for $r \geq 3$. Assume $\|\tilde{\theta}\|_1 < \epsilon$ is sufficiently small and ω is determined from $\tilde{\theta}$ through (2.1). Then for any a, there exists constant C_r only depending on r such that*

$$\left\| \omega_\alpha PV \int_a^{a+2\pi} \frac{J(\alpha')q_1[\omega^{[3]}](\alpha,\alpha')d\alpha'}{(\omega(\alpha)-\omega(\alpha'))\, q_1[\omega](\alpha,\alpha')} \right\|_r$$
$$\leq C_r \left\{ \|\omega_\alpha^{[3]}\|_r \left[\|J\|_r + \|J\|_0 \|\tilde{\theta}\|_{r+1} \exp\left(C_r\|\tilde{\theta}\|_r\right)\right] + \|\omega_\alpha^{[3]}\|_{r+1} \exp\left(C_r \|\tilde{\theta}\|_r\right)\right\}.$$

Proof. We note that

$$\omega_\alpha \mathrm{PV} \int_a^{a+2\pi} \frac{J(\alpha')q_1[\omega^{[3]}](\alpha,\alpha')d\alpha'}{(\omega(\alpha)-\omega(\alpha'))\, q_1[\omega](\alpha,\alpha')} = -D_\alpha \int_a^{a+2\pi} \frac{J(\alpha')q_1[\omega^{[3]}](\alpha,\alpha')}{q_1[\omega](\alpha,\alpha')} d\alpha' + \frac{\omega_\alpha^{[3]}}{\omega_\alpha} \mathrm{PV} \int_a^{a+2\pi} \frac{J(\alpha')\omega_\alpha(\alpha)d\alpha'}{\omega(\alpha)-\omega(\alpha')}.$$

We rely on Lemmas 3.12 and 3.14, as well as Cauchy Schwartz inequality, and Banach algebra property of $\|\cdot\|_r$ norm for $r \geq 1$ to complete the proof. □

Lemma 3.17. *Suppose for $r \geq 2$, $z \in H_p^r$ corresponds to $\tilde{\theta} \in \dot{H}^{r-1}$ through (2.1) and (2.2) and $\|\tilde{\theta}\|_1 < \epsilon_1$, where ϵ_1 is small enough for Propositions 2.4 and 3.9 to apply. Further assume $|L - 2\pi| \leq \frac{1}{2}$ and $y(0,t) \in S_M$. Then there exists $\Upsilon > 0$ such that if $0 \leq \beta < \Upsilon$, then $\mathcal{K}[z] : H_p^0 \to H_p^{r-2}$, and in particular, there are positive constants C_1 depending on r only such that*

$$\|\mathcal{K}[z]f\|_{r-2} \leq C_1 \|f\|_0 (1+\beta^2)(1+\|\omega_\alpha\|_{r-1}^{r-2}). \tag{3.18}$$

Further, $\mathcal{K}[z] : H_p^1 \to H_p^{r-1}$, and

$$\|\mathcal{K}[z]f\|_{r-1} \leq C_1 \|f\|_1 (1+\beta^2)(1+\|\omega_\alpha\|_{r-1}^{r-1}). \tag{3.19}$$

Proof. We will deal with $\mathcal{K}_1$ and $\mathcal{K}_2$ separately. By Lemma 6 in [2], we have

$$\|\mathcal{K}_1[z]f\|_{r-2} \leq C_1 \|f\|_0 (1+\|\omega_\alpha\|_{r-1}^{r-2}), \tag{3.20}$$

$$\|\mathcal{K}_1[z]f\|_{r-1} \leq C_1 \|f\|_1 (1+\|\omega_\alpha\|_{r-1}^{r-1}), \tag{3.21}$$

where the positive constants C_1 both depend on r.

Now consider $D_\alpha^{r-1}\mathcal{K}_2[z]f$, given by:

(3.22)
$$\frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} f(\alpha') D_\alpha^{r-1} \Big\{ \frac{\beta}{4} l_1\big(\frac{1}{4}\beta(z(\alpha)-z(\alpha'))\big) - \frac{\beta}{4} \tanh\big[\frac{\beta}{4}\big(z(\alpha)-z^*(\alpha')\big)\big]\Big\} d\alpha'$$
$$= \frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} f(\alpha') D_\alpha^{r-1} \frac{\beta}{4} l_1\big(\frac{1}{4}\beta(z(\alpha)-z(\alpha'))\big) d\alpha'$$
$$- \frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} f(\alpha') D_\alpha^{r-1} \frac{\beta}{4} \tanh\Big\{\frac{\beta}{4}\big[\big(z(\alpha)-z(\alpha')\big) + 2i\big(y(\alpha',t)-y(0,t)\big) + 2iy(0,t)\big]\Big\} d\alpha'.$$

Equation (3.22) involves upto $r-1$ derivative of z. From (3.14),

$$\big|z(\alpha)-z(\alpha')\big| = \frac{L}{2\pi} \Big| \int_{\alpha'}^{\alpha} e^{i\zeta + i\theta(\zeta)} d\zeta \Big| \leq \frac{L}{2} < 2\pi \tag{3.23}$$

$$z(\alpha,t) - z^*(\alpha',t) = \big(z(\alpha,t)-z(\alpha',t)\big) + 2i\big(y(\alpha',t)-y(0,t)\big) + 2iy(0,t). \tag{3.24}$$

From (3.23), (3.24) and $|y(0,t)| < M$, there exists $\Upsilon > 0$ small enough so that if $0 \leq \beta < \Upsilon < 1$, then $\Big|\beta\big(z(\alpha) - z(\alpha')\big)\Big| \leq \pi$, and $\Big|\beta\big[\big(z(\alpha) - z(\alpha')\big) + 2i\big(y(\alpha',t) - y(0,t)\big) + 2iy(0,t)\big]\Big| < C\beta$. Since l_1 and tanh analytic, we conclude that

$$\|\mathcal{K}_2[z]f\|_{r-1} \leq C_1\beta^2\|f\|_0\big(1 + \|\omega_\alpha\|_{r-1}^{r-1}\big), \tag{3.25}$$

where C_1 depends only on r. Combining (3.20), (3.21) and (3.25), we complete the proof. □

Note 3.18. *Note from (3.2) and (3.25), for $r \geq 1$ and $|L - 2\pi| < \frac{1}{2}$, by Lemma 3.5, it follows that*

$$\|\mathcal{G}_2[z]f\|_{r-1} \leq C_1\beta^2\|f\|_0 \exp\big(C_2\|\tilde{\theta}\|_{r-1}\big), \tag{3.26}$$

where C_1 and C_2 depend only on r.

Lemma 3.19. *(See Lemma 3.8 in [50]) If $f \in H_p^1$, and $\omega^{(1)}$, $\omega^{(2)}$ correspond respectively to $\tilde{\theta}^{(1)}$ and $\tilde{\theta}^{(2)}$, each in $\dot{H}^1$, with $\|\tilde{\theta}^{(1)}\|_1$, $\|\tilde{\theta}^{(2)}\|_1 < \epsilon_1$, then for sufficient small ϵ_1,*

$$\|\mathcal{K}_1[\omega^{(1)}]f - \mathcal{K}_1[\omega^{(2)}]f\|_0 \quad \leq \quad C_1\|f\|_0\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_1.$$

Suppose $\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^r$. Then for $r \geq 1$,

$$\begin{aligned} &\|\mathcal{K}_1[\omega^{(1)}]f - \mathcal{K}_1[\omega^{(2)}]f\|_r \\ \leq \;& C_1 \exp\Big(C_2\big(\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r\big)\Big)\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r\|f\|_1, \end{aligned}$$

while for for $r \geq 3$,

$$\begin{aligned} &\|\mathcal{K}_1[\omega^{(1)}]f - \mathcal{K}_1[\omega^{(2)}]f\|_r \\ \leq \;& C_1 \exp\Big(C_2\big(\|\tilde{\theta}^{(1)}\|_{r-1} + \|\tilde{\theta}^{(2)}\|_{r-1}\big)\Big)\Big(\big(\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r\big)\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_{r-1} \\ &\qquad + \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r\Big)\|f\|_1, \end{aligned}$$

where constants C_1 and C_2 depend on r only.

Lemma 3.20. *Let $0 \leq \beta < \Upsilon$. Let $f \in H_p^1$, and $z^{(1)}$, $z^{(2)}$ correspond respectively to $\big(\tilde{\theta}^{(1)}, L^{(1)}(t), \hat{\theta}^{(1)}(0;t)\big)$ and $\big(\tilde{\theta}^{(2)}, L^{(2)}(t), \hat{\theta}^{(2)}(0;t)\big)$ (see (2.2)). Further, assume $\|\tilde{\theta}^{(1)}\|_1$, $\|\tilde{\theta}^{(2)}\|_1 < \epsilon_1$, $|L^{(1)} - 2\pi| < \frac{1}{2}$, $|L^{(2)} - 2\pi| < \frac{1}{2}$ and $y^{(1)}(0,t) = \operatorname{Im} z^{(1)}(0,t)$, $y^{(2)}(0,t) = \operatorname{Im} z^{(2)}(0,t)$ belong to S_M. Then for ϵ_1 and Υ small enough for Proposition 2.4 and Lemma 3.17 to apply, there exists constant C_1 depending only on r so that*

$$\|\mathcal{G}_2[z^{(1)}]f - \mathcal{G}_2[z^{(2)}]f\|_0 \leq C_1\beta^2\|f\|_0\big(\|\theta^{(1)} - \theta^{(2)}\|_1 + |L^{(1)}(t) - L^{(2)}(t)| + |y^{(1)}(0,t) - y^{(2)}(0,t)|\big).$$

If $\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^r$, then for $r \geq 1$,

$$\begin{aligned} \|\mathcal{G}_2[z^{(1)}]f - \mathcal{G}_2[z^{(2)}]f\|_r \leq C_1\beta^2\|f\|_1 \exp\Big(C_2\big(\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r\big)\Big)\Big(\|\theta^{(1)} - \theta^{(2)}\|_r \\ + |L^{(1)}(t) - L^{(2)}(t)| + |y^{(1)}(0,t) - y^{(2)}(0,t)|\Big), \end{aligned}$$

for constants C_1 and C_2 depending on r only.

Further, if $L^{(1)}$ and $L^{(2)}$ correspond to the same area V through (B.4), then

$$|L^{(1)} - L^{(2)}| \leq C\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_1, \tag{3.27}$$

with C depending on area V alone.

Proof. Note Definition 3.2. The first part of the proof uses the regularity of functions l_1 and tanh away from the poles and uses (2.2) and Lemma 3.5; the second part uses (B.4) and Lemma 3.5, taking into account the implied lower bound in (3.13) for $\|\tilde{\theta}\|_1 < \epsilon_1$. See [1] for more details. □

Lemma 3.21. *(See Lemma 8 in [2]) For $\psi \in H_p^r$ with $r \geq 1$, the operator $[\mathcal{H}, \psi]$ is bounded from H_p^0 to H_p^{r-1}. And we have*

$$\|[\mathcal{H}, \psi] f\|_{r-1} \leq C \|f\|_0 \|\psi\|_r,$$

where C depends on r.

Lemma 3.22. *(See Lemma 3.10 in [50]) For $r > \frac{1}{2}$ and $\psi \in H_p^r$, the operator $[\mathcal{H}, \psi]$ is bounded from H_p^1 to H_p^r, and*

$$\|[\mathcal{H}, \psi] f\|_r \leq C \|f\|_1 \|\psi\|_r,$$

where C depends on r.

Lemma 3.23. *Assume $0 \leq \beta < \Upsilon$, $f \in H_p^1$ and let $z^{(1)}$ and $z^{(2)}$ correspond respectively to $\big(\tilde{\theta}^{(1)}, L^{(1)}(t), \hat{\theta}^{(1)}(0;t)\big)$ and $\big(\tilde{\theta}^{(2)}, L^{(2)}(t), \hat{\theta}^{(2)}(0;t)\big)$ (see (2.2)). Further, assume $\|\tilde{\theta}^{(1)}\|_1$, $\|\tilde{\theta}^{(2)}\|_1 < \epsilon_1$, $|L^{(1)} - 2\pi| < \frac{1}{2}$, $|L^{(2)} - 2\pi| < \frac{1}{2}$ and $y^{(1)}(0,t) = \operatorname{Im} z^{(1)}(0,t)$, $y^{(2)}(0,t) = \operatorname{Im} z^{(2)}(0,t)$ belong to S_M. Then for sufficient small ϵ_1 and Υ so that Proposition 2.4 and and Lemmas 3.17 and 3.20 apply, there exists constants C_1 so that*

$$\|\mathcal{G}[z^{(1)}]f - \mathcal{G}[z^{(2)}]f\|_0 \leq C_1 \|f\|_0 \Big\{ \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_1 + \beta^2 \Big[|L^{(1)}(t) - L^{(2)}(t)| + \|\theta^{(1)} - \theta^{(2)}\|_1 + |y^{(1)}(0,t) - y^{(2)}(0,t)| \Big] \Big\},$$

Furthermore, if $\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^r$, then for $r \geq 1$,

$$\|\mathcal{G}[z^{(1)}]f - \mathcal{G}[z^{(2)}]f\|_r \leq C_1 \|f\|_1 \exp\Big(C_2 \big(\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r \big) \Big) \Big\{ \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r + \beta^2 \Big[|L^{(1)}(t) - L^{(2)}(t)| + \|\theta^{(1)} - \theta^{(2)}\|_r + |y^{(1)}(0,t) - y^{(2)}(0,t)| \Big] \Big\},$$

where the constants C_1 and C_2 depend on r.

Proof. The proof follows from Lemmas 3.19, 3.20 and 3.22, once we note the relation (1.4). □

Proposition 3.24. *Assume $0 \leq \beta < \Upsilon$, z corresponds to $\big(\tilde{\theta}, L(t), \hat{\theta}(0;t)\big)$ through (2.1), (2.2) for $r \geq 3$ with $\tilde{\theta} \in \dot{H}^r$. Further assume $\|\tilde{\theta}\|_1 < \epsilon_1$, $|L - 2\pi| < \frac{1}{2}$ and $y(0,t) = \operatorname{Im} z(0,t)$ belongs to S_M, and $|u_0| < 1$. Then for sufficiently small ϵ_1 and Υ (so that Proposition 2.4 and Lemmas 3.17 and 3.20 apply), there exists unique solution $\gamma \in \{u \in H_p^{r-2} | \hat{u}(0) = 0\}$ satisfying (B.3). For constants C_0 and C, solution γ satisfy estimates*

$$\begin{aligned} \|\gamma\|_0 &\leq C_0 \big(\sigma \|\tilde{\theta}\|_2 + 1 \big), \\ \|\gamma\|_1 &\leq C \big(\sigma \|\tilde{\theta}\|_3 + 1 + \|\tilde{\theta}\|_0 \big). \end{aligned}$$

Let $z^{(1)}$ and $z^{(2)}$ correspond respectively to $\big(\tilde\theta^{(1)}, L^{(1)}(t), \hat\theta^{(1)}(0;t)\big)$ and $\big(\tilde\theta^{(2)}, L^{(2)}(t), \hat\theta^{(2)}(0;t)\big)$ (see (2.2)). Further assume $\|\tilde\theta^{(1)}\| < \epsilon_1$, $\|\tilde\theta^{(2)}\| < \epsilon_1$, $|L^{(1)} - 2\pi| < \frac{1}{2}$, $|L^{(2)} - 2\pi| < \frac{1}{2}$ and $y^{(1)}(0,t) = \operatorname{Im} z^{(1)}(0,t)$, $y^{(2)}(0,t) = \operatorname{Im} z^{(2)}(0,t)$ belong to S_M. Then for sufficient small ϵ_1 and Υ, the corresponding $\gamma^{(1)}$ and $\gamma^{(2)}$ determined from (B.3) satisfies

(3.28)
$$\|\gamma^{(1)} - \gamma^{(2)}\|_0 \le C\Big(\|\theta^{(1)} - \theta^{(2)}\|_2 + |L^{(1)}(t) - L^{(2)}(t)| + \beta^2 |y^{(1)}(0,t) - y^{(2)}(0,t)|\Big).$$

Further, if $\tilde\theta^{(1)}, \tilde\theta^{(2)} \in \dot H^r$, then the corresponding $(\gamma^{(1)}, U^{(1)}, T^{(1)})$ and $(\gamma^{(2)}, U^{(2)}, T^{(2)})$ determined from (B.3), (1.6) and (B.5) satisfy

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{r-2} \le C_1 \exp\big(C_2(\|\tilde\theta^{(1)}\|_r + \|\tilde\theta^{(2)}\|_r)\big)\Big(\|\theta^{(1)} - \theta^{(2)}\|_r + |L^{(1)}(t) - L^{(2)}(t)| + \beta^2 |y^{(1)}(0,t) - y^{(2)}(0,t)|\Big), \tag{3.29}$$

$$\|U^{(1)} - U^{(2)}\|_{r-2} \le C_1 \exp\big(C_2(\|\tilde\theta^{(1)}\|_r + \|\tilde\theta^{(2)}\|_r)\big)\Big(\| \theta^{(1)} - \theta^{(2)}\|_r + |L^{(1)}(t) - L^{(2)}(t)| + \beta^2 |y^{(1)}(0,t) - y^{(2)}(0,t)|\Big), \tag{3.30}$$

$$\|T^{(1)} - T^{(2)}\|_{r-1} \le C_1 \exp\big(C_2(\|\tilde\theta^{(1)}\|_r + \|\tilde\theta^{(2)}\|_r)\big)\Big(\| \theta^{(1)} - \theta^{(2)}\|_r + |L^{(1)}(t) - L^{(2)}(t)| + \beta^2 |y^{(1)}(0,t) - y^{(2)}(0,t)|\Big), \tag{3.31}$$

where C_1 and C_2 depend on r only.

Proof. Since $\mathcal{F}_1[\omega_0]\gamma = \hat\gamma(0) = 0$ (see 3.2 and Note 3.3), (B.3) implies

$$\left[I + a_\mu\left(\mathcal{F}[z] - \mathcal{F}_1[\omega_0]\right)\right]\gamma = \frac{2\pi\sigma}{L}\theta_{\alpha\alpha} + \frac{L}{\pi}\big(1 + \frac{\mu_2 u_0}{\mu_1+\mu_2}\big)\sin\big(\alpha + \theta(\alpha)\big). \tag{3.32}$$

Therefore, if $\tilde\theta \in \dot H^2$, then by Notes 3.3 and 3.18, Lemma 3.23 (note that Lemma 3.23 still holds for $\mathcal{G}_1$.) imply

$$\begin{aligned}\|\mathcal{F}[z]\gamma - \mathcal{F}_1[\omega_0]\gamma\|_0 &\le \|\mathcal{G}_2[z]\gamma\|_0 + \|\mathcal{G}_1[z]\gamma - \mathcal{G}_1[\omega_0]\gamma\|_0 \\ &\le C_1\big(\|\tilde\theta\|_1 + C_2\beta^2\big)\|\gamma\|_0,\end{aligned} \tag{3.33}$$

So, for sufficiently small ϵ_1 and $\Upsilon > 0$, if $\|\tilde\theta\|_1 \le \epsilon_1$ and $0 \le \beta < \Upsilon$, then

$$\left[1 + a_\mu\left(\mathcal{F}[z] - \mathcal{F}_1[\omega_0]\right)\right]^{-1}$$

exists and is bounded independent of any parameters. Therefore, it follows from (3.32) that

$$\|\gamma\|_0 \le C_0(\sigma\|\tilde\theta\|_2 + 1).$$

Further, by Note 3.18 and Lemma 3.23 again, we have

$$\|\mathcal{F}[z]\gamma - \mathcal{F}_1[\omega_0]\gamma\|_{r-2} \le C_1\big(\exp(C_2\|\tilde\theta\|_{r-2})\|\tilde\theta\|_{r-2} + \beta^2\exp(C_2\|\tilde\theta\|_{r-2})\big)\|\gamma\|_1,$$

where C_1 and C_2 depend only on r. Therefore, for $r \ge 3$, it follows from (B.3) that

$$\begin{aligned}\|\gamma\|_{r-2} \le{}& C\sigma\|\tilde\theta\|_r + C\big(1 + \|\tilde\theta\|_{r-3}\big) \\ &+ C_1\big(\exp(C_2\|\tilde\theta\|_{r-2})\|\tilde\theta\|_{r-2} + \beta^2\exp(C_2\|\tilde\theta\|_{r-2})\big)\|\gamma\|_1\end{aligned} \tag{3.34}$$

which C, C_1 and C_2 depend on r, which implies for sufficiently small ϵ_1 and Υ that

$$\|\gamma\|_1 \le C\sigma\|\tilde{\theta}\|_3 + C\big(1+\|\tilde{\theta}\|_0\big). \tag{3.35}$$

From (B.3), we obtain

$$\|\gamma^{(1)}-\gamma^{(2)}\|_{r-2} \le C\Big(\frac{|L^{(1)}-L^{(2)}|}{L^{(1)}L^{(2)}} + \frac{1}{L^{(2)}}\big(\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_r + |\hat{\theta}^{(1)}(0;t)-\hat{\theta}^{(2)}(0;t)|\big) + \Big\|\mathcal{F}[z^{(1)}]\gamma^{(1)}-\mathcal{F}[z^{(2)}]\gamma^{(2)}\Big\|_{r-2},$$

and using Lemma 3.23, we have

$$\begin{aligned}\|\mathcal{F}[z^{(1)}]\gamma^{(1)}-\mathcal{F}[z^{(2)}]\gamma^{(2)}\|_{r-2} &\le \Big\|\mathcal{F}[z^{(1)}](\gamma^{(1)}-\gamma^{(2)})-\mathcal{F}_1[\omega_0](\gamma^{(1)}-\gamma^{(2)})\Big\|_{r-2} \\ &\quad + \Big\|\mathcal{F}[z^{(1)}]\gamma^{(2)}-\mathcal{F}[z^{(2)}]\gamma^{(2)}\Big\|_{r-2} \\ &\le C_1\exp\big(C_2\|\tilde{\theta}^{(1)}\|_r\big)\Big(\|\tilde{\theta}^{(1)}\|_{r-2}+\beta^2\exp(C_2\|\tilde{\theta}^{(1)}\|_{r-2})\Big)\|\gamma^{(1)}-\gamma^{(2)}\|_1 \\ &\quad + \Big\|\mathcal{F}[z^{(1)}]\gamma^{(2)}-\mathcal{F}[z^{(2)}]\gamma^{(2)}\Big\|_{r-2}\end{aligned} \tag{3.36}$$

with C_1 and C_2 depending on r. Hence by Lemma 3.23 again, the fourth and fifth statements in the proposition follow.

From (1.6), it follows that

$$\begin{aligned}\|U^{(1)}-U^{(2)}\|_{r-2} &= \Big\|\frac{\pi}{L^{(1)}}\mathcal{H}[\gamma^{(1)}]-\frac{\pi}{L^{(2)}}\mathcal{H}[\gamma^{(2)}]\Big\|_{r-2}+\Big\|\frac{\pi}{L^{(1)}}\mathcal{G}[z^{(1)}]\gamma^{(1)}-\frac{\pi}{L^{(2)}}\mathcal{G}[z^{(2)}]\gamma^{(2)}\Big\|_{r-2} \\ &\quad + (|u_0|+1)\Big\|\cos\big(\alpha+\theta^{(1)}(\alpha)\big)-\cos\big(\alpha+\theta^{(2)}(\alpha)\big)\Big\|_{r-2},\end{aligned}$$

by Lemmas 3.5 and 3.23, it is easy to obtain (3.30).

Also from (B.5), we have

$$\|T^{(1)}-T^{(2)}\|_{r-1} \le \Big\|(1+\theta_{1,\alpha})U^{(1)}-(1+\theta_{2,\alpha})U^{(2)}\Big\|_{r-2},$$

by (3.30), we get (3.31). □

4. Global existence for near-circular translating bubble without side-walls ($\beta = 0$)

In this section, we consider bubble solutions in the absence of side walls ($\beta = 0$) for near-circular initial shapes. It is readily checked that a time-independent solution that satisfies (B.1), (B.3)-(B.6) is $\theta = 0$, $\gamma = 2\sin\alpha$, $u_0 = 0$, $V = \pi$[7] this describes a steady circular bubble translating along the positive x-axis in the laboratory frame with speed $2 + u_0 = 2$. The uniqueness of this steady state, at least locally in the neighborhood of this solution, is established in a more general context in the steady state analysis of §5 for $\beta \ge 0$. Note in that case steady bubbles are not circular and move along the positive x-axis in the lab frame with speed $2 + u_0(\beta)$.

However, if we overlook the equation for $\hat{\theta}_t(0;t)$ which only affects parametrization α of the boundary, the remaining equations in (B.1), (B.3)-(B.6) are seen to be

[7]This is consistent, as it must be, with our choice length scale $L = L^{(s)} = 2\pi$ as the perimeter length of a steady bubble.

satisfied even for $\theta = \theta^{(s)} \equiv \hat{\theta}(0;t)$, $\gamma = \gamma^{(s)} \equiv 2\sin\left(\alpha + \hat{\theta}(0;t)\right)$, with $u_0 = 0$ and $V = \pi$. Geometrically, this still corresponds to the same translating steady circular bubble, since the time dependence of $\hat{\theta}(0;t)$ does not affect the circular shape and the normal speed $U = 0$ at the interface, as it must be in the frame of the steady bubble.

In studying the time evolution of near-circular interface, it turns out to be more convenient to use the time-dependent $\gamma^{(s)}$ and define a perturbed vortex sheet strength $\Gamma(\alpha,t) \equiv \gamma(\alpha,t) - \gamma^{(s)}(\alpha,t)$.

Using (B.3) and the property $\mathcal{G}[\omega_0]\gamma^{(s)} = 0$ (see Note 3.3), it follows that

$$(4.1)\quad (I + a_\mu \mathcal{F}[\omega])\Gamma = -a_\mu \left[\mathcal{F}[\omega]\gamma^{(s)} - \mathcal{F}[\omega_0]\gamma^{(s)}\right] + \frac{2\pi - L}{L}\sigma\theta_{\alpha\alpha} + \sigma\theta_{\alpha\alpha} + \frac{L-2\pi}{\pi}\sin\left(\alpha+\theta\right) + 2\Big(\sin\left(\alpha+\theta\right) - \sin\left(\alpha + \hat{\theta}(0;t)\right)\Big).$$

Further, from expression for $\gamma^{(s)}$ and property $\mathcal{G}_1[\omega_0]\gamma^{(s)} = 0$ (see Note 3.3), the normal velocity U in (1.6) for $\beta = 0$ may be re-expressed as

$$(4.2)\quad U = \frac{\pi}{L}\mathcal{H}[\Gamma] + \mathrm{Re}\left[\frac{\pi}{L}\mathcal{G}[\omega] - \frac{1}{2}\mathcal{G}[\omega_0]\gamma^{(s)}\right] + \cos(\alpha+\theta) - \cos\left(\alpha + \hat{\theta}(0;t)\right).$$

Proposition 4.1. *If $\tilde{\theta} \in \dot{H}^r$ with $\|\tilde{\theta}\|_1 < \epsilon_1$ and $|\hat{\theta}(0;t)| < \infty$, then for sufficiently small ϵ_1, there exists a unique solution $\Gamma \in \{u \in H_p^{r-2} | \hat{u}(0) = 0\}$ for $r \geq 3$ satisfying (4.1). This solution Γ satisfies the estimates*

$$(4.3)\quad \|\Gamma\|_0 \leq C\|\tilde{\theta}\|_2,$$

$$(4.4)\quad \|\Gamma\|_{r-2} \leq C_1 \exp(C_2\|\tilde{\theta}\|_{r-2})\|\tilde{\theta}\|_r,$$

$$(4.5)\quad \left\|\Gamma - \sigma\frac{2\pi}{L}\theta_{\alpha\alpha}\right\|_{r-2} \leq C_1 \exp\left(C_2\|\tilde{\theta}\|_{r-2}\right)\|\tilde{\theta}\|_{r-1},$$

where C_1 and C_2 depend only on r.

Let $\Gamma^{(1)}$ and $\Gamma^{(2)}$ correspond to $(\tilde{\theta}^{(1)}, \hat{\theta}^{(1)}(0;t))$ and $(\tilde{\theta}^{(2)}, \hat{\theta}^{(2)}(0;t))$ respectively. Assume $\|\tilde{\theta}^{(1)}\|_1 < \epsilon_1$ and $\|\tilde{\theta}^{(2)}\|_1 < \epsilon_1$. If $\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^r$ with $r \geq 3$, then for sufficient small ϵ_1,

$$(4.6)\quad \|\Gamma^{(1)} - \Gamma^{(2)}\|_{r-2} \leq C_1 \exp\left(C_2(\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r)\right)\Big(\left\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\right\|_r + \left|\hat{\theta}^{(1)}(0;t) - \hat{\theta}^{(2)}(0;t)\right|\Big),$$

where C_1 and C_2 depend on r alone.

Proof. In statements (3.28) and (3.29) in Proposition 3.24, we take $\beta = 0$, $\gamma^{(2)} = \gamma$, $\tilde{\theta}^{(1)} = \tilde{\theta}$, $L^{(1)} = L$,

$$\gamma^{(2)} = \gamma^{(s)} = 2\sin\left(\alpha + \hat{\theta}(0;t)\right),\ \tilde{\theta}^{(2)} = 0,\ L^{(2)} = 2\pi$$

and use Lemma 3.23 to obtain statements (4.3) and (4.4).

(4.1) can be written as

$$\Gamma - \sigma\frac{2\pi}{L}\theta_{\alpha\alpha} = -a_\mu\left[\mathcal{F}[\omega]\gamma - \mathcal{F}[\omega_0]\gamma\right] + \frac{L-2\pi}{\pi}\sin\left(\alpha+\theta\right) + 2\Big(\sin\left(\alpha+\theta\right) - \sin\left(\alpha+\hat{\theta}(0;t)\right)\Big).$$

Hence, by Lemma 3.23 with $\beta = 0$, Lemmas 3.20 and 3.6 (see Note 3.7), we obtain (4.5).

The statement (4.6) follows in a similar manner from (3.29).

□

When there is no side wall effect ($\beta = 0$), it is readily checked from (B.1), (B.3)-(B.6) that $y(0,t)$[8] does not affect the evolution of $\tilde{\theta}$ or $\hat{\theta}(0;t)$. So, in this section we will ignore (B.2) all together, since translations do not affect the shape and if necessary, $y(0,t)$ can be calculated from (B.2) at the end.

The main result in this section is the following proposition:

Proposition 4.2. *For $\sigma > 0$, there exists $\epsilon > 0$ such that for $r \geq 3$, if $\|\mathcal{Q}_1\theta_0\|_r < \epsilon$, then there exists a unique solution $\left(\tilde{\theta}, \hat{\theta}(0;t)\right) \in C\left([0,\infty), \dot{H}^r \times \mathbb{R}\right)$ to the Hele-Shaw problem (B.1), (B.3)-(B.6) satisfying initial conditions (2.5). Further, $\|\tilde{\theta}\|_r$, $|\hat{\theta}(\pm 1;t)|$ and $|L - 2\pi|$ each decay exponentially as $t \to \infty$, $|\hat{\theta}(0;t)|$ remains finite. Thus the circular translating steady bubble is asymptotically stable for sufficiently small initial disturbances in the H_p^r space.*

Note 4.3. *Proof of Proposition 4.2 is given at the end of §4. Note also Proposition 4.2 and Lemma 2.7 imply Theorem 1.13.*

4.1. *A priori* **estimates.** Before we consider global solutions to the system (B.1), (B.3)-(B.6) with the initial condition (2.5), some additional estimates are needed for the terms that arise in the evolution equations.

Definition 4.4. *We define operator $\mathfrak{W}$ so that*

$$\mathfrak{W}[f](\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \gamma^{(s)}(\alpha') \frac{\int_{\alpha'}^{\alpha} \mathcal{Q}_0\left(f(\zeta)\omega_{0_\zeta}(\zeta)\right)d\zeta}{\omega_0(\alpha) - \omega_0(\alpha')} d\alpha'.$$

Lemma 4.5. *For $f \in H_p^k$, there exists constant C_1 only dependent on k so that*

$$\|\mathfrak{W}[f]\|_k \leq C_1 \|f\|_k$$

Proof. We take ω_{0_α} and $Q_0\left[f\omega_{0_\alpha}\right]$ to be $g(\alpha)$ and $f(\alpha)$ in Lemma 3.13 respectively and define $h = \gamma^{(s)}$. Note that for this choice, the condition $\hat{g}(0) = 0 = \hat{f}(0)$ as well as the lower bound constraint on $g = \omega_{0,\alpha} = e^{i\alpha}$ is satisified. The proof follows since $\|.\|_{L^\infty}$ bounds in α on $D_\alpha^j \mathfrak{W}[f]$ imply $\|.\|_j$ bounds in the Lemma statement. □

From (4.1), after some algebraic manipulation, it follows that

$$\Gamma(\alpha,t) = \frac{2\pi}{L}\sigma\theta_{\alpha\alpha} + \Gamma_L(\alpha,t) + N_1(\alpha,t) + N_2(\alpha,t) + N_3(\alpha,t), \tag{4.7}$$

where

$$\Gamma_L(\alpha,t) = 2\mathcal{Q}_0\theta(\alpha,t)\cos\left(\alpha + \hat{\theta}(0;t)\right) + \frac{L-2\pi}{\pi}\sin\left(\alpha + \hat{\theta}(0;t)\right) - a_\mu \operatorname{Re}\left(\frac{\partial}{\partial\alpha}\left\{\mathfrak{W}[\mathcal{Q}_0\theta](\alpha)\right\}\right), \tag{4.8}$$

(4.9)

$$N_1 = a_\mu \operatorname{Re}\Big(-\frac{1}{i}\mathcal{G}[\omega]\Gamma + \frac{1}{i}\mathcal{G}[\omega_0]\Gamma\Big) + \frac{L-2\pi}{\pi}\left(\sin(\alpha+\theta) - \sin(\alpha + \hat{\theta}(0;t))\right)$$
$$+ a_\mu \operatorname{Re}\left(i(e^{i\mathcal{Q}_0\theta} - 1)\left\{\frac{\omega_{0_\alpha}}{\omega_\alpha}\left[\mathcal{G}[\omega]\gamma^{(s)} - \mathcal{G}[\omega_0]\gamma^{(s)}\right] - 2\left(\frac{\omega_{0_\alpha}}{\omega_\alpha} - 1\right)\cos\left(\alpha + \hat{\theta}(0;t)\right)\right\}\right)$$
$$+ 2\Xi_s\left[Q_0\theta; \hat{\theta}(0;t)\right]$$

[8]We ignored in all cases $x(0,t) = \operatorname{Re} z(0,t)$ which does not affect the evolution of the shape function θ.

$$N_2 = -2a_\mu \operatorname{Re}\Big(\frac{1}{i}\frac{\partial}{\partial\alpha}\big\{\mathfrak{W}[\Xi_e[\mathcal{Q}_0\theta]](\alpha)\big\}\Big), \tag{4.10}$$

and

(4.11)
$$N_3 = \operatorname{Re}\left(\frac{a_\mu \omega_{0_\alpha}}{i\pi\omega_\alpha}\int_{\alpha-\pi}^{\alpha+\pi}\gamma^{(s)}(\alpha')\frac{q_1[\omega-\omega_0](\alpha,\alpha')}{q_1[\omega_0](\alpha,\alpha')}\left[\frac{q_2[\omega](\alpha,\alpha')}{q_1[\omega](\alpha,\alpha')}-\frac{q_2[\omega_0](\alpha,\alpha')}{q_1[\omega_0](\alpha,\alpha')}\right]\right)$$
$$+\operatorname{Re}\left(\frac{a_\mu\omega_{0_\alpha}}{i\pi}\left[\frac{1}{\omega_\alpha}-\frac{1}{\omega_{0_\alpha}}\right]\int_{\alpha-\pi}^{\alpha+\pi}d\alpha'\frac{\gamma^{(s)}(\alpha')q_1[\omega-\omega_0](\alpha,\alpha')\omega_{0_\alpha}(\alpha)}{q_1[\omega_0](\alpha,\alpha')\left[\omega(\alpha)-\omega_0(\alpha')\right]}\right).$$

Further, from (1.6) it follows that normal velocity

(4.12)
$$U(\alpha,t) = \frac{2\pi^2}{L^2}\sigma\mathcal{H}(\theta_{\alpha\alpha})(\alpha) + U_L(\alpha,t) + \frac{1}{2}\mathcal{H}\Big(N_1(\cdot)+N_2(\cdot)+N_3(\cdot)\Big)(\alpha) + N_4(\alpha),$$

where

$$U_L(\alpha,t) = \frac{1}{2}\mathcal{H}[\Gamma_L](\alpha,t) + \frac{L-2\pi}{L}\cos\big(\alpha+\hat{\theta}(0;t)\big) - \mathcal{Q}_0\theta\sin\big(\alpha+\hat{\theta}(0;t)\big)$$
$$-\operatorname{Re}\Big(\frac{1}{i}\frac{\partial}{\partial\alpha}\big(\mathfrak{W}[\mathcal{Q}_0\theta](\alpha)\big)\Big),$$

and

(4.13)
$$N_4(\alpha) = \operatorname{Re}\Big(\frac{\pi}{L}\mathcal{G}[\omega]\Gamma - \frac{1}{2}\mathcal{G}[\omega_0]\Gamma\Big) + \frac{2\pi-L}{L}\operatorname{Re}\Big(\frac{1}{2}\Big[\mathcal{G}[\omega]\gamma^{(s)} - \mathcal{G}[\omega_0]\gamma^{(s)}\Big]\Big)$$
$$+\operatorname{Re}\left((e^{i\mathcal{Q}_0\theta}-1)\left\{\frac{\omega_{0_\alpha}}{2\omega_\alpha}\big(\mathcal{G}[\omega]\gamma^{(s)}-\mathcal{G}[\omega_0]\gamma^{(s)}\big) - \left(\frac{\omega_{0_\alpha}}{\omega_\alpha}-1\right)\cos\big(\alpha+\hat{\theta}(0;t)\big)\right\}\right)$$
$$-\operatorname{Re}\left(\frac{\omega_{0_\alpha}}{2\pi}\mathrm{PV}\int_{\alpha-\pi}^{\alpha+\pi}\gamma^{(s)}(\alpha')\frac{q_1[\omega-\omega_0](\alpha,\alpha')}{q_1[\omega_0](\alpha,\alpha')}\Big(\frac{1}{\omega(\alpha)-\omega(\alpha')}-\frac{1}{\omega_0(\alpha)-\omega_0(\alpha')}\Big)d\alpha'\right)$$
$$+\frac{2\pi-L}{2L}\mathcal{H}[\Gamma-\frac{2\pi}{L}\sigma\theta_{\alpha\alpha}] + \operatorname{Re}\Big(\frac{\partial}{\partial\alpha}\big(\mathfrak{W}[\Xi_e[\mathcal{Q}_0\theta]](\alpha)\big)\Big) + \Xi_c\Big[Q_0\theta;\hat{\theta}(0;t)\Big]$$

Using (4.12) and (B.5), from (B.1) we obtain

$$\tilde{\theta}_t = \frac{2\pi}{L}\mathcal{Q}_1\big(U_\alpha + T(1+\theta_\alpha)\big) \tag{4.14}$$
$$= \mathcal{A}[\tilde{\theta}](\alpha,t) + \mathcal{A}_N[\tilde{\theta},\hat{\theta}(0;\cdot),L](\alpha,t) + \mathfrak{N}\big[\tilde{\theta},\hat{\theta}(0;\cdot)\big](\alpha,t),$$

where the operators $\mathcal{A}$ and $\mathcal{A}_N$ acting on real valued functions $\tilde{\theta}\in\dot{H}^r$ for $r\geq 3$ are defined by

$$\mathcal{A}[\tilde{\theta}](\alpha,t) = \sum_{k=2}^{\infty}e^{ik\alpha}\Big(-\sigma d(k)\hat{\theta}(k;t) + m(k)\hat{\theta}(k+1;t)\Big) + c.c., \tag{4.15}$$

(4.16)
$$\mathcal{A}_N[\tilde{\theta},\hat{\theta}(0;\cdot),L](\alpha,t) = \sum_{k=2}^{\infty}e^{ik\alpha}\left\{\left(\frac{-8\pi^3}{L^3}+1\right)\sigma\ d(k)\ \hat{\theta}(k;t) + e^{-i\hat{\theta}(0;t)}\left(\frac{2\pi}{L}-1\right)m(k)\hat{\theta}(k+1;t)\right\}$$
$$+\Big(e^{-i\hat{\theta}(0;t)}-1\Big)\sum_{k=2}^{\infty}e^{ik\alpha}m(k)\hat{\theta}(k+1;t) + c.c.,$$

where *c.c.* indicates complex conjugate of explicitly shown terms on the right side in each of (4.15), (4.16) [9] and

$$d(k) = \frac{1}{2}k(k^2-1)\ ,\ m(k) = (1+a_\mu)\frac{(k^2-1)(k+1)}{k(k+2)}, \tag{4.17}$$

and

$$\mathfrak{N}\big[\tilde{\theta},\hat{\theta}(0;\cdot)\big](\alpha,t) = \frac{2\pi}{L}\mathcal{Q}_1\Big\{\Big(\frac{1}{2}\mathcal{H}\Big(N_1(\cdot)+N_2(\cdot)+N_3(\cdot)\Big)(\alpha)+N_4(\alpha)\Big)_\alpha + N_5(\alpha)\Big\}, \tag{4.18}$$

where

$$\begin{aligned} N_5(\alpha) \quad = \quad & \int_0^\alpha \Big[\frac{1}{2}\mathcal{H}\Big(N_1(\cdot)+N_2(\cdot)+N_3(\cdot)\Big)(\alpha')+N_4(\alpha')\Big]d\alpha' \\ & -\frac{\alpha}{2\pi}\int_0^{2\pi}\Big[\frac{1}{2}\mathcal{H}\Big(N_1(\cdot)+N_2(\cdot)+N_3(\cdot)\Big)(\alpha)+N_4(\alpha)\Big]d\alpha \\ & +\int_0^\alpha \theta_\alpha(\alpha')U(\alpha')d\alpha' - \frac{\alpha}{2\pi}\int_0^{2\pi}\theta_\alpha(\alpha)U(\alpha)d\alpha \\ & +\Big(\int_0^\alpha \theta_\alpha(\alpha')U(\alpha')d\alpha' - \frac{\alpha}{2\pi}\int_0^{2\pi}\theta_\alpha(\alpha)U(\alpha)d\alpha\Big)\theta_\alpha(\alpha). \end{aligned} \tag{4.19}$$

It is straightforward to check from (4.17) that for any $k \geq 2$,

$$\frac{3}{8}k^3 \leq d(k) \leq \frac{1}{2}k^3\ ,\ \frac{9}{16}(1+a_\mu)k \leq m(k) \leq (1+a_\mu)k. \tag{4.20}$$

After some calculation, we also find from (B.1) that

$$\hat{\theta}_t(0;t) = \frac{1}{L}\int_0^{2\pi} T(\alpha,t)\Big(1+\theta_\alpha(\alpha,t)\Big)d\alpha = \mathfrak{N}_0[\tilde{\theta},\hat{\theta}(0;\cdot)](t), \tag{4.21}$$

where the functional $\mathfrak{N}_0$ of real valued $\big(\tilde{\theta}(\alpha,t),\hat{\theta}(0;t)\big)$ is defined by[10]

$$\begin{aligned} \mathfrak{N}_0\big[\tilde{\theta},\hat{\theta}(0;\cdot)\big](t) = & \int_0^{2\pi}\int_0^\alpha \Big(\big(\frac{2\pi^2}{L^3}-\frac{1}{4\pi}\big)\sigma\mathcal{H}(\theta_{\alpha\alpha})(\alpha') + \big(\frac{1}{L}-\frac{1}{2\pi}\big)U_L(\alpha')\Big)d\alpha' d\alpha \\ & -\pi\big(\frac{1}{L}-\frac{1}{2\pi}\big)\int_0^{2\pi}U_L(\alpha)d\alpha + \frac{1}{L}\int_0^{2\pi}N_5(\alpha)d\alpha + B_0\big[\tilde{\theta},\hat{\theta}(0;\cdot)\big](t), \end{aligned} \tag{4.22}$$

with the functional B_0 defined by

$$B_0\big[\tilde{\theta},\hat{\theta}(0;\cdot)\big](t) = \sum_{k=1}^\infty \Big(\frac{\sigma k}{2}\hat{\theta}(k;t) - e^{-i\hat{\theta}(0;t)}(1+a_\mu)\frac{k+1}{k(k+2)}\hat{\theta}(k+1;t)\Big) + c.c.. \tag{4.23}$$

With respect to the functional $B_0[\tilde{\theta}(\alpha,t),\hat{\theta}(0;t)]$, the following statement readily follows.

[9] Note that while L is shown as an independent argument of $\mathcal{A}_N$, in the evolution equation (4.14), itself, L is determined from $\tilde{\theta}$ through (B.4) and (2.1).

[10] Note that the Fourier component $\hat{\theta}(1;t)$ appearing in the summation is being determined indirectly from $\tilde{\theta}$ through (B.6) (see Proposition 2.4).

Lemma 4.6. *With $\tilde\theta \in \dot H_1$ and $\|\tilde\theta\|_2 < \epsilon$ sufficiently small, then*

$$\left| B_0\left[\tilde\theta(\alpha,t), \hat\theta(0;t)\right]\right| \le C\|\tilde\theta\|_2.$$

Further $B_0^{(1)}$ and $B_0^{(2)}$ correspond to respectively to $\left(\tilde\theta^{(1)}, \hat\theta^{(1)}(0;t)\right)$ and and $\left(\tilde\theta^{(2)}, \hat\theta^{(2)}(0;t)\right)$, then

$$\left|B_0^{(1)} - B_0^{(2)}\right| \le C\left\{\|\tilde\theta^{(1)}(\cdot,t) - \tilde\theta^{(2)}(\cdot,t)\|_2 + |\hat\theta^{(1)}(0;t) - \hat\theta^{(2)}(0;t)|\|\tilde\theta^{(1)}(\cdot,t)\|_1\right\}.$$

Proof. The proof follows easily from the expression (4.23), and Proposition 2.4 relating $\hat\theta(1;t)$ to $\tilde\theta$. □

We have the following estimates for the nonlinear terms N_j, $j = 1,\cdots,5$:

Lemma 4.7. *If for $r \ge 3$, $\tilde\theta \in \dot H^r$ and $\|\tilde\theta\|_1 < \epsilon_1$, then for sufficiently small ϵ_1, N_j, $j = 1,\cdots,5$, defined by (4.9), (4.10), (4.11), (4.12), (4.13) and (4.19) satisfy*

$$\|N_j\|_{r-1} \le C_1 \exp(C_2\|\tilde\theta\|_{r-1})\|\tilde\theta\|_{r-1}\|\tilde\theta\|_r, \tag{4.24}$$

where C_1 and C_2 depend only on r. Further let $N_j^{(1)}$ and $N_j^{(2)}$ correspond to $\left(\tilde\theta^{(1)}, \hat\theta^{(1)}(0;t)\right)$ and $\left(\tilde\theta^{(2)}, \hat\theta^{(2)}(0;t)\right)$ respectively, each in $\dot H^r \times \mathbb{R}$ with $\|\tilde\theta^{(1)}\|_1$ and $\|\tilde\theta^{(2)}\|_1 < \epsilon_1$. Then for sufficiently small ϵ_1,

(4.25)
$$\left\|N_j^{(1)} - N_j^{(2)}\right\|_{r-1} \le C_1 \exp\left(C_2\left(\|\tilde\theta^{(1)}\|_{r-1} + \|\tilde\theta^{(2)}\|_{r-1}\right)\right)\Big\{\left(\|\tilde\theta^{(1)}\|_{r-1} + \|\tilde\theta^{(2)}\|_{r-1}\right)$$
$$\times\left(\left\|\tilde\theta^{(1)} - \tilde\theta^{(2)}\right\|_r + \left|\hat\theta^{(1)}(0;t) - \hat\theta^{(2)}(0;t)\right|\right) + \left(\|\tilde\theta^{(1)}\|_r + \|\tilde\theta^{(2)}\|_r\right)\left\|\tilde\theta^{(1)} - \tilde\theta^{(2)}\right\|_{r-1}\Big\},$$

Proof. For estimating N_1 we use Lemmas 3.6 (see Note 3.7), 3.23, 3.20 (in particular (3.27) for $L^{(1)} = L$, $L^{(2)} = 2\pi$, the latter corresponding to $\tilde\theta = 0$) and Proposition 4.1. For N_2, we use Lemmas 3.6 (see Note 3.7) and 4.5. For N_3, we use Lemmas 3.5, 3.12, 3.14 and 3.16 together with Cauchy-Schwartz inequality to get the desired bound.

For (4.8), by Lemmas 3.20 and 4.5, we have

$$\|\Gamma_L\|_{r-3} \le C\|\tilde\theta\|_{r-3}. \tag{4.26}$$

For N_4 we rely on (4.26), Lemmas 3.6 (see Note 3.7), 3.20 (equation (3.27) in particular), 3.23, 4.5 and Proposition 4.1. N_5 uses bounds similar to N_j for $j = 1,\cdots,4$ as well as bounds on U (In Proposition 3.24, we choose $U^{(1)} = U$, $U^{(2)} = 0$, $\tilde\theta^{(1)} = \tilde\theta$, $\tilde\theta^{(2)} = 0$, and $L^{(1)} = L$, $L^{(2)} = 0$ in (3.30) to get the bound of U.). □

Corollary 4.8. *If for $r \ge 3$, $\tilde\theta \in \dot H^r$ and $\|\tilde\theta\|_1 < \epsilon_1$, then for sufficiently small ϵ_1, the function $\mathfrak{N}$, and the functional $\mathfrak{N}_0$, defined in (4.18) and (4.22) satisfy the following estimates*

$$\begin{aligned} \|\mathfrak{N}\|_{r-1} &\le C_1 \exp(C_2\|\tilde\theta\|_r)\|\tilde\theta\|_r\|\tilde\theta\|_{r+1}, \\ |\mathfrak{N}_0| &\le C_1 \exp(C_2\|\tilde\theta(\cdot,t)\|_3)\|\tilde\theta(\cdot,t)\|_3^2 + C_1\|\tilde\theta(\cdot,t)\|_2. \end{aligned} \tag{4.27}$$

where C_1 and C_2 depend only on r. Further, let $\left(\mathfrak{N}^{(1)}, \mathfrak{N}_0^{(1)}\right)$ and $\left(\mathfrak{N}^{(2)}, \mathfrak{N}_0^{(2)}\right)$ correspond to $\left(\tilde\theta^{(1)}, \hat\theta^{(1)}(0;t)\right)$ and $\left(\tilde\theta^{(2)}, \hat\theta^{(2)}(0;t)\right)$ respectively, each in $\dot H^r \times \mathbb{R}$ with

$\|\tilde{\theta}^{(1)}\|_1$ *and* $\|\tilde{\theta}^{(2)}\|_1 < \epsilon_1$. *Then for sufficiently small* ϵ_1,

$$\left\|\mathfrak{N}^{(1)} - \mathfrak{N}^{(2)}\right\|_{r-1} \le C_1 \exp\left(C_2\big(\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r\big)\right)\Big\{\big(\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r\big) \tag{4.28}$$
$$\times\left(\left\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\right\|_{r+1}+\left|\hat{\theta}^{(1)}(0;t) - \hat{\theta}^{(2)}(0;t)\right|\right)+\big(\|\tilde{\theta}^{(1)}\|_{r+1}+\|\tilde{\theta}^{(2)}\|_{r+1}\big)\left\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\right\|_r\Big\},$$

$$\left|\mathfrak{N}_0^{(1)}-\mathfrak{N}_0^{(2)}\right| \le C_1 \exp\left(C_2\big(\|\tilde{\theta}^{(1)}\|_3 + \|\tilde{\theta}^{(2)}\|_3\big)\right)\Big\{\left\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\right\|_3+\|\tilde{\theta}^{(1)}\|_3|\hat{\theta}^{(1)}(0;t)-\hat{\theta}^{(2)}(0;t)|\Big\}, \tag{4.29}$$

where C_1 *and* C_2 *depend on* r.

Proof. On using Lemmas 4.6 and 4.7, the the proof follows from the expressions of $\mathfrak{N}$ and $\mathfrak{N}_0$ in terms of $N_1, \cdots, N_5$. □

4.2. **Weighted Sobolev Space and Estimates.** For any surface tension σ, we choose the integer K depending on σ such that

(a) if $\sigma \ge 1$, then $K = 2$;
(b) if $0 < \sigma < 1$, then $K = \left[\sqrt{1 + \frac{6}{\sigma}}\right] + 1$.

We define the weight $w(\sigma, k)$ so that

$$w(\sigma, k) = \sigma^{K-|k|} \text{ for } 2 \le |k| \le K(\sigma), w(\sigma, k) = 1 \text{ for } |k| > K(\sigma). \tag{4.30}$$

Definition 4.9. *Let* $r \ge 0$. *We define a family of weighted Sobolev norm in* $\dot{H}^r$ *by*

$$\|u\|_{w,r}^2 = \sum_{k=2}^{\infty} w^2(\sigma, k)|k|^{2r}|\hat{u}(k)|^2 + \sum_{k=-2}^{-\infty} w^2(\sigma, k)|k|^{2r}|\hat{u}(k)|^2, \tag{4.31}$$

and the corresponding inner-product:

$$(v, u)_{w,r} = \sum_{k=2}^{\infty} w^2(\sigma, k)|k|^{2r}\hat{v}^*(k)\hat{u}(k) + \sum_{k=-2}^{-\infty} w^2(\sigma, k)|k|^{2r}\hat{v}^*(k)\hat{u}(k). \tag{4.32}$$

Note 4.10. *If* u *and* v *are real valued, be in* $\dot{H}^r$, *the inner-product reduces to*

$$(v, u)_{w,r} = 2\,\mathrm{Re}\left[\sum_{k=2}^{\infty} w^2(\sigma, k)|k|^{2r}\hat{v}^*(k)\hat{u}(k)\right]. \tag{4.33}$$

Remark. It is clear that for any fixed $\sigma > 0$, the two norms $\|\cdot\|_{w,r}$ and $\|\cdot\|_r$ are equivalent. □

The following two lemmas involve useful inner product estimates involving $\mathcal{A}$ and $\mathcal{A}_N$:

Lemma 4.11. *For any* $r \ge 0$ *and* $v \in \dot{H}^{r+3/2}$,

$$(v, -\mathcal{A}[v])_{w,r} \ge \frac{15\sigma}{64}\|v\|_{w,r+3/2}^2.$$

Proof. It is convenient to define

$$\delta = \sup_{k\ge 2} \frac{m(k)w(k,\sigma)}{\sigma d^{1/2}(k)d^{1/2}(k+1)w(k+1,\sigma)}.$$

Since $(1+a_\mu)\le 2$, it is not difficult to conclude from the explicit expressions of $d(k)$ and $m(k)$ that in all cases, $\delta \le \frac{3}{8}$. Then, it follows from Cauchy Schwartz inequality that

$$\sum_{k=2}^{\infty} k^{2r} w^{2k}(\sigma,k) m(k) \operatorname{Re}\left\{(\hat{v}^*(k)\hat{v}(k+1)\right\} \le \frac{3}{8}\sigma \sum_{k=2}^{\infty} k^{2r} w^{2k}(\sigma,k) d(k) |\hat{v}(k)|^2.$$

It follows that

$$(v, -\mathcal{A}[v])_{w,r} \ge \frac{5\sigma}{8} \sum_{k=2}^{\infty} k^{2r} w^{2k}(\sigma,k) d(k) |\hat{v}(k)|^2 \ge \frac{15\sigma}{64} \|v\|_{w,r+3/2}^2.$$

□

With respect to the operator $\mathcal{A}_N$, we have the following estimate:

Lemma 4.12. *For $r\ge 3$, assume real $f, f_1, f_2 \in \dot{H}^{r+3/2}$ and a, a_1, a_2, L, L_1, L_2 are real numbers satisfying constraint $|L-2\pi| \le \frac{1}{2}$, $|L_j - 2\pi| \le \frac{1}{2}$ for $j=1,2$. Then there exists constant C_r only dependent on r so that*

$$\|\mathcal{A}_N[f,a,L]\|_{w,r-3/2} \le C_r \sigma \left(|L-2\pi| \|f\|_{w,r+3/2} + |a| \|f\|_{w,r-1/2}\right),$$

$$\|\mathcal{A}_N[f_1,a_1,L_1] - \mathcal{A}_N[f_2,a_2,L_2]\|_{w,r-3/2} \le C_r \sigma \Big(|L_1-L_2| \|f_2\|_{w,r+3/2} + |a_1-a_2| \|f_2\|_{w,r-1/2}$$
$$+ |L_1-2\pi| \|f_1-f_2\|_{w,r+3/2} + |a_1| \|f_1-f_2\|_{w,r-1/2}\Big).$$

Proof. From the definition of $\mathcal{A}_N$, it follows that

$$\|\mathcal{A}_N[f,a,L]\|_{w,r-3/2}^2 \le 2\Big|1-\frac{8\pi^3}{L^3}\Big|^2 \sum_{k=2}^{\infty} \sigma^2 k^{2r-3} d^2(k) w^2(k,\sigma) |\hat{f}(k)|^2$$
$$+ 2\left(\Big|2\sin\frac{a}{2}\Big|^2 + \Big|\frac{2\pi}{L}-1\Big|^2\right) \sum_{k=2}^{\infty} k^{2r-3} m^2(k) w^2(k,\sigma) |\hat{f}(k+1)|^2$$
$$\le C_r \sigma^2 \Big(|L-2\pi|^2 \|f\|_{r+3/2}^2 + (|a|^2 + |L-2\pi|^2) \sup_{k\ge 2} \frac{m^2(k) w^2(k,\sigma)}{\sigma^2 (k+1)^2 w^2(k+1,\sigma)} \|f\|_{r-1/2}^2\Big)$$
$$\le C_r \Big(|L-2\pi|^2 \|f\|_{r+3/2}^2 + |a|^2 \|f\|_{r-1/2}^2\Big).$$

Therefore, from bounds on $d(k)$ and $m(k)$, it follows that

$$\|\mathcal{A}_N[f_1-f_2,a_1,L_1]\|_{w,r-3/2} \le C_r \sigma \left(|L_1-2\pi| \|f_1-f_2\|_{w,r+3/2} + |a_1| \|f_1-f_2\|_{w,r-1/2}\right).$$

Further, since

$$\mathcal{A}_N[f,a_1,L_1] - \mathcal{A}_N[f,a_2,L_2] = \sigma \left(\frac{8\pi^3}{L_2^3} - \frac{8\pi^3}{L_1^3}\right) \sum_{k=2}^{\infty} e^{ik\alpha} d(k) \hat{f}(k)$$
$$+ \left\{\left(\frac{2\pi}{L_1} - \frac{2\pi}{L_2}\right) + \frac{2\pi}{L_2}\left(e^{ia_1} - e^{ia_2}\right)\right\} \sum_{k=2}^{\infty} e^{ik\alpha} m(k) \hat{f}(k+1),$$

the results follow from the definition of $\|\cdot\|_{w,r}$ on using the restriction on L_1, L_2. □

4.3. **Linear Evolution and space-time estimates.**

Definition 4.13. *For $r \geq 3$, we define the space of real valued functions*

$$H^r_\sigma \equiv C\big([0,\infty), \dot{H}^r\big) \cap L^2\big([0,\infty), \dot{H}^{r+3/2}\big),$$

equipped with the norm $\| \cdot \|_{H^r_\sigma}$ defined by

$$\|u\|^2_{H^r_\sigma} = \sup_{t \geq 0} e^{t\sigma} \|u(\cdot,t)\|^2_{w,r} + \frac{\sigma}{4} \int_0^\infty e^{\sigma t} \|u(\cdot,t)\|^2_{w,r+3/2} dt.$$

We now consider linear evolution equation

$$v_t(\alpha,t) - \mathcal{A}[v](\alpha,t) = f(\alpha,t) \text{ with } v(\cdot,0) = v_0 \in \dot{H}^r, \tag{4.34}$$

where $f \in H^{r-3}_\sigma$.

Lemma 4.14. *(A priori linear energy estimates) Suppose $r \geq 3$, $f \in H^{r-3}_\sigma$ and Then a solution $v(.,t) \in \dot{H}^r$ to (4.34) will satisfy the following energy inequality for any t:*

$$e^{\sigma t} \|v(\cdot,t)\|^2_{w,r} + \frac{\sigma}{4} \int_0^t e^{\sigma \tau} \|v(\cdot,\tau)\|^2_{w,r+3/2} d\tau \leq \|v_0\|^2_{w,r} + \frac{128}{3\sigma^2} \|f\|^2_{H^{r-3}_\sigma},$$

and thus

$$\|v\|^2_{H^r_\sigma} \leq \|v_0\|^2_{w,r} + \frac{128}{3\sigma^2} \|f\|^2_{H^{r-3}_\sigma}$$

Proof. Taking the $(\cdot,\cdot)_{w,r}$ inner-product on both sides of (4.34) with v, we obtain

$$\frac{d}{dt} \|v\|^2_{w,r} - 2\left(v(\cdot,t), \mathcal{A}[v]\right)_{w,r} = 2\left(v(\cdot,t), f(\cdot,t)\right)_{w,r}. \tag{4.35}$$

From Lemma 4.11, this implies

$$\frac{d}{dt} \|v\|^2_{w,r} + \frac{15\sigma}{32} \|v(\cdot,t)\|^2_{w,r+3/2} \leq 2 \|v(\cdot,t)\|_{w,r+3/2} \|f(.,t)\|_{w,r-3/2}.$$

Noting that

$$|k|^{r+3/2} \geq 2^{1/2} |k|^{r+1} \geq 2|k|^{r+1/2} \geq 2^{3/2} |k|^r \text{ for } k \geq 2$$

implies that

$$\|v\|_{w,r+3/2} \geq 2^{1/2} \|v\|_{w,r+1} \geq 2\|v\|_{w,r+1/2} \geq 2^{3/2} \|v\|_{w,r}.$$

It follows that on using Cauchy Schwartz inequality,

$$\frac{d}{dt} \|v\|^2_{w,r} + \sigma \|v\|^2_{w,r} + \frac{\sigma}{4} \|v(\cdot,t)\|^2_{w,r+3/2} \leq \frac{32}{3\sigma} \|f(\cdot,t)\|^2_{w,r-3/2}.$$

Integration gives the desired energy inequality. Noting that this is true for any t, and using the definition of $\| \cdot \|_{H^r_\sigma}$, we obtain the given bounds on $\|v\|_{H^r_\sigma}$. □

Remark. Proof of existence of a solution to the linear equation (4.34) for given real valued $f \in H^{r-3}_\sigma$ and the initial condition $v_0 \in \dot{H}^r$, satisfying the given conditions follows in a standard manner. Note that we can introduce a sequence of Galerkin approximants $v_N(\alpha,t)$ containing a finite number of Fourier modes. This will satisfy the energy bounds in Lemma 4.14, independent of N. These approximants clearly solve linear ODEs for which the unique solutions exist globally. In the Hilbert space $L^2\left([0,S], H^{r+3/2}\right)$, there exists a subsequence of $v_N \to v$ weakly. Therefore for almost all $t \in [0,S]$, this subsequence denoted again by $v_N(\cdot,t) \to v(\cdot,t)$ strongly in $\dot{H}^r$. From the energy bound, the limit $v(\cdot,t)$ is bounded in $\dot{H}^r$ for any $t \in [0,S]$, and $v \in L^2([0,S],H^{r+3/2})$. It is also easy to check that the limiting solution satisfies (4.34) in a classical sense for sufficiently large r. This proves existence of a global classical solution for any t noting that $r \geq 3$ since r is arbitrary. The uniqueness of this solution follows from the energy bound itself. □

Definition 4.15. *It is convenient to define a linear operator $e^{t\mathcal{A}}$ so that*

$$v = e^{t\mathcal{A}} v_0$$

is the unique solution $v(\alpha,t) \in \dot{H}^r$ satisfying (4.34) for $f=0$, with the initial condition $v(\alpha,0)=v_0$.

Note 4.16. *It is easily seen that $e^{t\mathcal{A}}$ is a semi-group. Further, using Duhammel principle, the solution $v(\alpha,t) \in \dot{H}^r$ satisfying (4.34) for $v_0=0$ may be expressed as*

$$v(\alpha,t) = \int_0^t e^{(t-\tau)\mathcal{A}} f(\alpha,\tau) d\tau. \tag{4.36}$$

Remark. The energy bounds in Lemma 4.14 imply that

$$\|e^{t\mathcal{A}} v_0\|_{H^r_\sigma} \leq \|v_0\|_{w,r}, \quad \left\| \int_0^t e^{(t-\tau)\mathcal{A}} f(\cdot,\tau) d\tau \right\|_{H^r_\sigma} \leq \frac{8\sqrt{2}}{\sqrt{3}\sigma} \|f\|_{H^{r-3}_\sigma}. \tag{4.37}$$

□

4.4. **Nonlinear evolution, contraction map and proof of Proposition 4.2.** We express the evolution equation (4.14) in the integral form:

$$\tilde{\theta}(\alpha,t) = e^{t\mathcal{A}}\tilde{\theta}_0 + \int_0^t d\tau e^{(t-\tau)\mathcal{A}} \left\{ \mathfrak{N}[\tilde{\theta}(\cdot,\tau),\hat{\theta}(0;\tau)] + \mathcal{A}_N[\tilde{\theta}(\cdot,\tau),\hat{\theta}(0;\tau),L(\tau)] \right\} \equiv \mathcal{S}_1\left[\tilde{\theta},\hat{\theta}(0;\cdot)\right](\alpha,t), \tag{4.38}$$

$$\hat{\theta}(0;t) = \int_0^t \mathfrak{N}_0 \Big[\mathcal{S}_1(\alpha,\cdot),\hat{\theta}(0;\cdot)\Big](\tau) d\tau \equiv \mathcal{S}_2[\tilde{\theta},\hat{\theta}(0;\cdot)](t), \tag{4.39}$$

where $L=L(t)$ is determined in terms of $\tilde{\theta}(.,t)$ through (B.4) and (2.1).

Equations (4.38) and (4.39) will be the basis of a contraction mapping theorem for $(\tilde{\theta},\hat{\theta}(0;t))$ in an small ball in the space

$$\mathcal{D} \equiv H^r_\sigma \times \mathbf{C}[0,\infty)$$

equipped with the norm $\|\cdot\|_{\mathcal{D}}$ so that

$$\left\| \left(\tilde{\theta},\hat{\theta}(0;\cdot)\right) \right\|_{\mathcal{D}} = \|\tilde{\theta}\|_{H^r_\sigma} + \left|\hat{\theta}(0;\cdot)\right|_\infty. \tag{4.40}$$

First, we define a mapping in $\mathcal{D}$ by

$$\mathcal{S}\left[\tilde{\theta},\hat{\theta}(0;\cdot)\right] \equiv \begin{pmatrix} \mathcal{S}_1\left[\tilde{\theta},\hat{\theta}(0;\cdot)\right](\alpha,t) \\ \mathcal{S}_2\left[\tilde{\theta},\hat{\theta}(0;\cdot)\right](t) \end{pmatrix}.$$

Secondly, we estimate the nonlinear terms in the space H^r_σ.

Lemma 4.17. *For $r \geq 3$ and $\sigma > 0$, assume $\left(\tilde{\theta}(\alpha,t),\hat{\theta}(0;t)\right)$ satisfy the condition*

$$\|\tilde{\theta}\|_{H^r_\sigma} \leq \epsilon \ , \ |\hat{\theta}(0;\cdot)|_\infty \leq \epsilon. \tag{4.41}$$

Then for $\mathcal{A}_N[\tilde{\theta},\hat{\theta}(0;\cdot),L(\cdot)](\alpha,t)$, $\mathfrak{N}[\tilde{\theta},\hat{\theta}(0;\cdot)](\alpha,t)$ *and scalar* $\mathfrak{N}_0[\tilde{\theta},\hat{\theta}(0;\cdot)](t)$, *determined from (4.16), (4.18) and (4.22) respectively,*[11]

$$\big\|\mathcal{A}_N[\tilde{\theta},\hat{\theta}(0;\cdot),L]+\mathfrak{N}[\tilde{\theta},\hat{\theta}(0;\cdot)]\big\|_{H_\sigma^{r-3}} \le c_1\|\tilde{\theta}\|_{H_\sigma^r}\left(\|\tilde{\theta}\|_{H_\sigma^r}+|\hat{\theta}(0;\cdot)|_\infty\right),$$

$$\Big|\int_0^t \mathfrak{N}_0[\tilde{\theta},\hat{\theta}(0;\cdot)](\tau)d\tau\Big|_\infty \le c_2\|\tilde{\theta}\|_{H_\sigma^3}.$$

Further, if both $\left(\tilde{\theta}^{(1)}(\alpha,t),\hat{\theta}^{(1)}(0;t)\right)$ *and* $\left(\tilde{\theta}^{(2)}(\alpha,t),\hat{\theta}^{(2)}(0;t)\right)$ *satisfy (4.41), then the corresponding* $\left(\mathcal{A}_N^{(1)},\mathfrak{N}^{(1)},\mathfrak{N}_0^{(1)}\right)$ *and* $\left(\mathcal{A}_N^{(2)},\mathfrak{N}^{(2)},\mathfrak{N}_0^{(2)}\right)$ *satisfy*

$$\|\mathcal{A}_N^{(1)}-\mathcal{A}_N^{(2)}\|_{H_\sigma^{r-3}}+\|\mathfrak{N}^{(1)}-\mathfrak{N}^{(2)}\|_{H_\sigma^{r-3}} \le c_3\epsilon\left(\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_{H_\sigma^r}+\big|\hat{\theta}^{(1)}(0;\cdot)-\hat{\theta}^{(2)}(0;\cdot)\big|_\infty\right),$$

$$\Big|\int_0^t \left(\mathfrak{N}_0^{(1)} - \mathfrak{N}_0^{(2)}\right)d\tau\Big|_\infty \le c_4\left(\|\tilde{\theta}^{(1)}-\tilde{\theta}^{(2)}\|_{H_\sigma^3}+\epsilon\big|\hat{\theta}^{(1)}(0;\cdot)-\hat{\theta}^{(2)}(0;\cdot)\big|_\infty\right).$$

Proof. We note the bounds for $\mathcal{A}_N$, $\mathfrak{N}$ and $\mathfrak{N}_0$ in Lemma 4.12 and Corollary 4.8. It follows from the equivalence of $\|\cdot\|_r$ and $\|\cdot\|_{w,r}$ norms and the definition of $\|.\|_{H_\sigma^r}$ norm that

$$\begin{aligned}
&e^{\sigma t/2}\big\|\mathcal{A}_N[\tilde{\theta},\hat{\theta}(0;\cdot),L]+\mathfrak{N}[\tilde{\theta},\hat{\theta}(0;\cdot)]\big\|_{w,r-3}\\
&\quad\le Ce^{\sigma t/2}\big(\|\tilde{\theta}\|_{w,r}\|\tilde{\theta}\|_1+\|\tilde{\theta}\|_{w,r-1}\|\tilde{\theta}\|_{w,r-2}+|\hat{\theta}(0;\cdot)|_\infty\|\tilde{\theta}\|_{w,r-2}\big)\\
&\quad\le C\|\tilde{\theta}\|_{H_\sigma^r}\left(\|\tilde{\theta}\|_{H_\sigma^r}+|\hat{\theta}(0;\cdot)|_\infty\right).
\end{aligned}$$

Further, it follows that

$$\begin{aligned}
&\int_0^\infty e^{\sigma t}\|\mathcal{A}_N[\tilde{\theta},\hat{\theta}(0;t),L]+\mathfrak{N}[\tilde{\theta}(\cdot,t),\hat{\theta}(0;t)]\|^2_{w,r-3/2}dt\\
&\quad\le C\sup_t\left[e^{\sigma t}\|\tilde{\theta}(\cdot,t)\|^2_{w,r}+|\hat{\theta}(0;t)|^2\right]\int_0^\infty e^{\sigma t}\|\tilde{\theta}\|^2_{w,r+3/2}dt\\
&\quad\le C\|\tilde{\theta}\|^2_{H_\sigma^r}\left(\|\tilde{\theta}\|^2_{H_\sigma^r}+|\hat{\theta}(0;\cdot)|^2_\infty\right).
\end{aligned}$$

Therefore the bounds for $\|\mathcal{A}_N+\mathfrak{N}\|_{H_\sigma^{r-3}}$ follows. For $\mathfrak{N}_0$, we use Corollary 4.8 again to note

$$\Big|\int_0^t \mathfrak{N}_0[\tilde{\theta},\hat{\theta}(0;.)](\tau)d\tau\Big|_\infty \le C\int_0^\infty \big(\|\tilde{\theta}(\cdot,\tau)\|^2_{w,3}+\|\tilde{\theta}(\cdot,\tau)\|_{w,2}\big)d\tau \le c_2\|\tilde{\theta}\|_{H_\sigma^3}.$$

The statements for the differences of $\mathfrak{N}$, $\mathfrak{N}_0$ for different $(\tilde{\theta},\hat{\theta}(0;t))$ follow from parallel statements in Lemma 4.12 and Corollary 4.8. □

We have the following contraction properties in a ball

$$\mathcal{V}_\epsilon \equiv \big\{(u,v)\in\mathcal{D}\big|\|u\|_{H_\sigma^r}\le\epsilon, |v|_\infty\le\epsilon\big\}.$$

[11]Note that Γ and L appearing in the expressions are determined in terms of $\hat{\theta}$ and $\hat{\theta}(0;t)$ through (4.7) and (B.4) on using (2.1) and (2.2).

Lemma 4.18. *Let $\sigma > 0$, $r \geq 3$. Assume $\Big(\tilde{\theta}, \hat{\theta}(0;t)\Big) \in \mathcal{V}_\epsilon$ and c_1, c_2, c_3, c_4 are as defined in Lemma 4.17. If for sufficiently small ϵ, $\|\tilde{\theta}_0\|_{w,r} < \min\{\frac{\epsilon}{2}, \frac{\epsilon}{2c_1}\}$ and $\hat{\theta}(0;0) = 0$, then*

$$\mathcal{S}\big[\tilde{\theta}, \hat{\theta}(0;\cdot)\big] \in \mathcal{V}_\epsilon.$$

Further, if each of $\Big(\tilde{\theta}^{(1)}, \hat{\theta}^{(1)}(0;t)\Big)$ and $\Big(\tilde{\theta}^{(2)}, \hat{\theta}^{(2)}(0;t)\Big)$ belongs to $\mathcal{V}_\epsilon$, then there exists c_5 depending on $c_1, \cdots, c_4$, such that

$$\Big\|\mathcal{S}\big[\tilde{\theta}^{(1)}, \hat{\theta}^{(1)}(0;\cdot)\big] - \mathcal{S}\big[\tilde{\theta}^{(2)}, \hat{\theta}^{(2)}(0;\cdot)\big]\Big\|_{\mathcal{D}} \leq c_5 \epsilon \Big\|\big(\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}, \hat{\theta}^{(1)}(0;\cdot) - \hat{\theta}^{(2)}(0;\cdot)\big)\Big\|_{\mathcal{D}}.$$

Proof. Define $c_6 = \frac{2\sqrt{2}}{\sqrt{3\sigma}} c_1$. By (4.37) and Lemma 4.17, we have

$$\Big\|\mathcal{S}_1\big[\tilde{\theta}, \hat{\theta}(0;\cdot)\big]\Big\|_{H^r_\sigma} \leq \|\tilde{\theta}_0\|_{w,r} + c_6\big(\|\tilde{\theta}\|^2_{H^r_\sigma} + \|\tilde{\theta}\|_{H^r_\sigma} |\hat{\theta}(0;\cdot)|_\infty\big) \leq \epsilon,$$

if $c_6 \epsilon < \frac{1}{4}$. We also have

$$\Big|\mathcal{S}_2\big[\tilde{\theta}, \hat{\theta}(0;\cdot)\big]\Big|_\infty \leq c_2 \Big\|\mathcal{S}_1\big[\tilde{\theta}, \hat{\theta}(0;\cdot)\big]\Big\|_{H^3_\sigma} \leq c_2\Big(\|\tilde{\theta}_0\|_{w,r} + c_6\|\tilde{\theta}\|^2_{H^r_\sigma} + c_6\|\tilde{\theta}\|_{H^r_\sigma}|\hat{\theta}(0;\cdot)|_\infty\Big) \leq \epsilon,$$

if $c_2 c_6 \epsilon < \frac{1}{4}$.

The statements for the differences of $\mathcal{S}$, for different $(\tilde{\theta}, \hat{\theta}(0;t))$ follows from parallel statements in Lemma 4.17.

□

Note 4.19. *Constants c_1, c_2, c_3, c_4 and c_5 depend on σ.*

Proof of Proposition 4.2: If $c_5 \epsilon < 1$, then it is clear that the right sides of (4.38) and (4.39) define a contraction map in a small ball $\mathcal{V}_\epsilon$ in the Banach space $\mathcal{D}$. Therefore, there exists a unique solution $\big(\tilde{\theta}, \hat{\theta}(0;t)\big)$ satisfying equations (4.38) and (4.39), hence (B.1). The local uniqueness of solutions (see Appendix §7.2) implies that this is the only solution. The $e^{-\sigma t/2}$ exponential decay of $\tilde{\theta}$ and hence of θ implies that the steady circle is approached exponentially in time. The constraint condition (B.4) implies that $L - 2\pi$ decays exponentially.

Note 4.20. *It is easy to show that given any j, $\tilde{\theta}(\cdot, t) \in \dot{H}^{r+3j/2}$ for $t \geq j$ in the following manner. Since $(\tilde{\theta}, \hat{\theta}(0;t)) \in \mathcal{V}_\epsilon$ for some $r \geq 3$, there exists $t_0 \in [0,1]$ such that $\|\tilde{\theta}(\cdot, t_0)\|_{r+3/2} < \epsilon$. So we can restart clock at $t = t_0$ and prove global solution in $H^{r+3/2}_\sigma$. It follows that there exists $t_1 \in (t_0, t_0+1]$ so that $\|\tilde{\theta}(\cdot, t_1)\|_{r+3} < \epsilon$. By bootstrapping, we obtain $\tilde{\theta}(\cdot, t) \in \dot{H}^{r+3j/2}$ for $t \geq j$.*

Indeed more can be shown to be true. The contraction argument in Proposition 4.2 can be carried out for arbitrary sized initial condition over small sized time interval. Through bootstrapping and using Sobolev embedding theorem, we can conclude that the solution is in C^∞ in space for $t \in (0, S]$. The property of smoothing of initial conditions is similar to other dissipative equations like Navier-Stokes.

5. Steady translating bubble in the channel with sidewalls ($\beta > 0$)

For a steadily traveling bubble solution, in the frame of an appropriately moving bubble, we have to require the normal interface speed $U = 0$. This would imply (A.1) is automatically satisfied for a time-independent $\theta^{(s)}(\alpha)$ and $L = L^{(s)} =$

2π, where $z(\alpha) = z^{(s)}(\alpha)$ describes the geometry shape of the steady bubble and $\gamma(\alpha, t) = \gamma^{(s)}(\alpha)$ is determined in terms of θ through (A.2).

Earlier, we have shown that for the bubble with the invariant area,

$$\int_0^{2\pi} U(\alpha) d\alpha = 0. \tag{5.1}$$

Further, there is no loss of generality in the steady problem to choose $\hat{\theta}^{(s)}(0) = 0$ since this corresponds to a choice of origin for α, and make $\alpha = 0$ correspond to $y^{(s)}(0) = 0$. Thus, from (1.6), the steady bubble problem reduces to

(C.1)
$$\mathfrak{U}\left[\tilde{\theta}^{(s)}, u_0, \beta\right] \equiv \mathcal{Q}_0\Big(\frac{1}{2}\mathcal{H}(\gamma^{(s)}) + \frac{1}{2}\operatorname{Re}\left(\mathcal{G}[z^{(s)}]\gamma^{(s)}\right) + (u_0+1)\cos\left(\alpha + \theta^{(s)}(\alpha)\right)\Big) = 0,$$

with vortex sheet strength $\gamma^{(s)}$ and $\hat{\theta}^{(s)}(\pm 1)$ determined by

$$\Big(I + a_\mu \mathcal{F}[z^{(s)}]\Big)\gamma^{(s)} = 2\big(1 + \frac{\mu_2}{\mu_1+\mu_2}u_0\big)\sin\left(\alpha + \theta^{(s)}(\alpha)\right) + \sigma\theta^{(s)}_{\alpha\alpha}, \tag{C.2}$$

$$\int_0^{2\pi} \exp\Big(i\alpha + i\big(\hat{\theta}^{(s)}(-1)e^{-i\alpha} + \hat{\theta}^{(s)}(1)e^{i\alpha} + \tilde{\theta}^{(s)}(\alpha)\big)\Big) d\alpha = 0, \tag{C.3}$$

where $\theta^{(s)} = \tilde{\theta}^{(s)} + \hat{\theta}^{(s)}(1)e^{i\alpha} + \hat{\theta}^{(s)}(-1)e^{-i\alpha}$. Hence we seek solutions $(\tilde{\theta}^{(s)}, u_0, \beta) \in \dot{H}^r \times (-1, 1) \times (-\Upsilon, \Upsilon)$ [12].

Recently, [45], [46] and [47] obtained selection results for steady finger for small non-zero surface tension.

Remark. For $r \geq 3$, by Propositions 2.4 and 3.24, we know that $\|\mathfrak{U}\|_{r-2} \leq C$ with C depending on Υ and the diameter of $\mathcal{B}^r_\epsilon$. Hence, $\mathfrak{U}$ maps an open set of $H^r_p \times \mathbb{R}^2$ into the space H^{r-2}_p. □

Note 5.1. *We know that* $\mathfrak{U}[0, 0, 0](\alpha) = 0$ *with the corresponding vortex sheet strength* $\gamma^{(s)}(\alpha) = 2\sin\alpha$ *and* $\hat{\theta}^{(s)}(\pm 1) = 0$. *We also see that* $\frac{\partial \mathfrak{U}}{\partial u_0}[0, 0, 0](\alpha) = \frac{\mu_1}{\mu_1+\mu_2}\cos\alpha$ *and the Fréchet derivative* $\mathfrak{U}_{\tilde{\theta}^{(s)}}[0, 0, 0]h$ *(see Appendix § 7.3) for* $h \in \dot{H}^r$ *is given by:*

$$\mathfrak{U}_{\tilde{\theta}^{(s)}}[0, 0, 0]h = \frac{\sigma}{2}\mathcal{H}(h_{\alpha\alpha}) - i\sum_{k=1}^{\infty}(1 + a_\mu)\frac{k+1}{k+2}\hat{h}(k+1)e^{ik\alpha} + c.c.. \tag{5.2}$$

It is convenient to recast the steady state problem in a contraction mapping problem using smallness of β and the knowledge that $\left(\tilde{\theta}^{(s)}, \gamma^{(s)}\right) = (0, 2\sin\alpha)$ is the steady state solution for $\beta = 0$. We rewrite $\mathfrak{U} = 0$ as

$$\mathfrak{U}_{\tilde{\theta}^{(s)}}[0, 0, 0]\tilde{\theta}^{(s)} + \mathfrak{U}_{u_0}[0, 0, 0]u_0 + \frac{\beta^2}{2}\mathfrak{U}_{\beta\beta}[0, 0, 0] = \mathfrak{N}^{(s)}\left[\tilde{\theta}^{(s)}, u_0, \beta\right], \tag{5.3}$$

where

(5.4)
$$\begin{aligned}\mathfrak{N}^{(s)}\left[\tilde{\theta}^{(s)}, u_0, \beta\right] &= -\mathfrak{U}[\tilde{\theta}^{(s)}, u_0, \beta] + \mathfrak{U}_{\tilde{\theta}^{(s)}}[0, 0, 0]\tilde{\theta}^{(s)} + \mathfrak{U}_{u_0}[0, 0, 0]u_0 + \frac{\beta^2}{2}\mathfrak{U}_{\beta\beta}[0, 0, 0] \\ &= \mathfrak{A}[\tilde{\theta}^{(s)}](\alpha) + \mathfrak{B}[\tilde{\theta}^{(s)}, u_0] + \mathfrak{C}[\tilde{\theta}^{(s)}, u_0, \beta]\end{aligned}$$

[12] We choose small ϵ and Υ such that Proposition 2.4 can be applied in (C.3) and Proposition 3.24 can also be applied in (C.2).

with

$$\mathfrak{A}[\tilde{\theta}^{(s)}](\alpha) = \mathfrak{U}\big[\tilde{\theta}^{(s)}, 0, 0\big](\alpha) - \mathfrak{U}_{\tilde{\theta}^{(s)}}[0,0,0]\tilde{\theta}^{(s)}(\alpha),$$

$$\mathfrak{B}[\tilde{\theta}^{(s)}, u_0] = \mathfrak{U}\big[\tilde{\theta}^{(s)}, u_0, 0\big] - \mathfrak{U}\big[\tilde{\theta}^{(s)}, 0, 0\big] - \mathfrak{U}_{u_0}[0,0,0]u_0,$$

$$\mathfrak{C}[\tilde{\theta}^{(s)}, u_0, \beta] = \mathfrak{U}\big[\tilde{\theta}^{(s)}, u_0, \beta\big] - \mathfrak{U}\big[\tilde{\theta}^{(s)}, u_0, 0\big] - \frac{\beta^2}{2}\mathfrak{U}_{\beta\beta}[0,0,0].$$

It will be shown that $\mathfrak{N}^{(s)}$ is either nonlinear in $\Big(\tilde{\theta}^{(s)}, u_0\Big)$ or at least $O(\beta^4)$.

Lemma 5.2. *For any $r \geq 3$, let $\|\tilde{\theta}^{(s)}\|_r$ and u_0 sufficiently small, then there exists C independent of u_0 and $\tilde{\theta}^{(s)}$ so that*

$$\big\|\mathfrak{A}[\tilde{\theta}^{(s)}]\big\|_{r-1} \leq C\|\tilde{\theta}^{(s)}\|_{r-1}\|\tilde{\theta}^{(s)}\|_r.$$

Further, let $\mathfrak{A}^{(1)}$ and $\mathfrak{A}^{(2)}$ correspond to $\tilde{\theta}_1^{(s)}$ and $\tilde{\theta}_2^{(s)}$ respectively, each in $\dot{H}^r$. Then there exists C independent of β, u_0 and $\tilde{\theta}^{(s)}$ so that

$$\big\|\mathfrak{A}^{(1)} - \mathfrak{A}^{(2)}\big\|_{r-1} \leq C\big(\|\tilde{\theta}_1^{(s)}\|_{r-1}\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r + \|\tilde{\theta}_1^{(s)}\|_r\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_{r-1}\big).$$

Proof. We identify $\mathfrak{A}[\tilde{\theta}^{(s)}]$ as the nonlinear part of normal velocity U for $\beta = 0$ in 4.12). By Lemma 4.7, the statements of the Lemma follow. □

Lemma 5.3. *For any $r \geq 3$, let $\|\tilde{\theta}^{(s)}\|_r$ and u_0 sufficiently small, then there exists C independent of u_0 and $\tilde{\theta}^{(s)}$ so that*

$$\big\|\mathfrak{B}[\tilde{\theta}^{(s)}, u_0]\big\|_r \leq C|u_0|\|\tilde{\theta}^{(s)}\|_r.$$

Further, let $\mathfrak{B}^{(1)}$ and $\mathfrak{B}^{(2)}$ correspond to $(\tilde{\theta}_1^{(s)}, u_0^{(1)})$ and $(\tilde{\theta}_2^{(s)}, u_0^{(2)})$ respectively, each in $\dot{H}^r$. Then there exists C independent of β, u_0 and $\tilde{\theta}^{(s)}$ so that

$$\big\|\mathfrak{B}^{(1)} - \mathfrak{B}^{(2)}\big\|_r \leq C\big(|u_0^{(1)}|\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r + \|\tilde{\theta}_1^{(s)}\|_r|u_0^{(1)} - u_0^{(2)}|\big).$$

Proof. Let $\gamma^{(u_0)}$ correspond to $(\tilde{\theta}^{(s)}, u_0, 0)$, while $\gamma^{(u_0)}$ corresponds to $(\tilde{\theta}^{(s)}, 0, 0)$. Then by (1.6), we obtain

$$\mathfrak{B}[\tilde{\theta}^{(s)}, u_0] = \frac{1}{2}\mathcal{H}[\gamma^{(u_0)} - \gamma_0^{(u_0)}] + \frac{1}{2}\operatorname{Re}\big(\mathcal{G}[z^{(s)}](\gamma^{(u_0)} - \gamma_0^{(u_0)})\big) + u_0\left(\cos\big(\alpha + \tilde{\theta}^{(s)}(\alpha)\big) - \frac{\mu_1}{\mu_1+\mu_2}\cos\alpha\right). \tag{5.5}$$

For (C.2) and the relation between $\mathcal{F}$ and $\mathcal{G}$, we also have

$$\gamma^{(u_0)} - \gamma_0^{(u_0)} = -a_\mu\operatorname{Re}\big(\frac{1}{i}\mathcal{G}[z^{(s)}](\gamma^{(u_0)} - \gamma_0^{(u_0)}) - \frac{1}{i}\mathcal{G}[\omega_0](\gamma^{(u_0)} - \gamma_0^{(u_0)})\big) + 2u_0\frac{\mu_2}{\mu_1+\mu_2}\sin\big(\alpha + \tilde{\theta}^{(s)}\big). \tag{5.6}$$

By Lemma 3.23 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$), from (5.6), we have

$$\|\gamma^{(u_0)} - \gamma_0^{(u_0)}\|_1 \leq C(\|\tilde{\theta}^{(s)}\|_r\|\gamma^{(u_0)} - \gamma_0^{(u_0)}\|_1 + |u_0|\big).$$

Hence for sufficient small $\|\tilde{\theta}^{(s)}\|_r$, we have

$$\|\gamma^{(u_0)} - \gamma_0^{(u_0)}\|_1 \leq C|u_0|. \tag{5.7}$$

Plugging (5.6) into (5.5), we have

$$\begin{aligned}\mathfrak{B}[\tilde{\theta}^{(s)}, u_0] = &\frac{1}{2}\mathcal{H}\left[a_\mu \operatorname{Re}\big(\frac{1}{i}\mathcal{G}[z^{(s)}](\gamma^{(u_0)} - \gamma_0^{(u_0)}) - \frac{1}{i}\mathcal{G}[\omega_0](\gamma^{(u_0)} - \gamma_0^{(u_0)})\big)\right]\\ &+ \frac{1}{2}\operatorname{Re}\big(\mathcal{G}[z^{(s)}](\gamma^{(u_0)} - \gamma_0^{(u_0)}) - \mathcal{G}[\omega_0](\gamma^{(u_0)} - \gamma_0^{(u_0)})\big)\\ &+ u_0\left(\Big(\cos\big(\alpha + \tilde{\theta}^{(s)}(\alpha)\big) - \cos\alpha\Big) + \frac{\mu_2}{\mu_1+\mu_2}\mathcal{H}\Big[\sin\big(\eta + \tilde{\theta}^{(s)}\big) - \sin(\eta)\Big](\alpha)\right).\end{aligned}$$

Hence, by Lemmas 3.5, 3.23 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$) and (5.7), we have the first statement.

For the difference term, by Lemmas 3.5, 3.23 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$) and Proposition 3.24. ☐

Lemma 5.4. *For any $r \geq 3$, assume $\|\tilde{\theta}^{(s)}\|_r$, u_0 and β are sufficiently small. Then there exists C independent of β, u_0 and $\tilde{\theta}^{(s)}$ so that*

$$\big\|\mathfrak{C}[\tilde{\theta}^{(s)}, u_0]\big\|_r \leq C\big(\beta^2|u_0| + \beta^2\|\tilde{\theta}^{(s)}\|_r + \beta^4\big).$$

Further, suppose $\mathfrak{C}^{(1)}$ and $\mathfrak{C}^{(2)}$ correspond to $(\tilde{\theta}_1^{(s)}, u_0^{(1)}, \beta)$ and $(\tilde{\theta}_2^{(s)}, u_0^{(2)}, \beta)$ respectively, each in $\dot{H}^r$. Then there exists C independent of β, u_0 and $\tilde{\theta}^{(s)}$ so that

$$\big\|\mathfrak{C}^{(1)} - \mathfrak{C}^{(2)}\big\|_r \leq C\beta^2\big(\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r + |u_0^{(1)} - u_0^{(2)}|\big).$$

Proof. Suppose $\gamma_0^{(s)}$ satisfying (C.2) corresponds to $(\tilde{\theta}^{(s)}, u_0, 0)$. Then for (1.6),

(5.8)

$$\begin{aligned}\mathfrak{C}[\tilde{\theta}^{(s)}, u_0, \beta] = &\frac{1}{2}\mathcal{H}[\gamma^{(s)} - \gamma_0^{(s)}] + \frac{1}{2}\operatorname{Re}\big(\mathcal{G}_1[z^{(s)}](\gamma^{(s)} - \gamma_0^{(s)})\big) + \frac{1}{2}\operatorname{Re}\big(\mathcal{G}_2[z^{(s)}]\gamma^{(s)}\big) - \frac{\beta^2}{6}(1+a_\mu)\cos\alpha\\ = &\frac{1}{2}\mathcal{H}\left[\gamma^{(s)} - \gamma_0^{(s)} + a_\mu\frac{\beta^2}{12}\sin\eta\right](\alpha) + \frac{1}{2}\operatorname{Re}\big(\mathcal{G}[z^{(s)}](\gamma^{(s)} - \gamma_0^{(s)}) - \mathcal{G}_1[\omega_0](\gamma^{(s)} - \gamma_0^{(s)})\big)\\ &+ \frac{1}{2}\operatorname{Re}\big(\mathcal{G}_2[z^{(s)}]\gamma_0^{(s)} - \mathcal{G}_2[i\omega_0]\gamma_0^{(s)}\big) + \frac{1}{2}\operatorname{Re}\big(\mathcal{G}_2[i\omega_0]\big(\gamma_0^{(s)} - 2\sin\eta)(\alpha)\big)\\ &+ \frac{1}{2}\operatorname{Re}\big(\mathcal{G}_2[i\omega_0](2\sin\eta)(\alpha) - \frac{\beta^2}{6}\cos\alpha\big).\end{aligned}$$

For (C.2), we also have

(5.9)

$$\gamma^{(s)} - \gamma_0^{(s)} = -a_\mu \operatorname{Re}\big(\frac{1}{i}\mathcal{G}_1[z^{(s)}](\gamma^{(s)} - \gamma_0^{(s)}) - \frac{1}{i}\mathcal{G}_1[\omega_0](\gamma^{(s)} - \gamma_0^{(s)})\big) - a_\mu \operatorname{Re}\big(\frac{1}{i}\mathcal{G}_2[z^{(s)}]\gamma^{(s)}\big).$$

Proposition 3.24 gives us

$$\|\gamma^{(s)}\|_1 \leq C, \ \|\gamma_0^{(s)}\|_1 \leq C.$$

By Lemma 3.23 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$), Note 3.18 and (5.9), we have

$$\|\gamma^{(s)} - \gamma_0^{(s)}\|_r \leq C(\|\tilde{\theta}^{(s)}\|_r\|\gamma^{(s)} - \gamma_0^{(s)}\|_1 + \beta^2).$$

Hence for sufficient small $\|\tilde{\theta}^{(s)}\|_r$, we have

$$\|\gamma^{(s)} - \gamma_0^{(s)}\|_r \leq C\beta^2. \tag{5.10}$$

(5.9) can be rewritten as

(5.11)
$$\gamma^{(s)} - \gamma_0^{(s)} + a_\mu \frac{\beta^2}{6} \sin\alpha = -a_\mu \operatorname{Re}\big(\frac{1}{i}\mathcal{G}[z^{(s)}](\gamma^{(s)} - \gamma_0^{(s)}) - \frac{1}{i}\mathcal{G}_1[\omega_0](\gamma^{(s)} - \gamma_0^{(s)})\big)$$
$$- a_\mu \operatorname{Re}\big(\frac{1}{i}\mathcal{G}_2[z^{(s)}]\gamma_0^{(s)} - \frac{1}{i}\mathcal{G}_2[i\omega_0]\gamma_0^{(s)}\big) - a_\mu \operatorname{Re}\big(\frac{1}{i}\mathcal{G}_2[i\omega_0]\big(\gamma_0^{(s)} - 2\sin\eta)(\alpha)\big)$$
$$- a_\mu \operatorname{Re}\big(\frac{1}{i}\mathcal{G}_2[i\omega_0](2\sin\eta)(\alpha) - \frac{\beta^2}{6}\sin\alpha\big).$$

We see from (C.2) that

$$\gamma_0^{(s)} - 2\sin\alpha = -a_\mu \operatorname{Re}\big(\frac{1}{i}\mathcal{G}_1[z^{(s)}]\gamma_0^{(s)} - \frac{1}{i}\mathcal{G}_1[\omega_0]\gamma_0^{(s)}\big)$$
$$+ 2\Big(\sin(\alpha + \tilde{\theta}^{(s)}) - 2\sin\alpha\Big) + 2u_0 \frac{\mu_2}{\mu_1 + \mu_2}\sin(\alpha + \tilde{\theta}^{(s)}) + \sigma\tilde{\theta}^{(s)}_{\alpha\alpha}.$$

Hence by Lemmas 3.5 and 3.23 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$), we have from above

$$\|\gamma_0^{(s)} - 2\sin(\cdot)\|_1 \le C\big(\|\tilde{\theta}^{(s)}\|_r + |u_0|\big). \tag{5.12}$$

We know the first derivative of $\mathcal{G}_2[i\omega_0](2\sin\eta)(\alpha)$ with respect to β at $\beta = 0$ is equal to 0. On calculation,

$$\Big(\mathcal{G}_2[i\omega_0](2\sin\eta)(\alpha)\Big)_{\beta\beta}\Big|_{\beta=0} = \frac{e^{i\alpha}}{3}.$$

Hence for sufficiently small β, by Taylor expansion, we have

$$\Big\|\mathcal{G}_2[i\omega_0](2\sin\eta)(\alpha) - \frac{\beta^2}{6}e^{i\alpha}\Big\|_r \le C\beta^4. \tag{5.13}$$

By Lemmas 3.20, 3.23 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$), Note 3.18, (5.12) and (5.13), from (5.11) we get

$$\Big\|\gamma^{(s)} - \gamma_0^{(s)} + a_\mu\frac{\beta^2}{6}\sin(\cdot)\Big\|_r \le C\big(\beta^2\|\tilde{\theta}^{(s)}\|_r + \beta^2 u_0 + \beta^4\big). \tag{5.14}$$

Hence, by Lemma 3.23, (5.12), (5.13) and (5.14), the first statement is obtained.
The proof for the second statement follows similarly. □

Hence we have

Lemma 5.5. *For any $r \ge 3$, assume $\|\tilde{\theta}^{(s)}\|_r$, u_0 and β are sufficiently small. Then there exists C independent of β, u_0 and $\tilde{\theta}$ so that*

$$\|\mathfrak{N}^{(s)}\|_{r-1} \le C\left[|u_0|\|\tilde{\theta}^{(s)}\|_r + |u_0|\beta^2 + \beta^4 + \beta^2\|\tilde{\theta}^{(s)}\|_r + \|\tilde{\theta}^{(s)}\|_r\|\tilde{\theta}^{(s)}\|_{r-1}\right]. \tag{5.15}$$

Further, suppose $\mathfrak{N}_1^{(s)}$ and $\mathfrak{N}_2^{(s)}$ correspond to $(\tilde{\theta}_1^{(s)}, u_0^{(1)}, \beta)$ and $(\tilde{\theta}_2^{(s)}, u_0^{(2)}, \beta)$ respectively, each in $\dot{H}^r$. Then there exists C independent of β, u_0 and $\tilde{\theta}^{(s)}$ so that

$$\big\|\mathfrak{N}_1^{(s)} - \mathfrak{N}_2^{(s)}\big\|_{r-1} \le C\Big(\beta^2\big(\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r + |u_0^{(1)} - u_0^{(2)}|\big) + \|\tilde{\theta}_1^{(s)}\|_{r-1}\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r$$
$$+ \|\tilde{\theta}_1^{(s)}\|_r\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_{r-1} + |u_0^{(1)}|\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r + \|\tilde{\theta}_1^{(s)}\|_r|u_0^{(1)} - u_0^{(2)}|\Big).$$

Proof. Combining Lemmas 5.2, 5.3 and 5.4, the two statements are obtained. □

Definition 5.6. *We define the linear operator A on $u \in \dot{H}^r$ by*

$$Au = -\frac{\sigma}{2}u_{\alpha\alpha} - \sum_{k=2}^{\infty}(1+a_\mu)\frac{k+1}{k+2}\hat{u}(k+1)e^{ik\alpha} - \sum_{k=-2}^{-\infty}(1+a_\mu)\frac{k-1}{k-2}\hat{u}(k-1)e^{ik\alpha}. \tag{5.16}$$

Proposition 5.7. *For $r \geq 3$, the linear operator $A : \dot{H}^r \to \dot{H}^{r-2}$, is invertible. Further, $\|A^{-1}f\|_r \leq C_r\|f\|_{r-2}$, for any $f \in \dot{H}^{r-2}$.*

Proof. For any surface tension σ, there exists the integer $K > 2$ such that $n^2 \geq \frac{8}{\sigma}$ for any $|n| \geq K$. Let us define a family of the spaces $Z_r := \{u \in \dot{H}^r | \mathcal{Q}_K u = u\}$ with $r \geq 0$. We define the linear operator $A_K := \mathcal{Q}_K A$, which maps from Z_r to Z_{r-2}. The corresponding bilinear mapping $E_K : Z_1 \times Z_1 \to \mathbb{R}$ is defined by

$$E_K[u,v] = 2\,\mathrm{Re}\left(\sum_{k=K}^{\infty}\big[\frac{\sigma}{2}k^2\hat{u}(k) - (1+a_\mu)\frac{k+1}{k+2}\hat{u}(k+1)\big]\hat{v}(-k)\right)$$

for any $u, v \in Z_1$.

It is easy to see that there exist $a > 0$ such that

$$\big|E_K[u,v]\big| \leq a\|u\|_1\|v\|_1,$$

and

$$\begin{aligned} E_K[u,u] \quad &\geq \quad \frac{\sigma}{2}\|u\|_1^2 - 3\Big|\sum_{k=K}^{\infty}\hat{u}(k+1)\hat{u}(-k) + \sum_{k=-K}^{-\infty}\hat{u}(k-1)\hat{u}(-k)\Big| \\ &\geq \quad \frac{\sigma}{2}\|u\|_1^2 - 2\|u\|_0^2 \geq \frac{\sigma}{4}\|u\|_1^2. \end{aligned}$$

The last inequality follows from $\frac{\sigma}{4}n^2 \geq 2$, for $|n| \geq K$.

Hence by Lax-Milgram theorem, we see that for any $f \in \dot{H}^{r-2}$, there exists only one $u_K \in Z_1$ such that $E_K[u_K, v] = (\mathcal{Q}_K f, v)_{L^2}$ for any $v \in Z_1$ and so $A_K u_K = \mathcal{Q}_K f$ for some $u_K \in Z_1$. We also have

$$\begin{aligned} \|\mathcal{Q}_K f\|_{r-2}^2 \geq 2\sum_{k=K}^{\infty}\frac{\sigma^2}{4}k^{2r}|\hat{u}_K(k)|^2 - 4\sum_{k=K}^{\infty}k^{2r-2}\frac{(k+1)^2}{(k+2)^2}|\hat{u}_K(k+1)|^2 \\ \geq \frac{\sigma^2}{4}\|u_K\|_r^2 - 2\|u_K\|_{r-2}^2 \geq \frac{\sigma^2}{8}\|u_K\|_r^2, \end{aligned} \tag{5.17}$$

for $\frac{\sigma}{4}n^2 \geq 2$.

Let us consider the linear operator A. It can be written as

$$Au = \sum_{k=2}^{K-1}\big(\frac{\sigma}{2}k^2\hat{u}(k) - (1+a_\mu)\frac{k+1}{k+2}\hat{u}(k+1)\big)e^{ik\alpha} + A_K Q_k u + c.c.$$

for $u \in \dot{H}^r$.

For any $f \in \dot{H}^{r-2}$, there exists only one solution $u_K \in Z_r$ such that $A_K u_K = \mathcal{Q}_K f$. Then using u_K, we consider the following finite linear equation system for

$(b_{K-1}, b_{K-2}..b_2, b_{-2}, \cdots, b_{-K+1})^T$

$$
(5.18)\quad \begin{pmatrix} \frac{\sigma}{2}(K-1)^2 & 0 & 0 & \cdots \\ -\frac{K-1}{K}(1+a_\mu) & \frac{\sigma}{2}(K-2)^2 & 0 & \cdots \\ 0 & -(1+a_\mu)\frac{K-2}{K-1} & \frac{\sigma}{2}(K-3)^2 & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{pmatrix} \begin{pmatrix} b_{K-1} \\ b_{K-2} \\ b_{K-3} \\ . \\ . \\ b_{-K+1} \end{pmatrix}
$$

$$
= \begin{pmatrix} \hat{f}(K-1) + (1+a_\mu)\frac{K}{K+1}\hat{u}_K(K) \\ \hat{f}(K-2) \\ . \\ . \\ \hat{f}(-K+1) + (1+a_\mu)\frac{-K}{-K+1}\hat{u}_K(-K) \end{pmatrix}.
$$

It is easy to from the triangle structure see that there exists only one solution $(b_{K-1}, \cdots, b_2, b_{-2}, \cdots, b_{-K+1})$. Then we choose $u = \sum_{k=2}^{K-1} b_k e^{ik\alpha} + \sum_{k=-2}^{-K+1} b_k e^{ik\alpha} + u_K$ and $Au = f$. Since $u_K \in H_p^r$, we induce $u \in H_p^r$.

Hence, for any $f \in \dot{H}^{r-2}$, there is only one $u = A^{-1}f \in \dot{H}^r$. By (5.17), $\|A^{-1}f\|_r \le C_r \|f\|_{r-2}$. □

Proposition 5.8. *For any surface tension $\sigma > 0$, $r \ge 3$, and sufficiently small ϵ, there exists a neighborhood O of (0,0) and a ball $\mathcal{B}_\epsilon^r \subset \dot{H}^r$ such that $\tilde{\theta}^{(s)} : O \to \mathcal{B}_\epsilon^r$ with $\mathcal{Q}_1\mathfrak{U}\big[\tilde{\theta}^{(s)}(u_0,\beta), u_0, \beta\big] = 0$. Further, $\tilde{\theta}^{(s)}(u_0,\beta;\alpha)$ is odd with respect to α for any $(u_0,\beta) \in O$.*

Proof. We define the operator $\mathcal{T}$ by

$$\mathcal{T}\tilde{\theta}^{(s)} \equiv A^{-1}\mathcal{Q}_1\mathfrak{N}^{(s)}[\tilde{\theta}^{(s)}, u_0, \beta].$$

By Lemma 5.5 and Proposition 5.7, for sufficient small ϵ, there exists a neighborhood O of $(0,0)$, such that the operator $\mathcal{T}$ is the contraction map in the ball $\mathcal{B}_\epsilon^r$ for $(u_0,\beta) \in O$.

Hence, by contraction mapping theorem, there exist open sets $O \subset \mathbb{R}^2$ such that $\tilde{\theta}^{(s)} = \mathcal{T}\tilde{\theta}^{(s)}$ in the ball $\mathcal{B}_\epsilon^r \subset \dot{H}^r$ for $(u_0,\beta) \in O$. By (5.3), we have

$$\mathcal{Q}_1\mathfrak{U}\big[\tilde{\theta}^{(s)}(u_0,\beta;\alpha), u_0, \beta\big] = 0.$$

For any $(u_0,\beta) \in O$, we define $\eta(\alpha) = -\tilde{\theta}^{(s)}(u_0,\beta;-\alpha) - \hat{\theta}^{(s)}(-1)e^{i\alpha} - \hat{\theta}^{(s)}(1)e^{-i\alpha}$, $v(\alpha) = -(z^{(s)}(-\alpha))^*$, and $\xi(\alpha) = -\gamma^{(s)}(-\alpha)$. Then it is easy to check that

$$
\begin{aligned}
&\mathrm{Re}\Big(\frac{z_\alpha^{(s)}(-\alpha)}{2\pi i}\mathrm{PV}\int_0^{2\pi}\gamma^{(s)}(\alpha')K(-\alpha,\alpha')d\alpha'\Big) \\
&= -\mathrm{Re}\Big(\frac{v_\alpha(\alpha)}{2\pi i}\mathrm{PV}\int_0^{2\pi}\xi(\alpha')\Big\{\frac{\beta}{4}\coth\Big[\frac{\beta}{4}\big(v(\alpha)-v(\alpha')\big)\Big]-\frac{\beta}{4}\tanh\Big[\frac{\beta}{4}\big(v(\alpha)-v^*(\alpha')\big)\Big]\Big\}d\alpha'\Big).
\end{aligned}
$$

Hence, $\mathcal{Q}_1\mathfrak{U}[\mathcal{Q}_1\eta(\alpha), u_0, \beta] = \mathcal{Q}_1\mathfrak{U}[\tilde{\theta}^{(s)}(\alpha), u_0, \beta] = 0$ with $\xi(\alpha)$ satisfying (C.2).

Also by uniqueness, we have $\tilde{\theta}^{(s)}(u_0,\beta;\alpha) = \mathcal{Q}_1\eta(\alpha) \equiv -\tilde{\theta}^{(s)}(u_0,\beta;-\alpha)$. □

Note 5.9. *Note that the $\tilde{\theta}^{(s)}$ of Proposition 5.8 is not the steady state since we only required $\mathcal{Q}_1\mathfrak{U} = 0$ instead of $\mathcal{Q}_0\mathfrak{U} = \mathfrak{U} = 0$. Here u_0 is arbitrary. The additional condition $(\mathcal{Q}_0 - \mathcal{Q}_1)\mathfrak{U} = 0$ can be satisfied by constraining u_0 appropriately. The usefulness of this Proposition is to show any steady solution $\theta^{(s)}$ that actually* *satisfies* $\mathcal{Q}_0\mathfrak{U} = 0$ *must be an odd function since this is true for any sufficient small* u_0.

Definition 5.10. *We define a family of Banach spaces* $\{X_r\}_{r\geq 0}$ *by*

$$X_r = \{u \in \dot{H}^r | u(-\alpha) = -u(\alpha)\},$$
$$Y_r = \{u \in H_p^r \big| \mathcal{Q}_0 u = u, u(-\alpha) = u(\alpha)\}.$$

Remark. Proposition 5.8 shows us that the shape of the steady bubble must be symmetric with the center of the channel. Also $\mathfrak{U} : X_r \times \mathbb{R}^2 \to Y_{r-2}$. Hence it is reasonable to consider the solution $(\tilde{\theta}^{(s)}, u_0, \beta)$ to (C.1)-(C.3) in the space $X_r \times \mathbb{R}^2$. □

Proof of Theorem 1.14: Let $f = \mathfrak{N}^{(s)}[\tilde{\theta}^{(s)}, u_0, \beta] - \frac{\beta^2}{2}\mathfrak{U}_{\beta\beta}[0,0,0]$ and $g = A^{-1}\mathcal{H}(\mathcal{Q}_1 f)$. Actually it is easy to check that $f(-\alpha) = f(\alpha)$ and $g(-\alpha) = -g(\alpha)$ for $\tilde{\theta}^{(s)} \in X_r$.

We define an operator $\mathfrak{T}$ in $X_r \times \mathbb{R}$ by

$$\mathfrak{T}[\tilde{\theta}^{(s)}, u_0] = \Big(A^{-1}\mathcal{H}(\mathcal{Q}_1 f), 2\hat{f}(1) + \frac{\mu_1+\mu_2}{\mu_1}\big(\frac{4}{3} + \frac{4}{3}a_\mu\big) i\hat{g}(2)\Big)^T.$$

By Lemma 5.5 and Proposition 5.7, for sufficient small ϵ, there exist an open set $O_1 \subset \mathbb{R}$ and a ball $O_2 \subset X_r \times \mathbb{R}$ such that $\mathfrak{T}$ is the contraction map in the ball O_2 for any $\beta \in O_1$. Hence, by contraction mapping theorem, we have $(\tilde{\theta}^{(s)}, u_0)^T = \mathfrak{T}[\tilde{\theta}^{(s)}, u_0]$ for any $\beta \in O_1$. By (5.3), we have

$$\mathcal{Q}_0\mathfrak{U}\big[\tilde{\theta}^{(s)}(\beta;\alpha), u_0(\beta), \beta\big] = 0.$$

By Lemma 5.5 and Proposition 5.7, for sufficiently small ϵ and Υ, there exists C independent of ϵ and Υ, such that

$$\|\tilde{\theta}^{(s)}\|_r + |u_0| \leq C\beta^2. \tag{5.19}$$

We deduce from (C.2) that

$$\begin{aligned}\gamma^{(s)}(\alpha) =& 2\sin\alpha - a_\mu \operatorname{Re}\Big(\frac{z_\alpha^{(s)}}{\pi i}\mathrm{PV}\int_{\alpha-\pi}^{\alpha+\pi}\frac{\gamma^{(s)}(\alpha')}{z^{(s)}(\alpha) - z^{(s)}(\alpha')}d\alpha' - \frac{e^{i\alpha}}{\pi i}\mathrm{PV}\int_{\alpha-\pi}^{\alpha+\pi}\frac{\gamma^{(s)}(\alpha')}{\int_{\alpha'}^{\alpha} e^{i\zeta}d\zeta}d\alpha'\Big)\\ &- a_\mu \operatorname{Re}\Big(\frac{\omega_{s_\alpha}}{2\pi i}\sum_{n=1}^{\infty}\frac{2B_{2n}}{(2n)!}(-1)^n\beta^{2n}\int_0^{2\pi}\gamma^{(s)}(\alpha')\big(z^{(s)}(\alpha) - z^{(s)}(\alpha')\big)^{2n-1}d\alpha'\Big)\\ &+ 2\Big(\sin\big(\alpha + \theta^{(s)}(\alpha)\big) - \sin\alpha\Big) + \sigma\theta_{\alpha\alpha}^{(s)} + \frac{2\mu_2}{\mu_1+\mu_2}u_0\sin\big(\alpha + \theta^{(s)}(\alpha)\big),\end{aligned}$$

where B_n is nth Bernoulli number. By (5.19) and Lemma 3.23, we have

$$\|\gamma^{(s)} - 2\sin\alpha\|_{r-2} \leq C\beta^2,$$

where C depends on ϵ and Υ.

Remark. Since we consider the steady solution in $\dot{H}^r$ for $r \geq 3$, where r is arbitrary, by uniqueness shown in Theorem 1.14, the steady solution is in H^∞, and hence in C^∞. In fact from ellipticity of A, the solution is expected to be analytic. The result is consistent with analyticity results for arbitrary channel width in the small σ limit [47]. □

Note 5.11. *Actually for $\mu_2 = 0$, by formal expansion in correspondence to earlier calculation using conformal mapping* [36]*, we have*

$$\begin{aligned}
\theta^{(s)}(\alpha) &= \beta^4\Big(\frac{1}{54\sigma}\sin(3\alpha) + \frac{1}{72\sigma^2}\sin(2\alpha)\Big) + O(\beta^6),\\
u_0 &= -\frac{\beta^2}{6} + \beta^4\Big(\frac{7}{180} + \frac{1}{216\sigma^2}\Big) + O(\beta^6),\\
\gamma^{(s)}(\alpha) &= 2\sin\alpha - \frac{\beta^2}{6}\sin\alpha + \beta^4\Big(\Big(-\frac{19}{120} + \frac{1}{72\sigma^2}\Big)\sin(3\alpha) + \Big(\frac{1}{72} + \frac{7}{216\sigma^2}\Big)\sin\alpha\\
&\qquad + \frac{1}{54\sigma}\sin(4\alpha) - \frac{1}{54\sigma}\sin(2\alpha)\Big) + O(\beta^6).
\end{aligned}$$

For steady states, two fluid flows can be related to one fluid flow by transform variables [39]*.*

6. Evolution of symmetric bubble with sidewalls ($\beta > 0$)

Lemma 6.1. *If initial conditions satisfy the symmetry condition*

$$\theta_0(-\alpha) = -\theta_0(\alpha)\ ,\ y_0 = 0,$$

then the corresponding solution $(\theta(\alpha,t), L(t), y(0,t))$ in $H_p^r \times \mathbf{C}^1 \times \mathbf{C}^1$ to (A.1) and (1.8) satisfy symmetry condition for all time, i.e. $\theta(-\alpha,t) = -\theta(\alpha,t)$ and $y(0,t) = 0$. The corresponding vortex sheet strength $\gamma(\alpha,t)$, determined from (A.2) also obeys the symmetry condition $\gamma(-\alpha,t) = -\gamma(\alpha,t)$ and the bubble shape is symmetric about the channel centerline (x-axis).

Proof. If θ_0 is odd and $y(0,0) = 0$, it follows from (2.1) and (2.2), that $z^*(\alpha,0) = z(-\alpha,0)$ and we have a symmetric bubble to start with. The corresponding vortex sheet strength determined from (A.2) $\gamma(\alpha,t)$ is easily to be odd. Again, it is readily checked that that if $(\theta(\alpha,t), \gamma(\alpha,t), L(t), y(0,t))$ solve (A.1)-(A.3) and (1.8), then so does $(-\theta(-\alpha,t), -\gamma(-\alpha,t), L(t), -y(0,t))$. Since the initial condition is symmetric, it follows from local uniqueness of solution (see Appendix §7.2) that symmetry is preserved in time. From the geometric relation

$$z(\alpha,t) = \frac{iL(t)}{2\pi}\int_0^\alpha e^{i\alpha + i\theta(\alpha,t)}d\alpha + z(0,t),$$

symmetry about the x ($Re\ z$) axis follows. □

Remark. Symmetry implies $\hat{\theta}(0;t) = 0 = y(0,t)$. and so the evolution equation for $\hat{\theta}(0;t)$ in (B.1) and $y(0,t)$ in (1.8) can be ignored. For the symmetry bubble, we also have

$$\mathfrak{K}(\alpha,\alpha') = \frac{\beta}{2}\coth\Big[\frac{\beta}{2}\big(z(\alpha) - z(\alpha')\big)\Big].$$

Proposition 5.8 implies that the steady bubble solution $\big(\theta^{(s)}(\alpha), \gamma^{(s)}(\alpha)\big)$ are also odd functions of time. □

6.1. **Main results for the translating bubble in the strip.** In this section, we first state the main results for the translating bubble.

It is convenient to define

Definition 6.2.

$$\Gamma(\alpha,t) = \gamma(\alpha,t) - \gamma^{(s)}(\alpha),$$

$$\theta(\alpha,t) = \widetilde{\Theta}(\alpha,t) + \tilde{\theta}^{(s)}(\alpha) + \hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha}. \tag{6.1}$$

In this section, we will find solutions $\widetilde{\Theta}$ to satisfy (B.1) with initial condition with the initial condition

$$\widetilde{\Theta}(\alpha,0) = \widetilde{\Theta}_0(\alpha) \equiv \mathcal{Q}_1\left[\theta_0 - \theta^{(s)}\right](\alpha). \tag{6.2}$$

We will also consider the motion of the interface with small symmetric perturbation around the steady bubble. Since the bubble area is invariant with time, we take V to be the steady bubble area, i.e..

$$V = \frac{1}{2}\operatorname{Im}\int_0^{2\pi} z_\alpha^{(s)}(z^{(s)})^* d\alpha. \tag{6.3}$$

The main result in this section is the following proposition:

Proposition 6.3. *For $\sigma > 0$, there exist $\epsilon, \Upsilon > 0$ such that for $r \geq 3$, if $\|\widetilde{\Theta}(\cdot,0)\|_r < \epsilon$, $0 < \beta < \Upsilon$, then for initial shape symmetric about channel centerline, i.e. $\widetilde{\Theta}(-\alpha,0) = -\widetilde{\Theta}(\alpha,0)$, there exists a global solution $\widetilde{\Theta} \in \dot{H}_r$ to the Hele-Shaw initial value problem with initial condition (6.2). Furthermore, $\|\mathcal{Q}_0\Theta\|_r$ decays exponentially as $t \to \infty$. Thus the translating steady bubble is asymptotically stable for sufficiently small symmetric initial disturbances in the H_p^r space.*

Note 6.4. *Proposition 6.3 and Lemma 2.7 imply Theorem 1.16.*

6.2. **Evolution equation in integral form.** It is readily checked that $\Gamma(\alpha,t)$ satisfies

(6.4)

$$\begin{aligned}\big(I + a_\mu\mathcal{F}[z]\big)\Gamma = -a_\mu\mathcal{F}[z]\gamma^{(s)} + a_\mu\mathcal{F}[z^{(s)}]\gamma^{(s)} + \frac{2\pi - L}{L}\sigma\theta_{\alpha\alpha} + \sigma(\theta-\theta^{(s)})_{\alpha\alpha} \\ + \frac{L-2\pi}{\pi}\big(1 + \frac{\mu_2}{\mu_1+\mu_2}u_0\big)\sin\big(\alpha+\theta\big) \\ + 2\big(1 + \frac{\mu_2}{\mu_1+\mu_2}u_0\big)\Big(\sin\big(\alpha+\theta\big) - \sin\big(\alpha+\theta^{(s)}\big)\Big).\end{aligned}$$

Hence, we have

Proposition 6.5. *If $\widetilde{\Theta} \in \dot{H}^r$ with $\|\widetilde{\Theta}\|_1 < \epsilon_1$, and $0 \leq \beta < \Upsilon$ then for sufficiently small ϵ_1 and Υ, there exists a unique solution $\Gamma \in \{u \in H_p^{r-2} | \hat{u}(0) = 0\}$ for $r \geq 3$ satisfying (6.4). This solution Γ satisfies the estimates*

$$\begin{aligned}\|\Gamma\|_0 &\leq C\|\widetilde{\Theta}\|_2, \\ \|\Gamma\|_{r-2} &\leq C_1\exp(C_2\|\widetilde{\Theta}\|_{r-2})\|\widetilde{\Theta}\|_r,\end{aligned}$$

where C_1 and C_2 depend on r.

Let $\Gamma^{(1)}$ and $\Gamma^{(2)}$ correspond to $\widetilde{\Theta}^{(1)}$ and $\widetilde{\Theta}^{(2)}$ respectively. Assume $\|\widetilde{\Theta}^{(1)}\|_1 < \epsilon_1$ and $\|\widetilde{\Theta}^{(2)}\|_1 < \epsilon_1$. If $\widetilde{\Theta}^{(1)}, \widetilde{\Theta}^{(2)} \in \dot{H}^r$ with $r \geq 3$, then for sufficient small ϵ_1,

$$\|\Gamma^{(1)} - \Gamma^{(2)}\|_{r-2} \leq C_1\exp\big(C_2(\|\widetilde{\Theta}^{(1)}\|_r + \|\widetilde{\Theta}^{(2)}\|_r)\big)\big\|\widetilde{\Theta}^{(1)} - \widetilde{\Theta}^{(2)}\big\|_r \tag{6.5}$$

where C_1 and C_2 depend on r alone.

Proof. In Proposition 3.24, we take $\gamma^{(2)} = \gamma$, $\tilde{\theta}^{(1)} = \tilde{\theta}$, $L^{(1)} = L$,

$$\gamma^{(2)} = \gamma^{(s)},\ \tilde{\theta}^{(2)} = \tilde{\theta}^{(s)},\ L^{(2)} = 2\pi$$

and use Lemma 3.23 to obtain the first two statements. The statement (6.5) follows in a similar manner from (3.29).

□

The evolution equation (B.1) translates into the following equation for Θ:

$$\widetilde{\Theta}_t(\alpha, t) = \frac{2\pi}{L}\mathcal{Q}_1\big(U_\alpha + T(1+\theta_\alpha)\big) = \mathcal{A}\big[\widetilde{\Theta}\big] + \mathcal{L}_\beta[\widetilde{\Theta}] + \mathcal{N}[\widetilde{\Theta}]. \tag{6.6}$$

where L is determined from (B.4) with V determined from (6.3).

We can integrate the evolution equation (6.6) and rewrite it as the following integral equation:

$$\widetilde{\Theta}(\alpha, t) = e^{t\mathcal{A}}\widetilde{\Theta}_0 + \int_0^t e^{(t-\tau)\mathcal{A}}\left(\mathcal{L}_\beta[\widetilde{\Theta}] + \mathcal{N}[\widetilde{\Theta}]\right)(\alpha,\tau)d\tau \equiv \mathcal{R}[\widetilde{\Theta}](\alpha, t). \tag{6.7}$$

We will eventually show that $\mathcal{R}$ defines a contraction in a sufficiently small ball in the X_r space for $r \geq 3$. For that purpose we need some properties.

Proposition 6.6. *If for $r \geq 3$, $\widetilde{\Theta} \in \dot{H}^r$ with $\|\widetilde{\Theta}\|_1 < \epsilon_1$, and $0 \leq \beta < \Upsilon$, then for sufficiently small ϵ_1 and Υ, the functions $\mathcal{L}_\beta$, and $\mathcal{N}$, defined in Appendix (§7.3), satisfy the following estimates*

$$\begin{aligned}
\left\|\mathcal{L}_\beta\right\|_{r-1} &\leq C_1\beta^2 \exp\big(C_2\|\widetilde{\Theta}\|_r\big)\|\widetilde{\Theta}\|_r,\\
\left\|\mathcal{N}\right\|_{r-1} &\leq C_1 \exp\big(C_2\|\widetilde{\Theta}\|_r\big)\|\widetilde{\Theta}\|_r\|\widetilde{\Theta}\|_{r+1},
\end{aligned}$$

where C_1 and C_2 depend only on r. Further, let $\left(\mathcal{L}_\beta^{(1)}, \mathcal{N}^{(1)}\right)$ and $\left(\mathcal{L}_\beta^{(2)}, \mathcal{N}^{(2)}\right)$ correspond to $\widetilde{\Theta}^{(1)}$ and $\widetilde{\Theta}^{(2)}$ respectively, each in $\dot{H}^r$ with $\|\widetilde{\Theta}^{(1)}\|_1$ and $\|\widetilde{\Theta}^{(2)}\|_1 < \epsilon_1$. Then for sufficiently small ϵ_1,

$$\begin{aligned}
\left\|\mathcal{L}_\beta^{(1)} - \mathcal{L}_\beta^{(2)}\right\|_{r-1} &\leq C_1\beta^2\exp\Big(C_2\big(\|\widetilde{\Theta}^{(1)}\|_r + \|\widetilde{\Theta}^{(2)}\|_r\big)\Big)\left\|\widetilde{\Theta}^{(1)} - \widetilde{\Theta}^{(2)}\right\|_r,\\
\left\|\mathcal{N}^{(1)} - \mathcal{N}^{(2)}\right\|_{r-1} &\leq C_1\exp\Big(C_2\big(\|\widetilde{\Theta}^{(1)}\|_r + \|\widetilde{\Theta}^{(2)}\|_r\big)\Big)\Big\{\big(\|\widetilde{\Theta}^{(1)}\|_r + \|\widetilde{\Theta}^{(2)}\|_r\big)\left\|\widetilde{\Theta}^{(1)} - \widetilde{\Theta}^{(2)}\right\|_{r+1}\\
&\quad + \big(\|\widetilde{\Theta}^{(1)}\|_{r+1} + \|\widetilde{\Theta}^{(2)}\|_{r+1}\big)\left\|\widetilde{\Theta}^{(1)} - \widetilde{\Theta}^{(2)}\right\|_r\Big\},
\end{aligned}$$

where C_1 and C_2 depend on r.

Proof. On using Lemmas 3.6 (see Note 3.7), 3.12, 3.14, 3.16, 3.23, 3.20 and Proposition 6.5, the proof follows from the expressions of $\mathcal{L}_\beta$ and $\mathcal{N}$. □

Remark. It is easily to check that $\big(\mathcal{L}_\beta[\widetilde{\Theta}] + \mathcal{N}[\widetilde{\Theta}]\big)(-\alpha) = -\big(\mathcal{L}_\beta[\widetilde{\Theta}] + \mathcal{N}[\widetilde{\Theta}]\big)(\alpha)$.
□

6.3. **Contraction properties of $\mathcal{R}$ and global existence for symmetric disturbances.**

Note 6.7. *For the linear evolution equation (4.34), if f and v_0 are odd with respect to α, then by uniqueness of the linear equation (4.34), $v(-\alpha, t) = -v(\alpha, t)$.*

First, by Proposition 6.6, we have

Lemma 6.8. *Assume $0 \le \beta < \Upsilon$. Suppose for $r \ge 3$ $\widetilde{\Theta}(\alpha, t) \in X_r$ satisfy the condition $\|\widetilde{\Theta}\|_{H^r_\sigma} \le \epsilon$. Then for $\mathcal{L}_\beta[\widetilde{\Theta}](\alpha, t)$ and $\mathcal{N}[\widetilde{\Theta}](\alpha, t)$ determined from the Appendix (§7.3), as ϵ and Υ are small enough, we have*

$$\left\|\mathcal{L}_\beta[\widetilde{\Theta}] + \mathcal{N}[\widetilde{\Theta}]\right\|_{H^{r-3}_\sigma} \le C\|\widetilde{\Theta}\|_{H^r_\sigma}\left(\|\widetilde{\Theta}\|_{H^r_\sigma} + \beta^2\right).$$

Further, if both $\widetilde{\Theta}^{(1)}(\alpha, t)$ and $\widetilde{\Theta}^{(2)}(\alpha, t)$ satisfy (6.7), then the corresponding $\left(\mathcal{L}^{(1)}_\beta, \mathcal{N}^{(1)}\right)$ and $\left(\mathcal{L}^{(2)}_\beta, \mathcal{N}^{(2)}\right)$ satisfy

$$\|\mathcal{L}^{(1)}_\beta - \mathcal{L}^{(2)}_\beta\|_{H^{r-3}_\sigma} + \|\mathcal{N}^{(1)} - \mathcal{N}^{(2)}\|_{H^{r-3}_\sigma} \le C(\epsilon + \beta^2)\|\widetilde{\Theta}^{(1)} - \widetilde{\Theta}^{(2)}\|_{H^r_\sigma}.$$

Hence, by Lemmas 4.14 and 6.8, we have

Lemma 6.9. *Assume $0 \le \beta < \Upsilon$. Let $r \ge 3$, $\|\widetilde{\Theta}_0\|_{w,r} < \frac{\epsilon}{2}$ and $\widetilde{\Theta} \in X_r$ with $\|\widetilde{\Theta}\|_{H^r_\sigma} \le \epsilon$. For sufficiently small ϵ and Υ, the operator $\mathcal{R}$ defined in (6.7) satisfies the following estimate:*

$$\left\|\mathcal{R}\big[\widetilde{\Theta}\big]\right\|_{H^r_\sigma} \quad \le \quad C\epsilon.$$

Further, if $\|\widetilde{\Theta}^{(1)}\|_{H^r_\sigma} \le \epsilon$ and $\|\widetilde{\Theta}^{(2)}\|_{H^r_\sigma} \le \epsilon$, then

$$\left\|\mathcal{R}\big[\widetilde{\Theta}^{(1)}\big] - \mathcal{R}\big[\widetilde{\Theta}^{(2)}\big]\right\|_{H^r_\sigma} \quad \le \quad C\epsilon\|\widetilde{\Theta}^{(1)} - \widetilde{\Theta}^{(2)}\|_{H^r_\sigma}.$$

Further, $\mathcal{R}[\widetilde{\Theta}](-\alpha) = -\mathcal{R}[\widetilde{\Theta}](\alpha)$.

Proof of Proposition 6.3: If $C\epsilon < 1$, then it is clear that the right side of (6.7) define a contraction map in an ϵ ball in the Banach space $X_r \cap H^r_\sigma$. Therefore, there exists a unique solution $\widetilde{\Theta}$ satisfying the equation (6.7), hence (B.1). The local uniqueness of solutions (see Appendix §7.2) implies that this is the only solution. The $e^{-\sigma t/2}$ exponential decay of $\widetilde{\Theta}$ and hence of Θ implies that the steady symmetric translating bubble is approached exponentially in time. The constraint condition (B.4) shows that $L - 2\pi$ decays exponentially.

Acknowledgements: Partial support for this research was provided by the U.S. National Science (DMS-0733778, DMS-0807266).

We also like to thank Dr. Xuming Xie for valuable comments and suggestions.

7. APPENDIX

7.1. **Proof of Lemma 3.6.** Consider $j_0 = 1$ firstly. Let $F(u) = uh(u)$. Then $h(u)$ is also an entire function of order 1.

$$\left\|F\big(u(\cdot)\big)\right\|_\infty \le C_1 \exp(C_2\|u\|_\infty)\|u\|_\infty \le C_1 \exp\left(C_2\|u\|_1\right)\|u\|_1. \tag{7.1}$$

We see

$$\|D_\alpha F(u(\cdot))\|_0 = \|u_\alpha D_u F\|_0 \le C_1 \exp\left(C_2\|u\|_1\right)\|u\|_1.$$

For $k \geq 2$, by Banach Algebra property, we also have

$$\left\| D_\alpha F\big(u(\alpha)\big) \right\|_{k-1} \leq C \|D_\alpha u\|_{k-1} \|D_u F(u(\alpha))\|_{k-1} \leq C\|u\|_k \sum_{j=1}^{\infty} |a_j| j \|u\|_{k-1}^{j-1} \leq C_1 \|u\|_k \exp\big(C_2 \|u\|_{k-1}\big). \tag{7.2}$$

Hence, by (7.1) and (7.2), we have for $k \geq 2$,

$$\left\| F\big(u(\cdot)\big) \right\|_k \leq C_1 \|u\|_k \exp\big(C_2 \|u\|_{k-1}\big), \tag{7.3}$$

with C_1 and C_2 depending only on k.

Let $F(u) = u^2 g(u)$. Then $g(u)$ is also an entire function of order 1.

$$\left\| F\big(u(\cdot)\big) \right\|_\infty \leq C \exp(\|u\|_\infty) \|u\|_\infty^2 \leq C \exp\left(\|u\|_1\right) \|u\|_1^2.$$

And $D_u F(u)$ is the entire function of order 1 with $j_0 = 1$, so for $k \geq 2$, by Bananch Algebra and (7.3), we have

$$\left\| D_\alpha F\big(u(\cdot)\big) \right\|_{k-1} \leq C \|u_\alpha\|_{k-1} \|D_u F(u(\alpha))\|_{k-1} \leq C_1 \|u\|_k \|u\|_{k-1} \exp\big(C_2 \|u\|_{k-1}\big)$$

with C_1 and C_2 depending only on k. Hence, for $k \geq 2$,

$$\left\| F\big(u(\cdot)\big) \right\|_k \leq C_1 \|u\|_{k-1} \|u\|_k \exp\big(C_2 \|u\|_{k-1}\big), \tag{7.4}$$

with C_1 and C_2 depending only on k.

By the same technique, we obtain the difference results.

7.2. **Local uniqueness of Hele-Shaw bubble solutions.**

We have the local uniqueness theorem for the system (B.1)-(B.6) as follows:

Theorem 7.1. *Let $0 \leq \beta < \Upsilon$ and $|u_0| < 1$, where Υ is small enough for Lemmas 3.17, 3.23 and Proposition 3.24 to apply. Let $\big(\tilde{\theta}_1(\alpha, t), \hat{\theta}_1(0; t), y_1(0, t)\big)$ and $\big(\tilde{\theta}_2(\alpha, t), \hat{\theta}_2(0; t), y_2(0, t)\big)$ be solutions of the system (B.1)-(B.6) with the same initial condition (2.5) in the space $C\left([0, S], \mathcal{B}_\epsilon^r \times \mathbb{R} \times S_M\right)$ with $r \geq 4$. Suppose $\|\tilde{\theta}_1\|_1 < \epsilon_1$ and $\|\tilde{\theta}_2\|_1 < \epsilon_2$ such that $|L_1 - 2\pi| < \frac{1}{2}$ and $|L_2 - 2\pi| < \frac{1}{2}$ by (3.27). Then for sufficient small ϵ_1 and Υ, the two solutions are the same in $\dot{H}^2 \times \mathbb{R} \times S_M$.*

Proof. We define the energy function $E^d(t)$ for the difference of two solutions by

$$E^d(t) = \frac{1}{2} \int_0^{2\pi} \left(D_\alpha^2 \tilde{\theta}_1 - D_\alpha^2 \tilde{\theta}_2\right)^2 d\alpha + \frac{1}{2}\big(\hat{\theta}_1(0; t) - \hat{\theta}_2(0; t)\big)^2 + \frac{1}{2}\big(y_1(0, t) - y_2(0, t)\big)^2. \tag{7.5}$$

Taking derivatives on both sides with respect to t, and using (B.1)-(B.6), we have using (1.6)

$$
\begin{aligned}
(7.6)\quad \frac{dE^d(t)}{dt} &= \int_0^{2\pi} D_\alpha^2\big(\tilde\theta_1-\tilde\theta_2\big)D_\alpha^3\mathcal{Q}_1\big(\frac{2\pi}{L_1}U_1-\frac{2\pi}{L_2}U_2\big)d\alpha\\
&+\int_0^{2\pi} D_\alpha^2\big(\tilde\theta_1-\tilde\theta_2\big)D_\alpha\mathcal{Q}_1\big(\frac{2\pi}{L_1}(1+\theta_{1,\alpha})U_1-\frac{2\pi}{L_2}(1+\theta_{2,\alpha})U_2\big)d\alpha\\
&+\int_0^{2\pi} D_\alpha^2\big(\tilde\theta_1-\tilde\theta_2\big)D_\alpha^2\mathcal{Q}_1\big(\frac{2\pi}{L_1}T_1\theta_{1,\alpha}-\frac{2\pi}{L_2}T_2\theta_{2,\alpha}\big)d\alpha\\
&+\big(\hat\theta_1(0;t)-\hat\theta_2(0;t)\big)\int_0^{2\pi}\big[\frac{2\pi}{L_1}T_1(1+\theta_{1,\alpha})-\frac{2\pi}{L_2}T_2(1+\theta_{2,\alpha})\big]d\alpha\\
&+\big(y_1(0,t)-y_2(0,t)\big)\big[-U_1(0,t)\sin\big(\theta_1(0,t)\big)+U_2(0,t)\sin\big(\theta_2(0,t)\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{2\pi} D_\alpha^2\big(\tilde\theta_1-\tilde\theta_2\big)(D_\alpha^3+D_\alpha)\mathcal{Q}_1\big(\frac{2\pi}{L_1}U_1-\frac{2\pi}{L_2}U_2\big)d\alpha\\
&+\int_0^{2\pi} D_\alpha^2\big(\tilde\theta_1-\tilde\theta_2\big)D_\alpha\mathcal{Q}_1\big(\frac{2\pi}{L_1}\theta_{1,\alpha}U_1-\frac{2\pi}{L_2}\theta_{2,\alpha}U_2\big)d\alpha\\
&+\int_0^{2\pi} D_\alpha^2\big(\tilde\theta_1-\tilde\theta_2\big)D_\alpha^2\mathcal{Q}_1\big(\frac{2\pi}{L_1}T_1\theta_{1,\alpha}-\frac{2\pi}{L_2}T_2\theta_{2,\alpha}\big)d\alpha\\
&+\big(\hat\theta_1(0;t)-\hat\theta_2(0;t)\big)\int_0^{2\pi}\big[\frac{2\pi}{L_1}T_1(1+\theta_{1,\alpha})-\frac{2\pi}{L_2}T_2(1+\theta_{2,\alpha})\big]d\alpha\\
&+\big(y_1(0,t)-y_2(0,t)\big)\big[-U_1(0,t)\sin\big(\theta_1(0,t)\big)+U_2(0,t)\sin\big(\theta_2(0,t)\big)\big]=I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

By (1.6), we have

$$
\begin{aligned}
I_1 \quad = \quad & \int_0^{2\pi} D_\alpha^2\big(\tilde\theta_1-\tilde\theta_2\big)(D_\alpha^3+D_\alpha)\mathcal{Q}_1\big(\frac{2\pi^2}{L_1^2}\mathcal{H}[\gamma_1]-\frac{2\pi^2}{L_2^2}\mathcal{H}[\gamma_2]\big)d\alpha\\
&+\int_0^{2\pi} D_\alpha^2\big(\tilde\theta_1-\tilde\theta_2\big)(D_\alpha^3+D_\alpha)\mathcal{Q}_1\Big(\frac{2\pi^2}{L_1^2}\operatorname{Re}\big(\mathcal{G}[z_1]\gamma_1\big)-\frac{2\pi^2}{L_2^2}\operatorname{Re}\big(\mathcal{G}[z_2]\gamma_2\big)\Big)d\alpha\\
&+(u_0+1)\int_0^{2\pi} D_\alpha^2\big(\tilde\theta_1-\tilde\theta_2\big)(D_\alpha^3+D_\alpha)\mathcal{Q}_1\Big(\frac{2\pi}{L_1}\cos\big(\alpha+\theta_1(\alpha)\big)-\frac{2\pi}{L_2}\cos\big(\alpha+\theta(\alpha)\big)\Big)d\alpha.
\end{aligned}
$$

Using (B.3) and by Lemma 3.23 and Proposition 3.24, we have

$$
\begin{aligned}
I_1 = -\sigma \int_0^{2\pi} D_\alpha^3 \big(\tilde\theta_1 - \tilde\theta_2\big) \Lambda D_\alpha^3 \big(\frac{4\pi^3}{L_1^3}\tilde\theta_1 - \frac{4\pi^3}{L_2^3}\tilde\theta_2\big) d\alpha + \sigma \int_0^{2\pi} D_\alpha^2 \big(\tilde\theta_1 - \tilde\theta_2\big) \Lambda D^2 \big(\frac{4\pi^3}{L_1^3}\tilde\theta_1 - \frac{4\pi^3}{L_2^3}\tilde\theta_2\big) d\alpha \\
+ \big(1 + \frac{\mu_2}{\mu_1+\mu_2} u_0\big) \int_0^{2\pi} D_\alpha^2 \big(\tilde\theta_1 - \tilde\theta_2\big) \Lambda D_\alpha^2 \mathcal{Q}_1 \Big(\frac{2\pi}{L_1} \sin\big(\alpha + \theta_1(\alpha)\big) - \frac{2\pi}{L_2} \sin\big(\alpha + \theta_2(\alpha)\big)\Big) d\alpha \\
- a_\mu \int_0^{2\pi} D_\alpha^2 \big(\tilde\theta_1 - \tilde\theta_2\big) \Lambda D_\alpha^2 \mathcal{Q}_1 \Big(\frac{2\pi^2}{L_1^2} \mathcal{F}[z_1]\gamma_1 - \frac{2\pi^2}{L_2^2} \mathcal{F}[z_2]\gamma_2 \Big) d\alpha \\
+ \big(1 + \frac{\mu_2}{\mu_1+\mu_2} u_0\big) \int_0^{2\pi} D_\alpha^2 \big(\tilde\theta_1 - \tilde\theta_2\big) \Lambda \mathcal{Q}_1 \Big(\frac{2\pi}{L_1} \sin\big(\alpha + \theta_1(\alpha)\big) - \frac{2\pi}{L_2} \sin\big(\alpha + \theta_2(\alpha)\big)\Big) d\alpha \\
- a_\mu \int_0^{2\pi} D_\alpha^2 \big(\tilde\theta_1 - \tilde\theta_2\big) \Lambda \mathcal{Q}_1 \Big(\frac{2\pi^2}{L_1^2} \mathcal{F}[z_1]\gamma_1 - \frac{2\pi^2}{L_2^2} \mathcal{F}[z_2]\gamma_2 \Big) d\alpha \\
+ \int_0^{2\pi} D_\alpha^2 \big(\tilde\theta_1 - \tilde\theta_2\big) (D_\alpha^3 + D_\alpha) \mathcal{Q}_1 \Big(\frac{2\pi^2}{L_1^2} \operatorname{Re}\big(\mathcal{G}[z_1]\gamma_1\big) - \frac{2\pi^2}{L_2^2} \operatorname{Re}\big(\mathcal{G}[z_2]\gamma_2\big)\Big) d\alpha \\
+ (u_0 + 1) \int_0^{2\pi} D_\alpha^2 \big(\tilde\theta_1 - \tilde\theta_2\big) (D_\alpha^3 + D_\alpha) \mathcal{Q}_1 \Big(\frac{2\pi}{L_1} \cos\big(\alpha + \theta_1(\alpha)\big) - \frac{2\pi}{L_2} \cos\big(\alpha + \theta_2(\alpha)\big)\Big) d\alpha \\
\leq -\sigma \int_0^{2\pi} D_\alpha^3 \big(\tilde\theta_1 - \tilde\theta_2\big) \Lambda D_\alpha^3 \big(\frac{4\pi^3}{L_1^3}\tilde\theta_1 - \frac{4\pi^3}{L_2^3}\tilde\theta_2\big) d\alpha \\
+ C \|\tilde\theta_1 - \tilde\theta_2\|_2 \Big(\|\theta_1 - \theta_2\|_3 + \beta |y_1(0,t) - y_2(0,t)| \Big),
\end{aligned}
$$

where C depends on ϵ. For I_2, I_3, I_4 and I_5, by (3.30) and (3.31) in Proposition 3.24, we obtain

$$
\begin{aligned}
I_2 + I_3 + I_4 + I_5 \leq C \|\tilde\theta_1 - \tilde\theta_2\|_2 \Big(\|\theta_1 - \theta_2\|_3 + \beta |y_1(0,t) - y_2(0,t)| \Big) \\
+ C \big|\hat\theta_1(0;t) - \hat\theta_2(0;t)\big| \Big(\|\theta_1 - \theta_2\|_3 + \beta |y_1(0,t) - y_2(0,t)| \Big) \\
+ \big|y_1(0,t) - y_2(0,t)\big| \big\| U_1(\cdot,t) \sin\big(\cdot + \theta_1(\cdot,t)\big) - U_2(\cdot . t) \sin\big(\cdot + \theta_2(\cdot,t)\big)\big\|_1 \\
\leq C \|\tilde\theta_1 - \tilde\theta_2\|_2 \Big(\|\theta_1 - \theta_2\|_3 + \beta |y_1(0,t) - y_2(0,t)| \Big) \\
+ C \big|\hat\theta_1(0;t) - \hat\theta_2(0;t)\big| \Big(\|\theta_1 - \theta_2\|_3 + \beta |y_1(0,t) - y_2(0,t)| \Big) \\
+ \big|y_1(0,t) - y_2(0,t)\big| \Big(\|\theta_1 - \theta_2\|_3 + \beta |y_1(0,t) - y_2(0,t)| \Big),
\end{aligned} \tag{7.7}
$$

where C depends on ϵ. Actually, combining the estimates for I_1, I_2, I_3, I_4 and I_5, by Cauchy inequality, we have

$$\frac{dE^d(t)}{dt} \leq C E^d(t).$$

That is

$$E^d(t) \leq E^d(0) e^{Ct}.$$

Hence, $E^d(t) = 0$ if $E^d(0) = 0$. □

7.3. **The Fréchet derivative $\mathfrak{U}_{\tilde{\theta}^{(s)}}[0,0,0]$ in §5.** From (C.2), $\gamma^{(s)}$ is the result of an operator acting on $(\tilde{\theta}^{(s)}, u_0, \beta)$. From substituting $\tilde{\theta}^{(s)} = \epsilon h$ and taking the ϵ derivative at $\epsilon = 0$ and using Proposition 2.4, we have

(7.8)
$$\mathfrak{U}_{\tilde{\theta}^{(s)}}[0,0,0]h = \frac{1}{2}\mathcal{H}\big[\gamma^{(s)}_{\tilde{\theta}^{(s)}}[0,0,0]h\big](\alpha) + i\sum_{k=1}^{\infty}\frac{1}{k+2}\hat{h}(k+1)e^{ik\alpha} - h(\alpha)\sin\alpha + c.c..$$

From (C.2) and Proposition 2.4, we have

$$\begin{aligned}\gamma^{(s)}_{\tilde{\theta}^{(s)}}[0,0,0]h(\alpha) &= 2(1+a_\mu)h(\alpha)\cos\alpha + \sigma h_{\alpha\alpha}(\alpha) + 2a_\mu\mathcal{H}[h\sin\alpha](\alpha)\\ &\quad - a_\mu\sum_{k=1}^{\infty}\frac{2}{k+2}\hat{h}(k+1)e^{ik\alpha} + c.c..\end{aligned} \tag{7.9}$$

Hence, combining (7.8) and (7.9), using $\mathcal{H}^2 = -I$, we obtain

$$\begin{aligned}\mathfrak{U}_{\tilde{\theta}^{(s)}}[0,0,0]h &= \frac{\sigma}{2}\mathcal{H}\big[h_{\alpha\alpha}\big](\alpha) + (1+a_\mu)i\sum_{k=1}^{\infty}\frac{1}{k+2}\hat{h}(k+1)e^{ik\alpha}\\ &\quad - (1+a_\mu)h(\alpha)\sin\alpha + (1+a_\mu)\mathcal{H}\big[h\cos\alpha\big](\alpha) + c.c.\\ &= \frac{\sigma}{2}\mathcal{H}\big[h_{\alpha\alpha}\big](\alpha) - i(1+a_\mu)\sum_{k=1}^{\infty}\frac{k+1}{k+2}\hat{h}(k+1)e^{ik\alpha} + c.c..\end{aligned}$$

7.4. **Expressions for $\mathcal{L}_\beta$ and $\mathcal{N}$.**

Definition 7.2. *We define the function*
$$\omega_s(\alpha) = \int_0^\alpha e^{i\tau + i\theta^{(s)}(\tau)}d\tau.$$

$$\begin{aligned}\mathcal{L}_\beta\big[\widetilde{\Theta}\big](\alpha,t) &= \mathcal{Q}_1\Big\{\Big(\frac{1}{2}\mathcal{H}\Big(\mathcal{L}_{\beta_1}\big[\widetilde{\Theta}\big]\Big)(\alpha,t) + \mathcal{L}_{\beta_2}\big[\widetilde{\Theta}\big](\alpha,t)\Big)_\alpha + \mathcal{L}_{\beta_3}\big[\widetilde{\Theta}\big](\alpha,t)\Big\},\\ \mathcal{N}\big[\widetilde{\Theta}\big](\alpha,t) &= \frac{2\pi}{L}\mathcal{Q}_1\Big\{\Big(\frac{1}{2}\mathcal{H}\Big(\mathcal{N}_1\big[\widetilde{\Theta}\big]\Big)(\alpha,t) + \mathcal{N}_2\big[\widetilde{\Theta}\big](\alpha,t)\Big)_\alpha + \mathcal{N}_3\big[\widetilde{\Theta}\big](\alpha,t)\Big\}\\ &\quad + \frac{2\pi - L}{L}\Big\{\sum_{k=2}^{\infty}(1+a_\mu)\frac{(k^2-1)(k+1)}{k(k+2)}\widehat{\Theta}(k+1)e^{ik\alpha}\\ &\quad - \sum_{k=-2}^{-\infty}(1+a_\mu)\frac{(k^2-1)(k-1)}{k(k-2)}\widehat{\Theta}(k-1)e^{ik\alpha} + \mathcal{L}_\beta\big[\widetilde{\Theta}\big](\alpha,t)\Big\},\end{aligned}$$

where

$$\begin{aligned}&\mathcal{L}_{\beta_1}\big[\widetilde{\Theta}\big](\alpha)\\ =\ & a_\mu\,\mathrm{Re}\Big(-\frac{1}{i}\mathcal{G}[z^{(s)}]\Gamma\Big) + a_\mu\,\mathrm{Re}\Big(-\frac{1}{i}\mathcal{G}_1[z](\gamma^{(s)} - 2\sin\alpha) + \frac{1}{i}\mathcal{G}_1[\omega_s](\gamma^{(s)} - 2\sin\alpha)\Big)\\ & - a_\mu\,\mathrm{Re}\Big(z_\alpha\mathcal{K}_2[z]\gamma^{(s)}(\alpha) - i\omega_{s_\alpha}\mathcal{K}_2[z^{(s)}]\gamma^{(s)}(\alpha)\Big) - 4a_\mu D_\alpha\,\mathrm{Re}\Big(\mathfrak{B}[\Theta](\alpha) - \mathfrak{W}[\Theta](\alpha)\Big)\\ & + \frac{L-2\pi}{\pi}\big(\sin(\alpha+\theta^{(s)}) - \sin\alpha\big) + 2\Theta\big(\cos(\alpha+\theta^{(s)}) - \cos\alpha\big) + \frac{2\pi - L}{L}\sigma\theta^{(s)}_{\alpha\alpha}\\ & + \frac{L-2\pi}{\pi}\frac{\mu_2}{\mu_1+\mu_2}u_0\sin\big(\alpha+\theta\big) + 2\frac{\mu_2}{\mu_1+\mu_2}u_0\Big(\sin\big(\alpha+\theta\big) - \sin\big(\alpha+\theta^{(s)}\big)\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{\beta_2}\big[\widetilde{\Theta}\big] = \frac{2\pi-L}{2L}\mathcal{H}[\gamma^{(s)}-2\sin\alpha'] + \mathrm{Re}\Big(\frac{1}{2}\mathcal{G}[z^{(s)}]\Gamma\Big) + \mathrm{Re}\Big(\frac{\pi}{L}\mathcal{G}_1[z]\big(\gamma^{(s)}(\alpha)-2\sin\alpha\big)\\ -\frac{1}{2}\mathcal{G}_1[\omega_s]\big(\gamma^{(s)}(\alpha)-2\sin\alpha\big)\Big) + 2\,\mathrm{Re}\Big(-\omega_\alpha\mathcal{K}_2[z]\gamma^{(s)}(\alpha)+\omega_{s_\alpha}\mathcal{K}_2[z^{(s)}]\gamma^{(s)}(\alpha)\Big)\\ +\frac{2\pi-L}{L}\,\mathrm{Re}\Big(\mathcal{G}_1[\omega_s]\sin\alpha\Big) + \mathrm{Re}\Big(2i\frac{\partial}{\partial\alpha}\mathfrak{B}[\Theta](\alpha) - 2i\frac{\partial}{\partial\alpha}\mathfrak{W}[\Theta](\alpha)\Big)\\ + u_0\big[\cos\big(\alpha+\theta(\alpha)\big)-\cos\big(\alpha+\theta^{(s)}(\alpha)\big)\big],\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{\beta_3}\big[\widetilde{\Theta}\big] \;=\; & \Big(\int_0^\alpha \theta_\alpha^{(s)}(\alpha')\big(\frac{2\pi^2}{L^2}\sigma\mathcal{H}(\Theta_{\alpha\alpha})(\alpha')+\mathcal{L}\big[\widetilde{\Theta}\big](\alpha')\big)d\alpha'\\ & -\frac{\alpha}{2\pi}\int_0^{2\pi}\theta_\alpha^{(s)}(\alpha)\big(\frac{2\pi^2}{L^2}\sigma\mathcal{H}(\Theta_{\alpha\alpha})(\alpha')+\mathcal{L}\big[\widetilde{\Theta}\big](\alpha')\big)d\alpha\Big)(1+\theta_\alpha^{(s)})\\ & +\theta_\alpha^{(s)}\Big(\int_0^\alpha\big(\frac{2\pi^2}{L^2}\sigma\mathcal{H}(\Theta_{\alpha\alpha})(\alpha')+\mathcal{L}\big[\widetilde{\Theta}\big](\alpha')\big)d\alpha' - \frac{\alpha}{2\pi}\int_0^{2\pi}\mathcal{L}\big[\widetilde{\Theta}\big](\alpha)d\alpha\Big)\\ & +\Big\{\int_0^\alpha\big(1+\theta_\alpha^{(s)}(\alpha')\big)\Big[\frac{1}{2}\mathcal{H}\Big(\mathcal{L}_{\beta_1}\big[\widetilde{\Theta}\big](\cdot)\Big)(\alpha')+\mathcal{L}_{\beta_2}\big[\widetilde{\Theta}\big](\alpha')\Big]d\alpha'\\ & -\frac{\alpha}{2\pi}\int_0^{2\pi}\big(1+\theta_\alpha^{(s)}(\alpha)\big)\Big[\frac{1}{2}\mathcal{H}\Big(\mathcal{L}_{\beta_1}\big[\widetilde{\Theta}\big](\cdot)\Big)(\alpha)+\mathcal{L}_{\beta_2}\big[\widetilde{\Theta}\big](\alpha)\Big]d\alpha\Big\}(1+\theta_\alpha^{(s)}),\end{aligned}$$

$$\begin{aligned}\mathcal{N}_1\big[\widetilde{\Theta}\big] = a_\mu\,\mathrm{Re}\Big(-\frac{1}{i}\mathcal{G}[z]\Gamma+\frac{1}{i}\mathcal{G}[z^{(s)}]\Gamma\Big)+\frac{L-2\pi}{\pi}\Big(\sin(\alpha+\theta)-\sin\big(\alpha+\theta^{(s)}(\alpha)\big)\Big)\\ +2\Big(\sin(\alpha+\theta)-\sin\big(\alpha+\theta^{(s)}(\alpha)\big)-\Theta\cos\big(\alpha+\theta^{(s)}(\alpha)\big)\Big)-2a_\mu\,\mathrm{Re}\Big(\frac{1}{i}\frac{\partial}{\partial\alpha}\big\{\mathfrak{B}[\Xi_e[\Theta]](\alpha)\big\}\Big)\\ +2a_\mu\,\mathrm{Re}\Big(i(e^{i\Theta}-1)\Big\{\frac{\omega_{s_\alpha}}{\omega_\alpha}\big(\mathcal{G}_1[\omega]\sin\alpha-\cos\alpha\big)-\big(\mathcal{G}_1[\omega_s]\sin\alpha-\cos\alpha\big)\Big\}\Big)\\ +2a_\mu\,\mathrm{Re}\Big(\frac{\omega_{s_\alpha}}{\pi i}\mathrm{PV}\int_{\alpha-\pi}^{\alpha+\pi}\sin(\alpha')\frac{q_1[\omega-\omega_s](\alpha,\alpha')}{q_1[\omega_s](\alpha,\alpha')}\Big(\frac{1}{\omega(\alpha)-\omega(\alpha')}-\frac{1}{\omega_s(\alpha)-\omega_s(\alpha')}\Big)d\alpha'\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{N}_2\big[\widetilde{\Theta}\big] = \frac{2\pi-L}{2L}\mathcal{H}[\Gamma-\frac{2\pi}{L}\sigma\Theta_{\alpha\alpha}]+\mathrm{Re}\Big(\frac{\pi}{L}\mathcal{G}[z]\Gamma-\frac{1}{2}\mathcal{G}[z^{(s)}]\Gamma\Big)\\ +\frac{2\pi-L}{L}\,\mathrm{Re}\Big(\mathcal{G}_1[\omega]\sin\alpha-\mathcal{G}_1[\omega_s]\sin\alpha\Big)+\mathrm{Re}\Big(\frac{\partial}{\partial\alpha}\big(\mathfrak{B}[\Xi_e[\Theta]](\alpha)\big)\Big)\\ +\mathrm{Re}\Big((e^{i\Theta}-1)\big\{\frac{\omega_{s_\alpha}}{\omega_\alpha}\big(\mathcal{G}_1[\omega]\sin(\alpha)-\cos\alpha\big)-\big(\mathcal{G}_1[\omega_s]\sin(\alpha)-\cos\alpha\big)\big\}\Big)\\ +\Big(\cos\big(\alpha+\theta(\alpha)\big)-\cos\big(\alpha+\theta^{(s)}\big)+\Theta\sin\big(\alpha+\theta^{(s)}\big)\Big)\\ -\mathrm{Re}\Big(\frac{\omega_{0_\alpha}}{\pi}\mathrm{PV}\int_{\alpha-\pi}^{\alpha+\pi}\sin(\alpha')\frac{q_1[\omega-\omega_s](\alpha,\alpha')}{q_1[\omega_s](\alpha,\alpha')}\cdot\Big(\frac{1}{\omega(\alpha)-\omega(\alpha')}-\frac{1}{\omega_s(\alpha)-\omega_s(\alpha')}\Big)d\alpha'\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{N}_3\big[\widetilde{\Theta}\big] &= \Big\{ \int_0^{\alpha} \big(1+\theta_\alpha^{(s)}(\alpha')\big)\Big[\frac{1}{2}\mathcal{H}\Big(\mathcal{N}_1\big[\widetilde{\Theta}\big](\cdot)\Big)(\alpha') + \mathcal{N}_2\big[\widetilde{\Theta}\big](\alpha')\Big]d\alpha' \\ &\quad -\frac{\alpha}{2\pi}\int_0^{2\pi} \big(1+\theta_\alpha^{(s)}(\alpha)\big)\Big[\frac{1}{2}\mathcal{H}\Big(\mathcal{N}_1\big[\widetilde{\Theta}\big](\cdot)\Big)(\alpha) + \mathcal{N}_2\big[\widetilde{\Theta}\big](\alpha)\Big]d\alpha \\ &\quad + \int_0^{\alpha} \Theta_\alpha(\alpha')U(\alpha')d\alpha' - \frac{\alpha}{2\pi}\int_0^{2\pi}\Theta_\alpha(\alpha)U(\alpha)d\alpha\Big\}(1+\theta_\alpha^{(s)}) \\ &\quad + \Big(\int_0^{\alpha}\Theta_\alpha(\alpha')U(\alpha')d\alpha' - \frac{\alpha}{2\pi}\Theta_\alpha(\alpha)U(\alpha)d\alpha\Big)\Theta_\alpha(\alpha).\end{aligned}$$

$$\mathcal{L}_\Gamma\big[\widetilde{\Theta}\big](\alpha) = 2\Theta(\alpha,t)\cos\alpha + \frac{L-2\pi}{\pi}\sin\alpha - 4a_\mu \,\mathrm{Re}\Big(\frac{\partial}{\partial\alpha}\big\{\mathfrak{W}[\Theta](\alpha)\big\}\Big),$$

$$\mathcal{L}\big[\widetilde{\Theta}\big] = \frac{1}{2}\mathcal{H}[\mathcal{L}_\Gamma](\alpha,t) + \frac{L-2\pi}{L}\cos\alpha - \mathcal{Q}_0\theta\sin\alpha + \mathrm{Re}\Big(i\frac{\partial}{\partial\alpha}\big(\mathfrak{W}[\Theta](\alpha)\big)\Big).$$

References

[1] Ye, J. (2010) Global Existence of translating Hele-Shaw bubble for arbitrary nonzero surface tension. Ph.D Thesis, The Ohio State University.

[2] Ambrose, D. M. (2002) Well-posedness of vortex sheets with surface tension. PhD thesis, Duke university.

[3] Ambrose, D. M. (2003) Well-posedness of vortex sheets with surface tension. *SIAM J. Math. Anal.* **35**, no.1, 211-244.

[4] Ambrose, D. N. (2004) Well-posedness of two-phase Hele-Shaw flow without surface tension. *Euro. Jnl of Applied Mathematics* **15**, no. 5, 597-607.

[5] Ambrose, D. N. & Masmoudi, N (2005) The zero surface tension limit of two-dimensional water waves. *Comm. Pure Appl. Math.* **58**, no. 10, 1287-1315.

[6] Baker, G., Meiron, D. and Orszag, S. (1982) Generalized vortex methods for free-surface flow problems. *J. Fluid Mech.* **123**, 477-501.

[7] Beale, J. T., Hou,T. and Lowengrub, J. (1993) Growth rates for the linearized motion of fluid interfaces away from equilibrium. *Comm. Pure Appl. Math.* **46**, no. 9, 1269-1301.

[8] Bensimon, D., Kandanoff, L. P., Liang, S., Shraiman, B. I. and Tang, C. (1986) Viscous flow in two dimensions. *Rev. Mod. Phys.* **32**, 977.

[9] Constantin, P. & Pugh, M. (1993) Global solutions for small data to the Hele-Shaw problem. *Nonlinearity* **6**, no. 3, 393-415.

[10] Degregoria, A. J. & Schwartz, L. W. (1986) A boundary-integral method for two-phase displacement in a Hele-Shaw cell. *J. Fluid Mech.* **164**, 383-400.

[11] Duchon, J. & Robert, R. (1984) Évolution d'une interface par capillarité et diffusion de volume. *Ann. l'Inst. H. Poinaré* **1**, no.5, 361-378.

[12] Escher, J. & Matioc, B. V. (2008) On periodic Stokesian Hele-Shaw flows with surface tension. *Euro. Jnl of Applied Mathematics* **19**, no. 6, 717-734.

[13] Escher, J. & Simonett, G. (1997) Classical solutions for Hele-Shaw models with surface tension. *Advances in Diff. Eq.* **2**, no. 4, 619-642.

[14] Escher, J. & Simonett, G. (1998) A center manifold analysis for the Mullins-Sekerka model. *J. Differential Equations* **143**, no. 2, 267-292.

[15] Fokas, A.S. & Tanveer, S., (1998), A Hele-Shaw problem and the second Painleve' transcendent, *Math. Proc. Cambridge Philos. Soc.*, **124**, no.1, 169-191.

[16] Friedman, A. & Tao, Y. (2003) Nonlinear stability of the Musktat problem with capillary pressure at the free boundary. *Nonlinear Analysis* **53**, 45-80.

[17] Hohlov, E. Yu. (1990) Time-dependent free boundary problems: the explicit solution. *MIAN Preprint*, no. **14**. Steklov Institute, Moscow.

[18] Homsy, G. M. (1987) Viscous fingering in porous media. *Ann. Rev. Fluid Mech.* **19**, 271-311.

[19] Hou, T., Lowengrub, J. and Shelley, M. (1994) Removing the stiffness from interfacial flows with surface tension. *J. Comput. Phys.* **114**, no. 2, 312-338.

[20] Hou, T., Lowengrub, J. and Shelley, M. (1997) The long-time motion of vortex sheets with surface tension. *Phys. Fluids* **9**, no. 7, 1933-1954.

[21] Howison, S.D (1986), Cusp development in Hele-Shaw flow with a free surface, *SIAM J. Appl. Math.* **46**, no. 1, 20-26.

[22] Howison, S. D. (1992) Complex variable methods in Hele-Shaw moving boundary problems. *Eur. J. Appl. Maths* **3**, no. 3, 209-224.

[23] Howison, S. D. (2000) A note on the two-phase Hele-Shaw problem. *J. Fluid Mech.* **409**, 243-249.

[24] Kessler, D. A., Koplik, J. and Levine, H. (1988) Patterned Selection in fingered growth phenomena. *Adv. Phys.* **37**, n0. 3, 255-339.

[25] Maclean, J. W. & Saffman, P. G. (1981) The effect of surface tension on the shape of fingers in the Hele-Shaw cell. *J. Fluid Mech.* **102**, 455-469.

[26] Majda, A. and Bertozzi, A. *Voricity and Incompressible Flow.* Cambridge University Press, Cambridge, UK, 2002.

[27] Pelce, P. (1988) *Dynamics of Curved Fronts.* Academic.

[28] Prokert, G. (1998) Existence results for Hele-Shaw flow driven by surface tension. *Euro. Jnl of Applied Mathematics* **9**, 195-221.

[29] Prüss, J. and Simonett, G. (2008) Stability of equilibria for the Stefan problem with surface tension. *SIAM J. Math. Anal.* **40**, no. 2, 675-698.

[30] Reed, Michael and Simon, Barry *Methods of Modern Mathematical Physics, Vol I: Functional Analysis, revised and enlarged Edition.*

[31] Muskhelishvili, N. I. *Singular Integral Equations: Boundary Problems of Function Theory and Their Applications to Mathematical Physics.* Dover, New York, second edition, 1992.

[32] Saffman, P.G. and Taylor, G.I. (1958) The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous fluid. *Proc. Roy. Soc. London. Ser. A* **245**, 312-329.

[33] Saffman, P.G. (1986) Viscous fingering in a Hele-Shaw cell. *J. Fluid Mech.* **173**, 73-94.

[34] Saffman, P.G. *Vortex Dynamics*. Cambridge University Press, Cambridge, UK, the first paperback edition, 1995.

[35] Su, J. Z. (2001) On the existence of finger solutions of the Hele-Shaw equation. *Nonlinearity* **14**, no. 1, 153-166.

[36] Tanveer, S. (1986) The effect of surface tension in the shape of a Hele-Shaw cell bubble. *Phys. Fluids* **29**, no. 11, 3537-3548.

[37] Tanveer, S. (1987), Analytic theory for the linear stability of Saffman-Taylor finger.*Phys. Fluids* **30**, no. 8, 2318-2329.

[38] Tanveer, S. & Saffman, P. G. (1987) Stability of a bubbles in a Hele-Shaw cell.*Phys. Fluids* **30**, 2624-2635.

[39] Tanveer, S. & Saffman, P.G. (1988) The effect of nonzero viscosity ratio on the stability of fingers and bubbles in a Hele-Shaw cell. *Phys. Fluids* **31**, no. 11, 3188-3198.

[40] Tanveer, S. (1989) Analytic theory for the determination velocity and stability of bubbles in a Hele-Shaw cell, Part II: Stability. *Theoret. Comput. Fluid Dynamics* **1**, no. 3, 165-177.

[41] Tanveer, S. (1991) Viscous Displacement in a Hele-Shaw cell. In *Asymptotics Beyond all orders* (ed. H. Segur, S. Tanveer and H. Levine). Plenum.

[42] Tanveer, S. (2000) Surprise in viscous fingering. *J. Fluid Mech.*, **409**, 273-308.

[43] Tao, T. (2007) A quantitative formulation of the global regularity problem for the periodic Navier-Stokes equation. Submitted, http://arxiv.org/abs.0710.1604v4-1.pdf

[44] Taylor, M. *Partial Differential Equations III: Nonlinear Equations*, Volume 117 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996.

[45] Xie, X. & Tanveer, S. (2003) Rigorous results in steady finger selection in viscous fingering. *Arch. Ration. Mech. Anal.* **166**, no. 3, 219-286.

[46] Tanveer, S. & Xie, X. (2003) Analyticity and nonexistence of classical steady Hele-Shaw fingers. *Communications on Pure and Applied Mathematics* **56**, no.3, 353-402.

[47] Xie, X. (2009) Nonexistence of classical steady Hele-Shaw bubble. *Nonlinear Anal.* **70**, no. 3, 1217-1238.

[48] Xu, J. J., (1991) Globally unstable oscillatory modes in viscous fingering. *Eur. J. Appl. Maths* **2**, 105.

[49] Xu, J. J., (1996) Interfacial instabilities and fingering formation in Hele-Shaw flow.*IMA Journal of Applied Mathematics* **57**, no. 2, 101-135.

[50] Ye, J. and Tanveer, S. (2008) Global solutions for two-phase Hele-Shaw bubble for a near-circular initial shape. Submitted, http://arXiv.org/abs/0810.2980