# AN INTEGRAL EQUATION APPROACH TO SMOOTH 3-D NAVIER-STOKES SOLUTION

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June 23, 2008

Abstract. We summarize a recently developed integral equation approach to tackling the long-time existence problem for smooth solution $v(x,t)$ to the 3-D Navier-Stokes equation in the context of a periodic box problem with smooth time-independent forcing and initial condition $v_0$. Using an inverse Laplace transform of $\hat{v}(k,t) - \hat{v}_0$ in $1/t$, we arrive at an integral equation for $\hat{U}(k,p)$, where $p$ is inverse-Laplace dual to $1/t$ and $k$ is the Fourier variable dual to $x$. The advantage of this formulation is that the solution $\hat{U}$ to the integral equation is known to exist *a priori* for $p \in \mathbb{R}^+$ and the solution is integrable and exponentially bounded at $\infty$.

Global existence of NS solution in this formulation is reduced to an asymptotics question. If $\|\hat{U}(.,p)\|_{l^1(\mathbb{Z}^3}$ has subexponential bounds as $p \to \infty$, then global existence to NS follows. Moreover, if $f = 0$, then the converse is also true in the following sense: if NS has global solution, then there exists $n \geq 1$ for which the the inverse Laplace transform of $\hat{v}(k,t) - \hat{v}_0$ in $1/t^n$ necessarily decays as $q \to \infty$, where $q$ is the inverse-Laplace dual to $1/t^n$.

We also present refined estimates of the exponential growth when the solution $\hat{U}$ is known on a finite interval $[0, p_0]$. We also show that for analytic $v^{[0]}$ and $f$, with finitely many nonzero Fourier-coefficients, the series for $\hat{U}(k,p)$ in powers of $p$ has a radius of convergence independent of initial condition and forcing; indeed the radius gets bigger for smaller viscosity. We also show that the the integral equation can be solved numerically with controlled errors.

Preliminary numerical calculations for Kida ([14]) initial conditions, though far from being optimized, and performed on a modest interval in the accelerated variable $q$ show decay in $q$.

## 1. Introduction

The global existence of smooth solution to the incompressible Navier-Stokes (NS) system

$$v_t + (v \cdot \nabla)v = -\nabla P + \nu \Delta v \ + \ f \ , \text{ with } \nabla \cdot v = 0 \ , \text{ and } v(x,0) = v^{[0]}(x) \tag{1.1}$$

for $x \in \Omega \subset \mathbb{R}^3$ for smooth $f$ and $v^{[0]}$ has remained a formidable open mathematical problem for almost three quarters of a century, despite extensive research (see for example monographs [11], [26], [5], [10]). Smooth solutions are known to exist only on a time interval $[0,T]$ where $T$ scales inversely with a suitable norm of $v^{[0]}$ and $f$. Global existence can be proved only when $f$ and $v^{[0]}$ are sufficiently small or $\nu$ is sufficiently large, *i.e.* the Reynolds number sufficiently small. This problem remains unsolved even for $f = 0$ (no forcing) or even without the mathematical

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complication of no-slip boundary on $\partial\Omega$, as is the case when $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3[0,2\pi]$ ($[0,2\pi]^3$ periodic box).

Global existence is not simply a mathematical issue. It has much wider impact, particularly if singular solutions exist. It is known [3] for instance that singularities can only occur when vorticity and more generally $\nabla v$ blow up. This means that if solutions were to develop singularities, the relevance of Navier-Stokes as a model of actual fluid flow becomes questionable, as linear stress-strain relation, incompressibility, and even the continuum hypothesis implicit in NS derivation become doubtful. Viscous regularization of inviscid Burger approximation determines the size of a shock-layer once a shock is formed, and analogously it may be expected that if smooth NS solutions were to become singular, regularizing effects such as departure from a linear stress-strain relation will determine the smallest time and space scales observed in fluid flow–*i.e.* there will be dependence on parameters other than those present in NS. This can profoundly affect our understanding of small scale turbulence and indeed of turbulent mixing, which is the theme of the present conference. In fact, some 75 years back, Leray [16], [17], [18] was motivated to study weak solutions of 3-D NS, conjecturing that turbulence was related to blow up of smooth solutions.

The typical method used in the mathematical analysis of NS, and of more general PDEs, is the so-called energy method. For NS, the energy method involves *a priori* estimates on the Sobolev $\mathbb{H}^m$ norms$^{(1)}$ of $v$. It is known that if $\|v(\cdot,t)\|_{\mathbb{H}^1}$ is bounded, then so are all the higher order energy norms $\|v(\cdot,t)\|_{\mathbb{H}^m}$ if they are bounded initially. The condition on $v$ has been further weakened [3] to $\int_0^t \|\nabla \times v(\cdot,t)\|_{L^\infty} dt < \infty$. Other controlling norms are known [21], [22], [15], [24] to exist; however, lack of a global conservation law involving a controlling norm is believed to be the primary obstacle [25] in the way of a global existence proof.

Globally, only weak solutions (possibly non-unique) are known to exist, as a result of Leray's pioneering work [16], [17], [18]. In that setting, $\nabla v$ can blow-up on a small set in space-time$^{(2)}$. However, when $f = 0$ (no forcing) in the periodic box problem, a time $T_c$ may be estimated in terms of the $\|v^{[0]}\|_{H^1}$ so that any weak Leray solution becomes smooth again for $t > T_c$.

Numerical solutions to (1.1), while generating physical evidence on the nature of fluid flow, do not address the existence issue either. Indeed, truncation errors in a usual numerical scheme depend on derivatives of $v$ and these are not known to exist beyond an initial time interval.

In this paper, we summarize the salient points of a new approach [9], [12] towards Navier-Stokes existence that can be generalized to a wider class of PDE initial value problems. The method is fundamentally different from the usual PDE approach to existence and uniqueness. It originates from ideas of Borel summation which associates a unique function to a divergent series under some conditions (see [2] for a general exposition and [13] which relates to N-S). In the context of NS equation,

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$^{(1)}$Here $\|.\|_{H^m}$ norm is essentially the sum of $L_2$ norms of a function and all its space derivaties upto order $m$

$^{(2)}$The 1-D Hausdorff measure of the set of blow-up points in space-time is known to be zero [4] consider a formal asymptotic expansion for small $t$,

$$v(x,t) \sim v^{[0]}(x) + \sum_{m=1}^{\infty} v^{[m]}(x) t^m. \tag{1.2}$$

By plugging in this formal expansion into Navier-Stokes equation (1.1) and matching like powers of $t$ we arrive(3) at a set of recurrence relation (as in (4.21) ) that completely determines all coefficients $v^{[m]}$ in terms of $v^{[0]}$, $f$ and its $x$-derivatives, assuming they exist. For analytic initial data $v^{[0]}$ and forcing $f$, it has been shown[9] that this series can be Borel-summed into an actual solution to the Navier-Stokes equation. The expression for the Borel sum of (1.2) is given as a Laplace transform

$$v(x,t) = v^{[0]}(x) + \int_0^{\infty} e^{-p/t} U(x,p) dp \tag{1.3}$$

where $U(x,p)$ satisfies an integral equation (IE) (see (2.8) in the sequel, where $\hat{U}(k,p) = \mathcal{F}\left[U(.,p)\right](k)$ is the Fourier transform of $U$). It has been shown [9], [12] that there exists a unique solution to this IE within the class of locally integrable functions, exponentially bounded in $p$ (like $e^{\alpha p}$), uniformly in $x$. Indeed, $U(x,p)$ generates a smooth solution to NS through the relation (1.3). The existence time interval for NS solution is $\left[0, \alpha^{-1}\right)$, where $\alpha$ is the growth exponent of the associated IE solution. If the IE solution $U$ does not grow with $p$ or grows at most subexponentially, then global existence of NS follows.

Borel summation has been used in other contexts including nonlinear ODEs [27], [6] as well as difference equations. It has also been used in obtaining PDE solution in parts of the complex plane [19]. We will not emphasize the theoretical aspects of Borel summation, beyond pointing out that it is limited to analytic initial data and forcing. Our generalization, leading to the integral equation representation (2.8) is much more general and transcends these restrictions. Further, while the asymptotic expansion (1.2) is only valid for $t \ll 1$, its Borel sum makes sense much more generally. Indeed, if the solution of the integral equation, which is known to exist *a priori*, does not grow as $p \to \infty$ or even grows sub-exponentially, then global existence of NS solution follows. Thus, in this representation, the global existence of a time evolving PDE becomes a problem of asymptotics for the associated integral equation.

Further, as will be discussed later, it is sometimes advantageous to use a more general Laplace transform representation:

$$v(x,t) = v^{[0]}(x) + \int_0^{\infty} e^{-q/t^n} U_{acc}(x,q) dq \tag{1.4}$$

for $n \geq 1$. The $q$-representation will be referred to as the accelerated representation; $U(.,p) \to U_{acc}(.,q)$ transformation is expressed through an explicit integral formula introduced by Ecalle in the context of ODEs.

(3)We assume that $f$ is either time-independent or has a convergent power series expansion in $t$ for small $t$. Further pressure $P$ is eliminated by projection onto the space of divergence free-field.

## 2. Navier-Stokes Equation and Integral equation

We denote by "$\hat{\ }$", the Fourier transform by "$\hat{*}$" the Fourier convolution ("$*$" is the Laplace convolution) and assume for simplicity that the forcing is time independent. We also denote Fourier transform by $\mathcal{F}$ and its inverse by $\mathcal{F}^{-1}$(4). For a $2\pi$ periodic box problem, in Fourier space, NS reads (see e.g. [5], [26], [11], [10])

$$\hat{v}_t + \nu|k|^2\hat{v} = -ik_jP_k\left[\hat{v}_j\hat{*}\hat{v}\right] + \hat{f}\ ,\ \ \hat{v}(k,0) = \hat{v}^{[0]}(k),\ \ \hat{v} = (\hat{v}_j)_{j=1,2,3} \tag{2.5}$$

where

$$P_k \equiv \left(1 - \frac{k(k\cdot)}{|k|^2}\right) \tag{2.6}$$

is the Fourier transform of the Hodge projection $P$, the projection on the space of divergence free vector-fields. We also follow the usual convention of summation over repeated indices. When $x \in \mathbb{T}^3[0,2\pi]^3$, we take $k = (k_1,k_2,k_3)$ an ordered triple of integers, *i.e.* $k \in \mathbb{Z}^3$, while if $x \in \mathbb{R}^3$ we would take $k \in \mathbb{R}^3$. Without loss of generality we can assume that the average velocity and force over a period are zero, implying $\hat{v}(0,t) = 0$ and $\hat{f}(0) = 0$.

We write $\hat{v} = \hat{v}^{[0]} + \hat{u}$ and apply the formal inverse Laplace transform in $1/t$ to the resulting equation to obtain

$$\begin{aligned} p\hat{U}_{pp} + 2\hat{U}_p + \nu|k|^2\hat{U} &= -ik_jP_k\left[v_j^{[0]}\hat{*}\hat{U} + \hat{U}_j\hat{*}\hat{v}^{[0]} + \hat{U}_j \overset{*}{*} \hat{U}\right] + \hat{v}^{[1]}(k)\delta(p) \\ &:= -ik_j\hat{H}_j(k,p) + \hat{v}^{[1]}(k)\delta(p) \end{aligned} \tag{2.7}$$

The solution to the homogeneous equation on the left side of (2.7) can be expressed in terms of the Bessel functions $J_1$ and $Y_1$. Using boundedness of $\hat{U}(k,p)$ at $p=0$ (which follows from $\hat{v}(k,0) = \hat{v}^{[0]}(k)$) one obtains the integral equation (IE) (see [9] for more details):

$$\hat{U}(k,p) = -ik_j\int_0^p \mathcal{G}(z,z')\hat{H}_j(k,p')dp' + \hat{U}^{[0]}(k,p) \equiv \mathcal{N}\left[\hat{U}\right](k,p)\ , \tag{2.8}$$

$$\text{where } \hat{U}^{[0]}(k,p) \equiv \frac{2J_1(z)}{z}\hat{v}^{[1]}(k)\ ,\ \ z = 2|k|\sqrt{\nu p},\ \ z' = 2k\sqrt{\nu p'}\ ,$$

$$\mathcal{G}(z,z') = \frac{\pi z'}{z}\left(J_1(z')Y_1(z) - Y_1(z')J_1(z)\right),$$

where $\overset{*}{*}$ denotes Fourier transform followed by Laplace transform and

$$\hat{v}^{[1]}(k) = -\nu|k|^2\hat{v}^{[0]} - ik_jP_k\left[\hat{v}_j^{[0]}\hat{*}\hat{v}^{[0]}\right] + \hat{f}(k) \tag{2.9}$$

### 2.1. Properties of the integral operator $\mathcal{N}$ and of the solution to IE (2.8).

For analysis of the IE, it is convenient to define a number of different spaces of functions and corresponding norms.

**Definition 2.1.** We denote by $l^1(\mathbb{Z}^3)$ the set of functions $\hat{f}$ of $k \in \mathbb{Z}^3$) such that(5)

$$\|\hat{f}(k)\|_{l^1(\mathbb{Z}^3)} = \sum_{k\in\mathbb{Z}^3}|\hat{f}(k)|$$

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(4) For periodic problem, $\mathcal{F}^{-1}$ is simply evaluation of a function based on its Fourier coefficients

(5) Since for NS the velocity and forcing have the property that $\hat{f}(0) = 0 = \hat{v}(0,t)$, the $k=0$ sum is left out in the $l^1$ sum.

Also, for analytic functions $f(x)$ whose series coefficients are possibly exponentially decaying functions in $\mathbb{Z}^3$, it is convenient to define in the Fourier space the norm:

$$\|\hat{f}\|_{\mu,\beta} = e^{\beta|k|}(1+|k|)^{\mu}|\hat{f}|$$

**Definition 2.2.** For $\alpha \geq 0$, we define define the norm $\|.\|_1^{(\alpha)}$ on functions of $(k,p)$, $k \in \mathbb{Z}^3$, $p$ real, with $p \geq 0$ (*i.e.* $p \in \mathbb{R}^+$):

$$\|\hat{U}\|_1^{(\alpha)} \equiv \int_0^\infty e^{-\alpha p}\left\{\sum_{k\in\mathbb{Z}^3}|\hat{U}(k,p)|\right\}dp = \int_0^\infty e^{-\alpha p}\|\hat{U}(\cdot,p)|_{l^1(\mathbb{Z}^3)}dp \tag{2.10}$$

**Definition 2.3.** We define $\mathcal{A}_1^{(\alpha)}$ to be the Banach space of functions $\hat{U}(k,p)$ that are $l^1(\mathbb{Z}^3)$ in $k \in \mathbb{Z}^3$ and absolutely integrable in $p$ with $\|\hat{U}\|_1^{(\alpha)} < \infty$.

**Definition 2.4.** For analytic $v^{[0]}$ and forcing $f$, it is convenient to define for $\beta \geq 0$, $\mu > 3$, the following space $\mathcal{A}$ of functions of $(k,p)$, bounded in $k$ and continuous in $p \in \mathbb{R}^+$ with

$$\|\hat{U}\| = \sup_{p\in\mathbb{R}^+} e^{-\alpha p}(1+p^2)\left[\sup_{k\in\mathbb{Z}}(1+|k|)^{\mu}e^{\beta|k|}|\hat{U}(k,p)|\right] = \sup_{p\in\mathbb{R}^+} e^{-\alpha p}(1+p^2)\|\hat{U}(\cdot,p)\|_{\mu,\beta}$$

We have the following theorems:

**Theorem 2.1.** *If $|k|^2\hat{v}_0, \hat{f} \in l^1$, then the integral equation (2.8) has a unique solution in the space $\mathcal{A}_1^{(\alpha)}$, if $\alpha$ is large enough. Taking the Laplace transform it follows that*

$$\hat{v}(k,t) = \hat{v}_0(k) + \int_0^\infty \hat{U}(k,p)e^{-p/t}dp \tag{2.11}$$

*satisfies the Navier-Stokes equation (2.5) in Fourier space. The generated Fourier series $v(x,t) = \mathcal{F}^{-1}[\hat{v}(.t)](x)$ is a classical solution to Navier-Stokes for $t \in \left(0, \frac{1}{\alpha}\right)$. If $\|\hat{v}_0\|_{\mu+2,\beta}$ and $\|f\|_{\mu,\beta}$ are finite for $\mu > 3$ and $\beta \geq 0$, then $\hat{U} \in \mathcal{A}$ for sufficiently large $\alpha$. Furthermore, in this case if $\beta > 0$, $\hat{U}$ is analytic for $p \in \mathbb{R}^+ \cup \{0\}$.*

**Outline of the proof:** The detailed proof of the theorem is given in Theorem 1 in [12], though a more general IE is considered; the Theorem 2.1 here corresponds to the special case $n = 1$ in [12]. The proof of the last statement for $n = 1$ is given in [9]. The key feature of the proof is the boundedness of $|k|\mathcal{G}$ for $z, z' \in \mathbb{R}^+$ for $z' \leq z$ (*i.e.* for $p' \leq p$), which follows from the properties of $J_1$ and $Y_1$. Therefore,

$$\|\mathcal{N}\left[\hat{U}\right](\cdot,p)\|_{l^1(\mathbb{Z}^3)} \leq \frac{C}{\sqrt{p}}\int_0^p\left[\|\hat{v}^{[0]}\|_{l^1}\|\hat{U}(\cdot,s)\|_{l^1} + \|\hat{U}(\cdot,s)\|_{l^1} * \|\hat{U}(\cdot,s)\|_{l^1}\right]ds + \|\hat{v}^{[1]}\|_{l^1}$$

and from the properties of Laplace convolutions we obtain

$$\|\mathcal{N}\left[\hat{U}\right]\|_1^{(\alpha)} \leq C\alpha^{-1/2}\|\hat{U}\|_1^{(\alpha)}\left(\|\hat{v}^{[0]}\|_{l^1} + \|\hat{U}\|_1^{(\alpha)}\right) + \frac{1}{\alpha}\|\hat{v}^{[1]}\|_{l^1},$$

and in a similar manner

$$\|\mathcal{N}\left[\hat{U}^{[1]}\right] - \mathcal{N}\left[\hat{U}^{[2]}\right]\|_1^{(\alpha)} \leq C\alpha^{-1/2}\|\hat{U}^{[1]} - \hat{U}^{[2]}\|_1^{(\alpha)}\left(\|\hat{v}^{[0]}\|_{l^1} + \|\hat{U}^{[1]}\|_1^{(\alpha)} + \|\hat{U}^{[2]}\|_1^{(\alpha)}\right)$$

It follows that for large enough $\alpha$, $\mathcal{N}$ is contractive with respect to $\|.\|_1^{(\alpha)}$ in the ball of radius $\frac{2}{\alpha}\|\hat{v}^{[1]}\|_{l^1}$. The transformations are easily undone to obtain a classical solution to the 3d NS equation for $t \in \left(0, \alpha^{-1}\right)^{(6)}$ satisfying the initial condition. Conversely, since a smooth solution to (1.1) is known to be unique, its Fourier transform must be expressible as (2.11), implying the solution is analytic in $t$ for Re $t^{-1} > \alpha$ for some $\alpha$. The inverse Laplace transform $\mathcal{L}^{-1}\left[\hat{v}(k,\tau^{-1}) - \hat{v}_0\right](p) = \hat{U}(k,p)$ must exist and satisfy IE (2.8) Therefore, the solution to (2.8) is unique, without any restriction on the ball size in the Banach space.

**Remark 2.5.** The main significance of Theorem 2.1 is not that there exists a unique smooth solution to 3-D Navier-Stokes locally in time. This has been a standard result for many years (see for instance [11], [26], [10], [5]). The connection with the integral equation (2.8) is more significant. Its solution $\hat{U}(k,p)$ exists for $p \in \mathbb{R}^+$. If this solution does not grow with $p$ or grows at most subexponentially, then 3-D NS will have global solution for the particular initial condition in question. So, in a sense the problem of global existence has become one of asymptotics. We will see later that this connection can be made stronger. Furthermore, a numerically discretized solution to the integral equation has rigorously controlled errors as will be discussed later unlike numerical schemes for N-S.

## 3. Revised estimates on the exponent $\alpha$

The existence interval $\left[0, \alpha^{-1}\right)$ for 3-D NS proved in Theorem 2.1 is suboptimal. It does not take into account the fact that initial data $v^{[0]}$ and forcing $f$ are real valued. (Blow up of Navier-Stokes solution for particular complex initial data is known [23].) Also, the estimates ignore possible cancellations in the integrals.

In the following, we address the issue of sharpening the estimates, in principle arbitrarily well, based on more detailed knowledge of the solution of the IE on a $p$-interval $[0, p_0]$. This knowledge may come from a computer assisted set of estimates, on *a priori* information based on optimal truncation of asymptotic series among others as explained in the next section. If this information shows that the solution is small for $p$ towards the right end of the interval, then $\alpha$ can be shown to be small. This in turn results in longer times of guaranteed existence, possibly global existence for $f = 0$ if this time exceeds $T_c$, the time after which it is known that a Leray weak solution becomes classical again because of long term effect of viscosity.

To get a mathematical sense of how such estimates are possible from the integral equation (2.8), define

$$\hat{U}^{(a)}(k,p) = \hat{U}(k,p) \text{ for } p \le p_0 \text{ and } 0 \text{ otherwise}$$

Define $\hat{W} = \hat{U} - \hat{U}^{(a)}$, where we see that $\hat{W}$ is nonzero only for $p > p_0$. Then, from (2.8) we have for $p > p_0$,

$$\hat{W}(k,p) = \hat{W}^{(0)}(k,p) - ik_j \int_{p_0}^{p} \mathcal{G}(z,z') \hat{H}_j^{(w)}(k,p') dp' \equiv \mathcal{N}^{(w)}\left[\hat{W}\right](k,p), \tag{3.12}$$

where

$$\hat{H}^{(w)}(k,p) = P_k \left[ \hat{v}_j^{[0]} \hat{*} \hat{W} + \hat{W}_j \hat{*} \hat{v}^{[0]} + \hat{W}_j \hat{*}_{*} \hat{U}^{(a)} + \hat{U}_j^{(a)} \hat{*}_{*} \hat{W} + \hat{W}_j \hat{*}_{*} \hat{W} \right] \tag{3.13}$$

$$\hat{W}^{(0)}(k,p) = 2\frac{J_1(z)}{z} \hat{v}^{[1]}(k) - ik_j \int_0^{\min p, 2p_0} \mathcal{G}(z,z') \hat{H}^{(a)}(k,p') dp', \tag{3.14}$$

(6) The solution immediately smooths out in $x$ for $t > 0$

and

$$\hat{H}^{(a)} = \hat{v}_j^{[0]} \hat{*} \hat{U}^{(a)} + \hat{U}_j^{(a)} \hat{*} \hat{v}^{[0]} + \hat{U}_j^{(a)} \hat{*}_{*} \hat{U}^{(a)} \tag{3.15}$$

We note that if the calculated $\hat{U}^{(a)}$ is seen to rapidly decrease in some subinterval $[p_d, p_0]$, then the inhomogeneous term $\hat{W}^{(0)}$ in the integral equation (3.12) becomes small. For sufficiently large $p_0$, the factor $\frac{1}{\sqrt{p}}$ multiplying integral term in (3.12) is also small for $p \geq p_0$. This ensures contractivity of operator $\mathcal{N}^{(w)}$ at a smaller $\alpha$. Precise statements on estimates on $\alpha$ based $\hat{U}^{(a)}$ is given below in Theorem 3.1.

3.0.1. *Sharper estimates.* Let $\hat{U}(k,q)$ be the solution of of (2.8) provided by Theorem 2.1. Define

$$\hat{U}^{(a)}(k,p) = \begin{cases} \hat{U}(k,p) & (0, p_0] \subset \mathbb{R}^+ \\ 0 & \text{otherwise} \end{cases} \tag{3.16}$$

$$\hat{U}^{(s)}(k,q) = -ik_j \int_0^p \mathcal{G}(p,p';k) \hat{H}_j^{(a)}(k,p')\, dp' + \hat{U}^0(k,p),$$
$$\hat{H}_j^{(a)}(k,p) = P_k \left[ \hat{v}_{0,j} \hat{*} \hat{U}^{(a)} + \hat{U}_j^{(a)} \hat{*} \hat{v}_0 + \hat{U}_j^{(a)} \hat{*}_{*} \hat{U}^{(a)} \right](k,p),$$

Using (3.16) we introduce the following functions of $\hat{U}^{(a)}(k,p)$, $\hat{v}_0$ and $\hat{f}$:

$$b := \alpha \int_{p_0}^{\infty} e^{-\alpha p} \|\hat{U}^{(s)}(\cdot,p)\|_{l^1}\, dp, \tag{3.17}$$

$$\epsilon_1 = \left[ C_1 + \int_0^{p_0} e^{-\alpha p'} C_2(p')\, dp' \right] \tag{3.18}$$

Finally, we let

$$C_0(k) = \sup_{p_0 \leq p' \leq p} \left\{ |\mathcal{G}(p,p';k)| \right\}, \qquad C_1 = 4 \sup_{k \in \mathbb{Z}^3} \left\{ |k| C_0(k) \right\} \|\hat{v}_0\|_{l^1},$$
$$C_2(q) = 4 \sup_{k \in \mathbb{Z}^3} \left\{ |k| C_0(k) \right\} \|\hat{U}^{(a)}(\cdot,p)\|_{l^1}, \qquad \epsilon = 2 \sup_{k \in \mathbb{Z}^3} \left\{ |k| C_0(k) \right\}.$$

**Theorem 3.1.** *(i) The exponential growth rate $\alpha$ of $\hat{U}$ is estimated in terms of the restriction of $\hat{U}$ to $[0, p_0]$ as follows.*

$$\text{If} \quad \alpha > \epsilon_1 + 2\sqrt{\epsilon b} \quad \text{then} \quad \int_0^{\infty} \hat{\|} U(\cdot,p) \|_{l^1} e^{-\alpha p} dp < \infty \tag{3.19}$$

**Remark 3.1.** It was argued in [12], in a slightly more general context, that the estimated $\epsilon_1 + \sqrt{2b\epsilon}$ is small if the solution $\hat{U}^{(a)}$ is small in some subinterval $[p_d, p_0]$. This implies imply a long interval $(0, \alpha^{-1})$ of guaranteed existence of a solution to (1.1).

## 4. Calculation of integral equation solution over $[0, p_0]$

The results in Theorem 3.1 rely on the knowledge of $\hat{U}$ for $p \in [0, p_0]$. There are two approaches to calculating the solution over $[0, p_0]$. One is based on using the power series of $\hat{U}(k,p)$ at $p = 0$, while the second approach relies on numerical solution on $[0, p_0]$. Unlike Navier-Stokes solution, the numerical soution to the integral equation (2.8) has errors that can be completely controlled.

4.1. **Power Series for $\hat{U}(k,p)$ and its radius of convergence.** When the initial data and forcing are analytic, the formal series (1.2) is useful since it implies

$$\hat{U}(k,p) = \sum_{m=1}^{\infty} \hat{v}_m(k) \frac{p^{m-1}}{m!} \tag{4.20}$$

Borel summability ensures[9] that the series (4.20) has a finite radius of convergence. The results in [9] only give a radius of convergence that depends on $v_0$ and $f$. However, we expect more to be true. We now show that in the special case when the initial condition has a finite number of Fourier modes, the radius of convergence is independent of the size of initial data and forcing. The argument below allows for $f$ to be a function of time as well, provided it is analytic in $t$ at $t=0$.

We note the small time expasion:

$$\widehat{v}(k,t) = \widehat{v}^{[0]}(k) + \sum_{m=1}^{\infty} \widehat{v}^{[m]}(k)\, t^m$$
$$\widehat{f}(k,t) = \widehat{f}^{[0]} + \sum_{m=1}^{\infty} \widehat{f}^{[m]}(k)\, t^m$$

where

$$\widehat{v}^{[m+1]} = \frac{1}{m+1}\left[\widehat{f}^{[m]} - \nu|k|^2\widehat{v}^{[m]} - ik_j P_k\left(\sum_{\ell=0}^{m} \widehat{v}_j^{[l]} \hat{*} \widehat{v}^{[m-\ell]}\right)\right], \qquad m \geq 0 \tag{4.21}$$

and recall that for small $p$,

$$\widehat{U}(k,p) = \sum_{m=0}^{\infty} \widehat{d}_m(k)\, p^m, \qquad \widehat{d}_m = \frac{\widehat{c}_{m+1}}{m!}. \tag{4.22}$$

If the power series converges for a large range of $p$, then, in principle, we can use it to compute $\widehat{U}$ as accurately as we wish. In order for this to be possible, though, we must understand how the radius of convergence of the series depends on $\nu$, the viscosity, and the size of the initial data $\widehat{v}^{[0]}$ and forcing $f$.

Taking the $\ell^1$-norm on both sides of (4.21) with respect to $k$ and writing

$$a_m = \|\widehat{v}^{[m]}\|_{\ell^1}, \qquad b_m = \|\widehat{f}^{[m]}\|_{\ell^1},$$

we obtain

$$a_{m+1} \leq \frac{1}{m+1}\left[b_m + \nu\big\||k|^2|\widehat{v}^{[m]}|\big\|_{\ell^1} + 2\sum_{\ell=0}^{m} \big\||k||\widehat{v}^{[\ell]}|\hat{*}|\widehat{v}^{[m-\ell]}|\big\|_{\ell^1}\right].$$

If the initial data $\widehat{v}^{[0]}$ has only a finite number of modes (which is the case here), we define $K_0 = \sup_{k\in \text{supp } v^{[0]}} |k| < \infty$. Then by induction, we can prove that $\sup_{k\in\text{supp } v^{[m]}} |k| = (m+1)K_0 < \infty$. It follows that

$$a_{m+1} \leq \frac{1}{m+1}\left[b_m + K_0^2\nu(m+1)^2 a_m + 2K_0(m+2)\sum_{\ell=0}^{m} a_\ell a_{m-\ell}\right]$$
$$\leq \frac{b_m}{m+1} + K_0^2\nu(m+1)a_m + 4K_0\sum_{\ell=0}^{m} a_\ell a_{m-\ell}.$$

Consider now the formal power series

$$y_0(t) := \sum_{m=0}^{\infty} \widetilde{a}_m t^m$$

where

$$\widetilde{a}_0 = a_0,$$

$$\widetilde{a}_{m+1} = \frac{b_m}{m+1} + K_0^2 \nu (m+1) \widetilde{a}_m + 4K_0 \sum_{\ell=0}^{m} \widetilde{a}_\ell \widetilde{a}_{m-\ell}, \qquad m \geq 0. \tag{4.23}$$

Clearly $a_m \leq \widetilde{a}_m$ for all $m \geq 0$, as it can be shown by a simple induction. If we multiply both sides of (4.23) by $t^m$ and sum over $m$ from 0 to $\infty$, then

$$\sum_{m=0}^{\infty} \widetilde{a}_{m+1} t^m = \sum_{m=0}^{\infty} \frac{b_m}{m+1} t^m + K_0^2 \nu \sum_{m=0}^{\infty} (m+1) \widetilde{a}_m t^m + 4K_0 \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \widetilde{a}_\ell \widetilde{a}_{m-\ell}\, t^m,$$

which shows that $y_0$ is a formal power series solution of the ODE

$$\frac{1}{t}(y - \widetilde{a}_0) = w + K_0^2 \nu (ty)' + 4K_0 y^2, \qquad \text{where} \qquad w(t) = \sum_{m=0}^{\infty} \frac{b_m}{m+1} t^m.$$

This equation can be written as

$$K_0^2 \nu t^2 y' + 4K_0 t y^2 + (K_0^2 \nu t - 1) y + (tw + \widetilde{a}_0) = 0,$$

or, with the change of variable $s = 1/t$,

$$-K_0^2 \nu y' + 4K_0 s^{-1} y^2 + (K_0^2 \nu s^{-1} - 1) y + (s^{-1} w + \widetilde{a}_0) = 0.$$

According to theory [8], any singularity of $\mathcal{B}[y]$ in Borel plane must exhibit itself as an exponentially small correction to $y_0$. Hence we write

$$y = y_0 + \delta$$

and find the ODE for $\delta$:

$$-K_0^2 \nu \delta' + 4K_0 s^{-1} (\delta^2 + 2z_0 \delta) + (K_0^2 \nu s^{-1} - 1) \delta = 0.$$

The dominant balance of this equation is

$$-K_0^2 \nu \delta' + \Big[ (8\widetilde{a}_0 K_0 + K_0^2 \nu) s^{-1} - 1 \Big] \delta = 0,$$

which gives the leading asymptotics of $\delta$:

$$\delta \sim e^{-K_0^{-2} \nu^{-1} s} s^{8\widetilde{a}_0 K_0^{-1} \nu^{-1} + 1}.$$

It follows from [8] that the radius of convergence of $\mathcal{B}[y]$, hence that of (4.22), is at least $K_0^{-2} \nu^{-1}$, and clearly this is independent of the size of the initial data and forcing.

4.2. **Control of numerical errors in a discretized scheme.** The errors in a numerical discretization scheme for 3-D Navier-Stokes cannot be readily controlled since these depend on derivatives of the classical solution; and these are not known to exist beyond some initial time interval. In contrast to Navier-Stokes PDE, the $p$ derivatives of solution $\hat{U}$ to the integral equation (2.8), are *a priori* bounded on any interval $[0, p_0] \subset \mathbb{R}^+$. Calculation of the numerical solution to the integral equation (2.8) with rigorous error control is relevant to Navier-Stokes solution in more than one way.

As argued earlier, the knowledge of solution over an interval $[0, p_0]$ provides a refined estimate of the exponent $\alpha$ for the solution $\hat{U}(k,p)$ to the IE, and indeed of the existence interval for the NS problem.

Further more, a numerical scheme to calculate solution to (2.8), which is analyzed in this section is interesting in its own right. It provides through Laplace Transform an alternate calculation of Navier-Stokes dynamics. Evidently, this method is not numerically efficient to determine $v(x,t)$ for fixed time $t$; nonetheless it may be advantageous in calculation of some long time averages involving $v$ and $\nabla v$ that occur in analysis of turbulent flow. These can sometimes be expressed as functionals of $\hat{U}$.

**Definition 4.1.** *We introduce $\mathcal{N}^{(N)}$ to be the integral operator based on Galerkin projection onto the space $k \in [-N, N]^3 \setminus \mathbf{0}$*

$$\mathcal{N}^{(N)}\left[\hat{V}\right](k,p) \;=\; \mathcal{P}_N \hat{U}^{[0]}(k,p) \;-\; ik_j \int_0^p \mathcal{G}(z,z';k)\mathcal{P}_N \hat{H}_j^{(N)}(p';k)dq' \tag{4.24}$$

*where $\mathcal{P}_N$ is the Galerkin projection to the space $k \in [-N,N]^3 \setminus \{0\}$ and*

$$\hat{H}_j^{(N)} = P_k\left[\hat{v}_j^{[0]} \hat{*} \hat{V} + \hat{V}_j \hat{*} \hat{v}^{[0]} + \hat{V}_j \hat{\underset{*}{*}} \hat{V}\right], \tag{4.25}$$

**Remark 4.2.** *The proof of existence and uniqueness of the solution to the integral equation $\hat{U}^{(N)} = \mathcal{N}^{(N)}\left[\hat{U}^{(N)}\right]$ is very similar to the one sketched for $\hat{U}$. It is also easily shown* [12] *that $\|\hat{U} - \hat{U}^{(N)}\|^{(\alpha)} = O(1/N)$ as $N \to \infty$.*

**Definition 4.3.** *We introduce a discrete operator $\mathcal{N}_\delta^{(N)}$ by*

$$\begin{aligned}\left\{\mathcal{N}_\delta^{(N)}\left[\hat{V}\right]\right\}(k,m\delta) = -ik_j \sum_{m'=0}^{m-1} w^{(1)}(m,m';k,\delta)\mathcal{P}_N \hat{H}_{j,\delta}^{(N)}(k,m'\delta') \\ + \hat{U}^{(0)}(k,m\delta) - ik_j w^{(1,1)}(m,k,\delta)\mathcal{P}_N \hat{H}_{j,\delta}^{(N)}(k,m\delta)\end{aligned} \tag{4.26}$$

*where $m$ is a positive integer,*

$$\begin{aligned}\hat{H}_{j,\delta}^{(N)}(k,m'\delta) = &\sum_{k' \in [-N,N]^3 \setminus \{0,k\}} P_k\left[\hat{v}_{0,j}(k')\hat{V}(k-k',m'\delta) + \hat{v}_0(k')\hat{V}(k-k',m'\delta)\right] \\ + &\sum_{k' \in [-N,N]^3 \setminus \{0,k\}, m''=0..,m'} P_k\left[\hat{V}_j(k',m''\delta)\hat{V}(k-k',(m'-m'')\delta)\right] w^{(2)}(m',m'';k,\delta),\end{aligned} \tag{4.27}$$

where we define $\hat{V}(k,0) = \hat{U}^{(0)}(k,0)$. The precise form of these functions and of the weights $w^{(1)}(m,m';k,\delta)$, $w^{(1,1)}(m,k,\delta)$ and $w^{(2)}(m',m'';k,\delta)$ will depend on the particular discretization scheme employed to calculate $\mathcal{N}^{(N)}\left[\hat{U}\right]$ In order to keep the discussion relevant to any reasonable uniform $\delta$ discretization of $p$, we do not specify the weights, aside from requiring consistency–that in the formal limit of $\delta \to 0$, the discrete operator $\mathcal{N}_\delta^{(N)}$ involving summation reduces to the integral operator $\mathcal{N}^{(N)}$. Based on the behavior of the kernel $\mathcal{G}$, consistency implies that

$$|k||w^{(1)}(m,m';k,\delta)| \le \frac{C_1\delta^{1/2}}{m^{1/2}} \tag{4.28}$$
$$|kw^{(1,1)}| \le C_{1,1}\delta(m\delta)^{-1/2}\ ,\ |w^{(2)}| \le C_2\delta\ ,\ |w^{(2,l)}| \le C_3\delta$$

**Definition 4.4.** *We let*

$$T_{E,\delta}^{(N)} = \mathcal{N}^{(N)}\hat{U}^{(N)} - \mathcal{N}_\delta^{(N)}\hat{U}^{(N)}$$

*be the truncation error due to q-discretization for a fixed number of Fourier modes, $[-N,N]^3$. The discretization is consistent (in the numerical analysis sense) if $T_{E,\delta}^{(N)}$ scales with some positive power of $\delta$ and involves a finite number of derivatives of $\hat{U}$.*

**Definition 4.5.** *Define the norm $\|\cdot\|^{(\alpha,\delta)}$ by*

$$\|\hat{f}\|^{(\alpha,\delta)} = \sup_{m\in\mathbb{Z}^+\cup\{0\}} (1+m^2\delta^2)e^{-\alpha m\delta}\|\hat{f}(\cdot,m\delta)\|_{l^1}$$

**Remark 4.6.** *More specific bounds on the truncation error depend on the specific numerical scheme. It is however standard for numerical quadratures to choose the weights $w^{(j)}$ so that integration in $p$ is exact for a piecewise polynomial of given order, $l$ for $p\in[0,p_0]$. For a general $\hat{V}(\cdot,p)$, the interpolation errors involve $l+1$ q-derivatives. Since $\hat{U}$ is analytic in $p$ for $p\in\mathbb{R}^+\cup\{0\}$, the derivatives of $\hat{U}$ are exponentially bounded for large $p$. It follows that $\|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)} \to 0$ as $\delta\to 0$.*

**Theorem 4.1.** *Consider a discretized integral equation consistent with (2.8), based on Galerkin truncation to $[-N,N]^3$ Fourier modes and uniform discretization in $p$,*

$$\hat{U}_\delta^{(N)} = \mathcal{N}_\delta^{(N)}\left[\hat{U}_\delta^{(N)}\right],$$

*see (4.26) below. Then, the error $\hat{U}-\hat{U}_\delta^{(N)}$ at the points $p=m\delta$, $m\ge 1\in\mathbb{Z}^+$ satisfies*

$$\|\hat{U}(\cdot,m\delta)-\hat{U}_\delta^{(N)}(\cdot,m\delta)\|_{l^1} \le \left[\|T_{E,N}+T_{E,\delta}^{(N)}\|^{(\alpha,\delta)}\right]\frac{e^{\alpha m\delta}\delta}{(1+m^2\delta^2)} \tag{4.29}$$

*for $m\ge 1$. In (4.1), $T_{E,N}$ is the truncation error due to Galerkin projection and $T_{E,\delta}^{(N)}$ is the truncation error due to the $\delta$-discretization in $p$ for a given $N$, where $\|T_{E,N}\|^{(\alpha,\delta)}\to 0$ as $N\to\infty$ for any $\delta$ and $\|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)}\to 0$ as $\delta\to 0$.*

**Remark 4.7.** *A detailed proof is given in* [12] *for the more general integral equation obtained with accelerated variable $q$ instead of $p$. Indeed, using an accelerated variable offers the advantage that the $\|T_{E,\delta}^{(N)}\|^{\alpha,\delta}\to 0$ as $\delta\to 0$, uniformly in $N$. This is not the case for $n=1$, i.e. when we use $p$.*

## 5. Acceleration

We have already established that at most subexponential growth of $\|\hat{U}(\cdot,p)\|_{l^1}$ implies global existence of a classical solution to (1.1).

We now look for a converse: suppose (1.1) has a global solution, is it true that $\hat{U}(\cdot,p)$ always is subexponential in $p$ ? The answer is no in general. Any complex singularity $t_s$ in the right-half complex $t$-plane of $v(x,t)$ produces exponential growth of $\hat{U}$ with rate $\mathrm{Re}\, t_s$ (oscillatory with a frequency $\mathrm{Im}\, t_s$), as it is seen by looking at the the asymptotics of inverse the Laplace transform.

However, when there is no forcing $f=0$, it can be proved (see Theorem 5.1 for precise statements) that given a global classical solution of (1.1), there is a $c>0$ so that for any $t_s$ we have $|\arg t_s|>c$. This means that for sufficiently large $n$, the function $v(x,\tau^{-1/n})$ has no singularity in the right-half $\tau$ plane. Then the inverse Laplace transform

$$U_{acc}(x,q)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{v(x,\tau^{-1/n})-v^{[0]}(x)\right\}e^{q\tau}d\tau \tag{5.30}$$

can be shown to decay as $q\to\infty$, reflecting the exponential decay of $v(x,t)$ for large $t$.

This means that it is advantageous to find $U_{acc}(x,q)$ so that the generalized Laplace transform representation

$$v(x,t)=v^{[0]}(x)+\int_0^{\infty}U_{acc}(x,q)e^{-q/t^n}dq \tag{5.31}$$

gives a solution to (1.1). The transformation from $U(x,p)$ to $U_{acc}(x,q)$ is referred to as acceleration and was first used in one variable by Écalle. In fact, there is an integral transformation that directly relate $U$ to $U_{acc}$, though this is not used in the analysis.

The resulting integral equation for $\hat{U}(k,q)$ has been analyzed [12] and results similar to Theorems 2.1 and 3.1 hold. Indeed, preliminary numerical calculations, described in §5.1 give encouraging results.
As mentioned in §5.1, while subexponential growth of $\hat{U}$ guarantees global solution to (1.1), the converse is not true if there exist purely complex right half $t$- plane singularities. The following theorem shows however that a converse is true, for $f=0$, *after suitable acceleration.* Thus the problem of global existence of (1.1) becomes a problem of asymptotics for solution $\hat{U}_{acc}$.

**Theorem 5.1.** *Assume that $\hat{v}_0\in l^1(\mathbb{Z}^3)$, $\nu>0$ and $\hat{f}=0$ (zero forcing). If NS has a global classical solution, then there exists $n$ large enough so that $U_{acc}(x,q)=O(e^{-Cq^{1/(n+1)}})$ as $q\to+\infty$, for some $C>0$ .*

The proof of this theorem in given in [12].

**Remark 5.1.** Together with Theorem 2.1, Theorem 5.1 shows that global existence is equivalent to an asymptotic problem. The solution to NS exists for all time if and only if $\hat{U}_{acc}$ decays for some $n\in\mathbb{Z}^+$.

5.1. **Preliminary Numerical Results.** The computations described here are for $n=1$ (unaccelerated case) and $n=2$. We present the details elsewhere[12]. The code is far from being optimized and the results are only presented over a modest interval in $p$ or $q$.

5.1.1. *Kida Flow.* Now we consider the Kida initial condition [14]

$$v_1(x_1,x_2,x_3,0) = \sin x_1(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3).$$

The high degree of symmetry (preserved in time) lowers the number of computational operations. We computed the solution for $\nu = 0.1$ and no forcing, for a Reynolds number $Re = 20\pi$. This is not very large, but we were mainly interested in testing the concepts developed. We used $q_0 = 10$ and used 128 points in each space dimension, *i.e.* $N = 64$, and step size $\Delta q = 0.05$. To investigate the growth of the solution $\hat{U}$ with $q$, we computed the $l^1$-norm

$$\|\hat{U}(\cdot,q)\|_{l^1,N} := \sum_{k\in[-N,N]^3} |\hat{U}(k,q)|$$

and plotted $\|\hat{U}_1(\cdot,q)\|_{l^1,128}$ vs. $q$ in Fig.1. For comparison we also included in Fig.1 a plot of the solution to the original (unaccelerated) equation, *i.e.* $q = p$ case.

Singularities in the right half plane, if present, come in conjugate pairs because of reality of the solution. These exponential growth (mixed with oscillations) of $\hat{U}(k,p)$, seen in Fig.1 (a). The oscillation parameters depend on $k$ while the growth rate is virtually insensitive to $k$. By monitoring the oscillation against the growth rate of each of the modes, we predicted that acceleration with $n = 2$ would eliminate singularities in the right-half $\tau = 1/t^n$ plane. This expectation is confirmed in Fig.1 (b), in which it is seen that $\|\hat{U}(.,q)\|_{l_1}$ decreases beyond some $q$. Actually, it has been shown in [12] that if (1.1) has a global solution and appropriate acceleration is used, then $\|\hat{U}(.,q)\|_{l^1}$ is $O(e^{-cq^{1/(n+1)}})$. Remarkably, though the interval of calculation is only modestly long, $[0,10]$, this asymptotic trend is clear in Fig.2. For large enough $q$, the low $k$ modes dominate, while for smaller $q$ more modes contribute to the norm, and this explains the damped oscillation in present in Fig.2. It is remarkable that a computation over a moderate $q$-interval can capture the large $q$-trend.

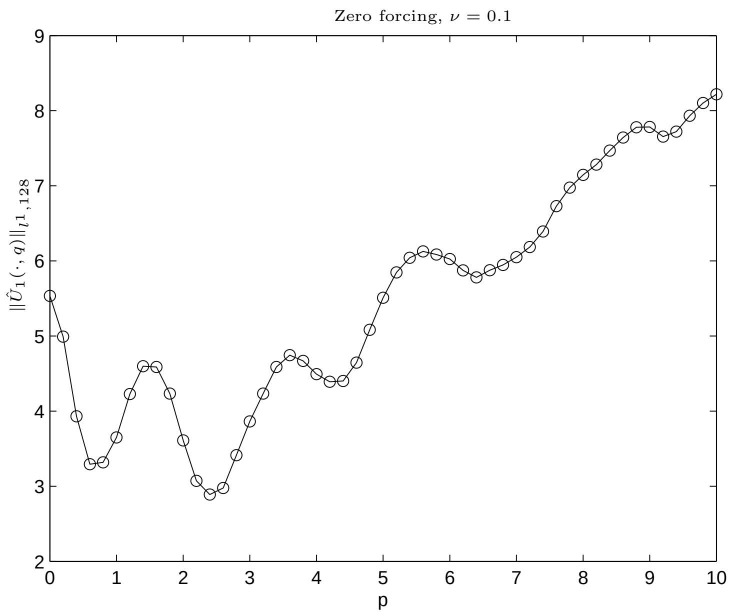
## 6. Discussion and Conclusion

We have shown here how an integral equation approach akin to Borel summation techniques is useful to the study of evolution PDEs, such as for 3-D incompressible Navier-Stokes. In this formulation, the PDE problem becomes equivalent to a nonlinear integral equations with unique solutions. These solutions are smooth and analytic in the dual variable providing control of the errors. Furthermore, global existence becomes an asymptotic problem. We also illustrate how to obtain better asymptotic estimates can be obtained if the solution for a finite interval $[0,p_0]$ is known. Preliminary numerical results are encouraging.

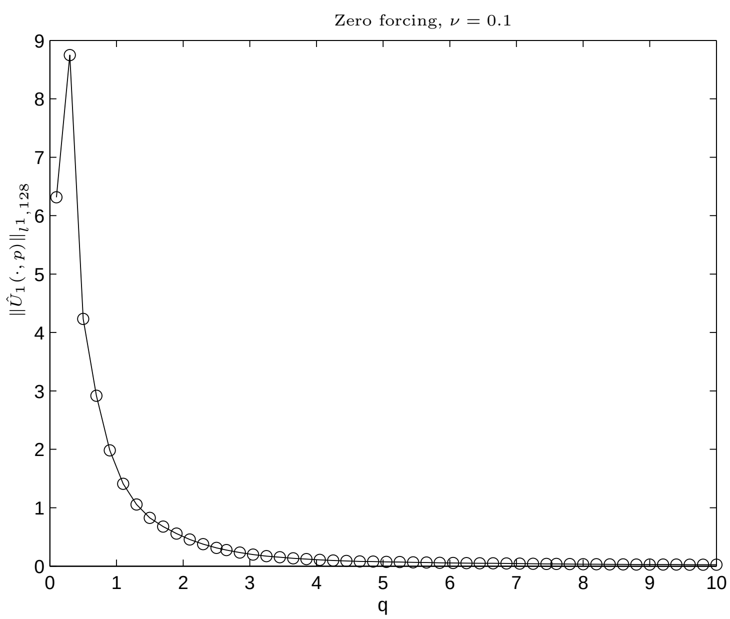
Indeed, the numerical approach outlined here, once fully optimized, is likely to be of interest in its own right since long-time averages one encounters in turbulence study may be more accurately computable in the dual variable representation rather than employing a DNS.

## 7. Acknowledgments.

This work was supported in part by the National Science Foundation (DMS-0406193, DMS-0601226, DMS-0600369 to OC and (DMS-0405837 to S.T). ST is also indebted to EPSRC and the Math Institute at Imperial for support during his year long visit to Imperial where this project got initiated.




(a)




(b)

FIGURE 1. For zero forcing and $\nu = 0.1$: (a). The original (unaccelerated) equation, $\|\hat{U}_1(\cdot, p)\|_{l^1,128}$ vs. $p$. (b). Accelerated equation with $n = 2$, $\|\hat{U}_1(\cdot, q)\|_{l^1,128}$ vs. $q$.

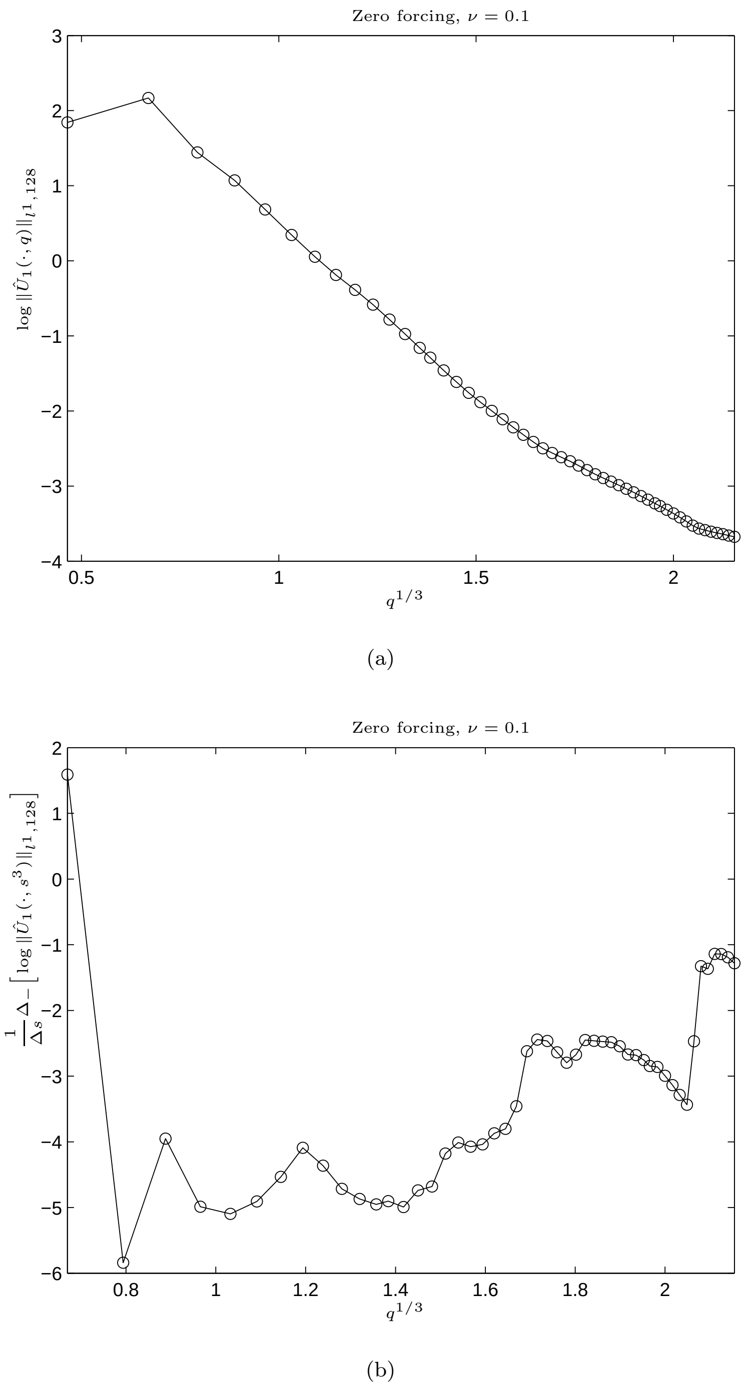



FIGURE 2. Asymptotic behavior of $\|\hat{U}_1(\cdot, q)\|_{l^1,128}$. (a). $\log \|\hat{U}_1(\cdot, q)\|_{l^1,128}$ vs. $q^{1/3}$. (b) $\dfrac{1}{\Delta s}\Delta_-\left[\log \|\hat{U}_1(\cdot, s^3)\|_{l^1,128}\right]$ vs. $s$, where $s = q^{1/3}$ and $\Delta_-$ is the backward difference operator in $s$.

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