Approximate analytical Tritronqué solution to P1 and rigorous estimates

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Basic Idea

Application of a very general but basic idea applicable to a whole class nonlinear problems written abstractly as $\mathcal{N}[u] = 0$.

Suppose, we determine some u_0 for which initial/BC are approximately satisfied and $\mathcal{N}[u_0] = R$ is small. Then $E = u - u_0$ satisfies

$$\mathcal{L}E = -R - \mathcal{N}_1[E] \ ,$$

where $\mathcal{L} = \mathcal{N}_u, \mathcal{N}_1[E] = \mathcal{N}[u_0 + E] - \mathcal{N}[u_0] - \mathcal{L}E$

If $\mathcal{L}$ can be suitably inverted, and nonlinearity $\mathcal{N}_1$ is regular, then we note E satisfies the weakly nonlinear equation,

$$E = E_0 - \mathcal{L}^{-1}R - \mathcal{L}^{-1}\mathcal{N}_1[E]$$

where E_0 solves $\mathcal{L}E_0 = 0$ and satisfies IC/BC.

Remarks

Inversion of this type needed in bounding $|u - u_0|$ in a problem of the type: $\mathcal{N}[u;\epsilon] = 0$ where $\mathcal{N}[u_0;0] = 0$

Previously, this idea used to determine errors in numerical solution to elliptic PDEs(Nakao *et al*, (2005); determine existence of Stokes Water Wave (Fraenkel, '07) using a rough u_0. Computer assisted proof by Kobayashi ('04)

Not recognized until recently is the determination of an accurate analytical quasi-solution u_0, together with efficient error determination.

Recent work in this direction: (Costin, Huang, Schlag, 2012), (Costin, Huang, T., 2012), (Costin, T., 2013), (Costin, Kim, T. 2014),(T. 2013) in problems arising in NLS, Proof of Dubrovin Conjecture for P-1, Blasius similarity solution, and water waves.

Painleve-1 and Tritronqué solution

Painleve-1 Equation:

$$y'' + 6y^2 - x = 0$$

Widely studied integrable ODE arising from reduction of many integrable PDEs. Dubbed as a 'nonlinear special function' (Clarkson)

Unique solution to P-1 with the following property termed as the Tritronqué (Boutroux)

$$y = \sqrt{\frac{x}{6}}\left[1 + o\left(x^{-5/8}\right)\right] \quad \text{as } x \to +\infty$$

Tritronqué properties studied before (Joshi & Kitaev ('01), Masoero ('11), Olver & Trogdon ('14), $\cdots$,) Dubrovin, Grava and Klein (2008) conjectured that the sector $\arg x \in \left(-\frac{4}{5}\pi, \frac{4}{5}\pi\right)$ is singularity free. Dubrovin conjecture proved recently (Costin, Huang & T., '14).

Past studies

Well-known Painleve solutions are single valued and meromorphic with singularity location determined by initial condition $y(x_0), y'(x_0)$ For $P-1$ singularities are double poles in $\mathbb{C}$.

Solution also characterized by $x_p, \hat{a}_2$ in the local representation:

$$y(x) = -\frac{1}{(x-x_p)^2} + (x-x_p)^2 \sum_{j=0}^{\infty} \hat{a}_j (x-x_p)^j \ , \tag{1}$$

$\hat{a}_0 = -\frac{x_p}{10}$, $\hat{a}_1 = -\frac{1}{6}$, $\hat{a}_3 = 0$ and for $n \geq 4$, $\hat{a}_n$ is determined from

$$\hat{a}_n = -\frac{6}{(n+5)(n-2)} \sum_{j=0}^{n-4} \hat{a}_k \hat{a}_{n-4-k} \tag{2}$$

Known that the closest x_p from the origin for the tritronqué is on the negative axis (Joshi & Kitaev, '01), and its location determined numerically.

Properties of tritronqué and open question

Singularties at larger distance can be rigorously estimated by adiabatic invariance of conserved quantities(Costin *et al*,'14).
When x_p is not particularly large, we are unaware of any method of rigorous analysis to confirm its location.

In general, while numerical methods have been used to calculate Painlev'e solutions to great accuracy (e.g. Fornberg), we are unaware of rigorous error determination.

Will find accurate tritronqué approximation with rigorous error bounds in some domain $D \subset \mathbb{C}$. The method is an extension of the methods used to prove Dubrovin Conjecture (Costin, Huang, T.)

Will be clear that the method is much more general and applicable to all solutions of all Painleve equations, and generally a broad class of nonlinear problems.

Key Steps

1. Coming up with a compact analytical representation of approximate solution y_0.

2. Proving that R is appropriate small and that boundary/initial conditions are satisfied to within small errors.

3. Finding bounds on $\mathcal{L}^{-1}$ good enough to apply contraction mapping theorem.

Definitions

Define $r = \frac{7}{10}$, $x_0 = -\frac{770766}{323285} = -2.384168..$, $\tau = \frac{x-(L+x_0+r)/2}{(L-x_0-r)/2}$.

Also, define $P(\zeta) = \sum_{k=0}^{17} a_n \zeta^n$, where where $a_0 = -x_0/10$,

$a_1 = -1/6$, $a_2 = \frac{19949}{321055}$, $a_3 = 0$ and

$$a_n = -\frac{6}{(n+5)(n-2)} \sum_{j=0}^{n-4} a_k a_{n-4-k} \text{ for } 17 \geq n \geq 4$$

Define $P_u(\tau) = \sum_{k=0}^{22} c_k \tau^k$ where $\mathbf{c} := (c_0, c_1, \cdots, c_{22})$ is given by

$$\mathbf{c} = \Big(\frac{335867}{539062}, \frac{419712}{989125}, -\frac{352463}{3539236}, \frac{60789}{1703279}, -\frac{132842}{11825541}, \frac{43961}{54574472}, \frac{39599}{12036926}, -\frac{213665}{48625258}, \frac{61644}{14973337}, -\frac{107283}{33444500}, \frac{44761}{18892011}, -\frac{28249}{13550715}, \frac{20641}{14839893}, \frac{13459}{92774551}, -\frac{4992}{34838093}, -\frac{11771}{8149937}, \frac{24115}{27631671}, \frac{42106}{39550107}, -\frac{21163}{32637441}, -\frac{9782}{15918509}, \frac{11581}{32652169}, \frac{14692}{88640147}, -\frac{12278}{123249611}\Big) \quad (3)$$

Definitions II

Further, take $b = \frac{4}{5}(24)^{1/4}$, $a = \frac{5}{2}b$,

$$N_0(x) = -\frac{4412401}{98304\sqrt{6}} x^{-19/2} \left[1 - \frac{1225}{90049\sqrt{6}} x^{-5/2} + \frac{30625}{2161176} x^{-5} \right]$$

$$\mathcal{G}_1(x) = x^{-5/8} \exp\left[-ibx^{5/4} \right] \ , \mathcal{G}_2(x) = x^{-5/8} \exp\left[ibx^{5/4} \right]$$

$$w_0 = \sum_{j=1}^{2} \frac{(-1)^j}{ia} \mathcal{G}_j(x) \int_{\infty}^{x} \mathcal{G}_{3-j}(y) y N_0(y) dy. = \mathrm{Re} \left\{ \int_0^{\infty} e^{-sbx^{5/4}} \mathcal{W}_0 \left(x^{5/4}, s \right) ds \right\} \ ,$$

$$\mathcal{W}_0(z,s) = -\frac{4412401\sqrt{6}}{368640 a z^7} \left((1+is)^{-15/2} - \frac{1225\sqrt{6}}{540294 z^2} \ (1+is)^{-19/2} + \frac{30625}{2161176 z^4} \ (1+is)^{-23/2} \right) \tag{4}$$

Note w_0 is known in terms of erf function

Further Definitions

We also define domains D_j for $j = 1, \cdots, 4$ with $D_1 = [5.5, \infty)$, $D_2 = [-0.49, 5.5)$, $D_3 = [x_0 + r, -0.49)$ and $D_4 = \{x \in \mathbb{C} : |x - x_0| = r, x \neq x_0 + r\}$. We define $D = D_1 \cup D_2 \cup D_3 \cup D_4$ (See Figure).

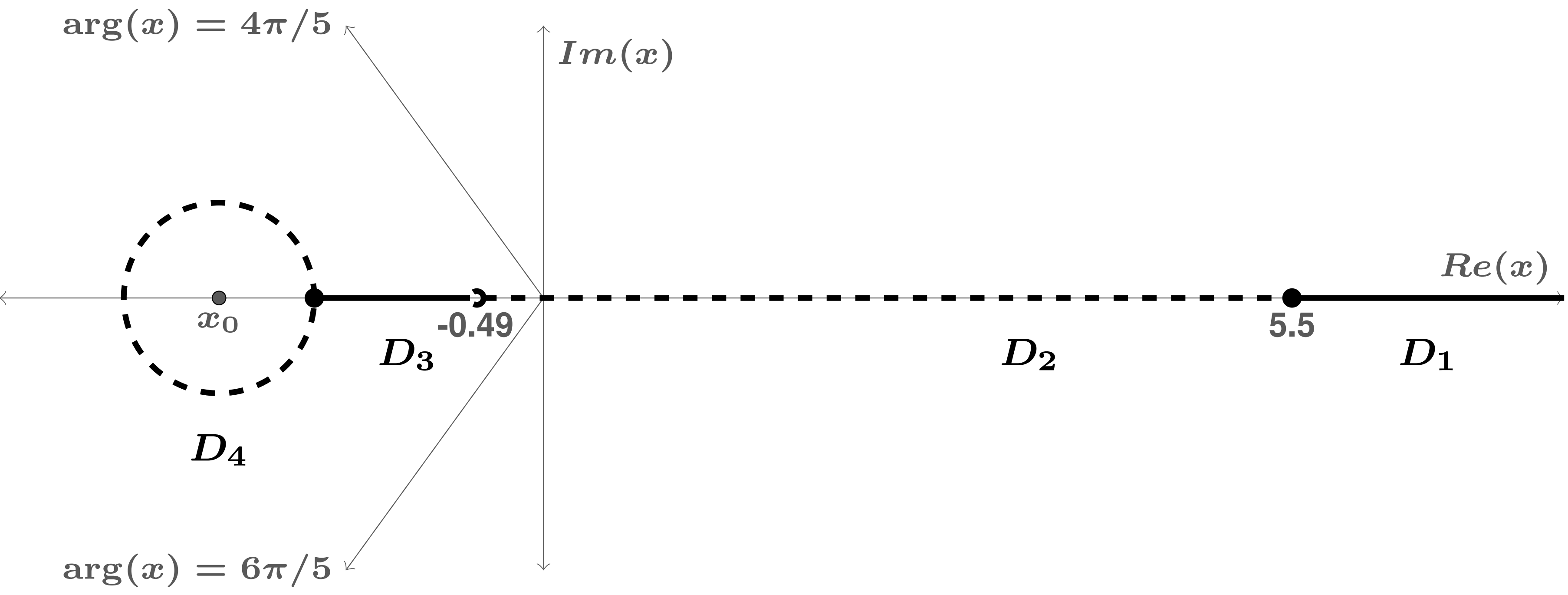

Figure 1: Sketch of Domain $D = D_1 \cup D_2 \cup D_3 \cup D_4$

Main Results

Theorem: *Let*

$$y_0(x) = \begin{cases} \sqrt{\frac{x}{6}}\left[1 + \frac{1}{8\sqrt{6}}x^{-5/2} - \frac{49}{768}x^{-5} + \frac{1225}{1536\sqrt{6}}x^{-15/2} + w_0(x)\right] & \text{on } D_1 \\ -\frac{1}{(x-x_0)^2} + P_u(\tau(x)) & \text{on } D_2 \cup D_3 \\ -\frac{1}{(x-x_0)^2} + (x-x_0)^2 P(x-x_0) & \text{on } D_4 \end{cases} \tag{5}$$

Then the tritronqée solution y *to P-1 has the representation*

$$y(x) = y_0(x) + E(x)\,,\ \text{where}\ \left|E(x)\right| \le 2.35 \times 10^{-5}\,,\ \left|E'(x)\right| \le 1.16 \times 10^{-4} \tag{6}$$

Moreover, y *has a unique double pole singularity at* $x = x_p \in \{\zeta : \zeta \in \mathbb{C}, |\zeta - x_0| < r\}$ *with*

$|x_p - x_0| \le 4.1 \times 10^{-6}$. *This is the closest singularity of the tritronquée solution from the origin.*

Crux of the Proof

In domain D_j, $j \geq 2$, $E = y - y_0$ satisfies

$$E'' + 12y_0 E = -R(x) - 6E^2(x) \tag{7}$$

G_1, G_2 are fundamental solutions to $G'' + 6y_0 G = 0$; hence

$$E(x) = \frac{1}{W}\int_{x_e}^{x} \left(G_2(x)G_1(t) - G_1(x)G_2(t)\right)\left(-R(t) - 6E^2(t)\right) dt \\ + E(x_e)G_1(x) + E'(x_e)G_2(x) =: \mathcal{N}[E] \tag{8}$$

with y_0 is chosen to make R small.

Bounds on G_1, G_2 obtained abstractly, or by using exact Green's function for a neighboring problem.

Small IC/BC and small R, guarantees small $E_0 = \mathcal{N}[0]$ and $\mathcal{N}$ is contractive in a small ball in $C(D_j)$. Smoothness of G_1, G_2, shows solution to be in $C^2(D_j)$.

Continuity of $y_0 + E$ and $y_0' + E'$ at x_e guarantees solution to be the tritronqué.

Determing y_0

In D_1, a few terms of the asymptotic series of the Tritronqué for large x used; but to obtain 10^{-12} accuracy for $x \geq 5.5$, needed to include w_0.

With y and y' at $x = 5.5$, projected numerical solution in $[x_0 + r, 5.5]$ to a truncated Chebyshev basis, after taking out $-1/(x - x_0)^2$ to obtain $P_u(\tau(x))$. This leads to $y_0 = -\frac{1}{(x-x_0)^2} + P_u(\tau(x))$ in $D_2 \cup D_3$.

In domain D_4 we used a truncated power-series representation, choosing x_0 and a_2 to satisfy continuity condition on y and y' at $x = r$.

Bounds for G_1, G_2 in Domain $D_2 = [-0.49, 5.5]$

Recall G_1, G_2 fundamenal solution to $G'' + 12y_0 G = 0$. It is easy to prove $y_0 > 0, y_0' > 0$ in D_2.

Lemma 0.1 *$\|G_1'\|_\infty \leq 3.391$, $\|G_1\|_\infty \leq 3.775$, $\|G_2'\|_\infty \leq 1$ and $\|G_2\|_\infty \leq 1.114$ on D_2.*

Proof: On multiplication by $2G_j'$, integration from $L = 5.5$ to x and gives

$$G_j'^2(x) + 12y_0(x)G_j^2(x) + 12\int_x^L y_0'(t)G_j^2(t)dt = G_j'^2(L) + 12y_0(L)G_j^2(L) \quad (9)$$

Using $y_0, y_0' > 0$, and initial conditions on G_j, above implies

$$G_1'(x)^2 + 12y_0(x)G_1(x)^2 \leq 12y_0(L), \quad (10)$$

$$G_2'(x)^2 + 12y_0(x)G_2(x)^2 \leq 1, \quad (11)$$

$|G_1'| \leq \sqrt{12y_0(L)} \leq 3.391$ and $|G_2'| \leq 1$ are immediate. To find bounds on G_1, G_2, it is convenient to partition D_2 into two intervals $[-0.49, \gamma_0)$ and $[\gamma_0, 5.5)$, where γ_0 will be chosen appropriately.

Bounds on G_1, G_2 in D_2–continued

Using bounds on $|G_1'|$,

$$|G_1(x)| \leq \frac{\sqrt{y_0(L)}}{\sqrt{y_0(x)}} \text{ when } \gamma_0 \leq x \leq L, \text{ and for } x \in (-0.49, \gamma_0),$$

$$|G_1(x)| \leq \int_x^{\gamma_0} |G_1'(x)| dx + |G_1(\gamma_0)| \leq (\gamma_0 - x)\sqrt{12 y_0(L)} + \frac{\sqrt{y_0(L)}}{\sqrt{y_0(\gamma_0)}}$$

Since y_0 is monotonically increasing, it follows from above that for any $x \in D_2$

$$\Big|G_1(x)\Big| \leq (\gamma_0 + 0.49)\sqrt{12 y_0(L)} + \frac{\sqrt{y_0(L)}}{\sqrt{y_0(\gamma_0)}} \tag{12}$$

Similarly, using bounds on G_2', we obtain for any $x \in D_2$,

$$\Big|G_2(x)\Big| \leq \frac{1}{\sqrt{12 y_0(\gamma_0)}} + (\gamma_0 + 0.49) \tag{13}$$

From explicit evaluation with $\gamma_0 = -\frac{16}{100}$, obtain lemma bounds.

Bounds on G_1, G_2 in domain D_4

$$Y_0(\zeta) := y_0(x_0+\zeta) = -\frac{1}{\zeta^2} + \zeta^2 P(\zeta) \tag{14}$$

where $\zeta = x - x_0$, $P(\zeta) = \sum_{j=0}^{17} a_j \zeta^j$ known.

We checked $\left|a_j\right| \le \frac{1}{2^j}$ for $0 \le j \le 17$.

Definition 0.2 *Define* $G_1(\zeta) = \sum_{n=0}^{\infty} A_n \zeta^{4+n}$, $G_2(\zeta) = \sum_{n=0}^{\infty} B_n \zeta^{n-3}$, *where* $A_0 = 1,\ A_1 = A_2 = A_3 = 0$, *and*

$$A_n = -\frac{12}{n(n+7)} \sum_{k=0}^{\min\{n-4,17\}} a_k A_{n-4-k} \quad , \quad \text{for } n \ge 4 \tag{15}$$

$B_0 = 1, B_1 = B_2 = B_3 = B_7 = 0$

$$\text{and for } n \ge 4, n \ne 7,\quad B_n = -\frac{12}{n(n-7)} \sum_{k=0}^{\min\{n-4,17\}} a_k B_{n-4-k} \tag{16}$$

G_1, G_2 are independent solutions to $G'' + 6Y_0G = 0$ with Wronskian -7.

Bounds on G_1, G_2 in D_4–II

Lemma 0.3 *For any integer $n \geq 1$,*

$$\left|A_n\right| \leq c_A\left(\frac{3}{4}\right)^n \quad , \quad \left|B_n\right| \leq c_B\left(\frac{3}{4}\right)^n , \tag{17}$$

where $c_A = 0.21$ and $c_B = 0.85$,

Proof: We checked the inequalities for the first twenty two coefficients $\{A_n, B_n\}_{n=1}^{22}$ through explicit calculations. Assume the inequality holds for $n \leq n_0$ for some $n_0 \geq 22$. Then, using bounds on a_k, and noting that recurrence relations no longer involves A_0, we obtain

$$\left|A_{n_0+1}\right| \leq\leq \frac{36 c_A}{(n_0+1)(n_0+8)} \left(\frac{4}{3}\right)^4 \left(\frac{3}{4}\right)^{n_0+1} \leq c_A \left(\frac{3}{4}\right)^{n_0+1} \tag{18}$$

So, the inequality holds for $n_0 + 1$. By induction it holds for all n. The same induction proof works for B_n after using $\frac{36}{(n_0+1)(n_0-6)} \left(\frac{4}{3}\right)^4 \leq 1$ for $n_0 \geq 22$. This immediately leads to bounds on G_1, G_2 and their derivatives in D_4.

Location of closest singularity x_p

Cauchy integral formula implies that the integral $-\frac{1}{2\pi i}\oint_{|\zeta|=r}\zeta y(x_0+\zeta)d\zeta$ equals to the number of singularities of $y(x)$ in $|x-x_0|<r$. From this observation, we calculate

$$\left|1+\frac{1}{2\pi i}\oint_{|\zeta|=r}\zeta y(x_0+\zeta)d\zeta\right| \le \left|1+\frac{1}{2\pi i}\oint_{|\zeta|=r}\zeta y_0(x_0+\zeta)d\zeta\right| + r^2\|y-y_0\|_\infty \le 1.2\times 10^{-5} \tag{19}$$

implying exactly one singulariy in $|x-x_0|<r$.

Also, Cauchy formula gives $x_p-x_0=-\frac{1}{4\pi i}\oint_{|\zeta|=r}\zeta^2 y(x_0+\zeta)d\zeta$, and hence

$$|x_p-x_0| \le \left|-\frac{1}{4\pi i}\oint_{|\zeta|=r}\zeta^2 E(\zeta)d\zeta\right| \le \frac{r^3}{2}\|E\|_\infty \le 4.1\times 10^{-6},$$

Conclusion

1. **We showed how a suitably accurate approximate solution u_0 can be constructed and used to determine rigorous error bounds.**
2. **ODE or systems of ODEs, including two point boundary value problems are easily amenable through this method. (Costin & T, ’13, Costin, Kim & T, ’14) Opens the opportunity for homoclinic-heteroclinic determination in higher dimension. Also, integro-differential equations are amenable to this approach.**
3. **PDE similarity blow up or spectral analysis in 1+1 dimension amenable to our type of analysis. (Costin *et al*, ’13.)**
4. **PDEs also fit into this approach, though the challenge is always to find a suitably compact representation; as otherwise, it becomes a computer assisted proof.**
5. **Papers available online. http://www.math.ohio-state.edu/$\sim$tanveer**