A new approach to regularity and singularity of PDEs including 3-D Navier Stokes equation

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Main idea

For an autonomous differential operator $\mathcal{N}$, consider

$$v_t = \mathcal{N}[v] \;,\; v(x,0) = v_0(x)$$

Formal small time expansion:

$$\tilde{v}(x,t) = v_0(x) + t v_1(x) + t^2 v_2(x) \;+\; ..,$$

where $v_1(x) = \mathcal{N}[v_0](x)$, $v_2 = \frac{1}{2}\left\{\mathcal{N}_v(v_0)[v_1]\right\}(x)$,..

Generically divergent if order of $\mathcal{N}$ is greater than 1

If divergent series can be resummed appropriately, then
$v = \sum_{\mathcal{B}} \tilde{v}$ can be shown to be a solution to IVP.

Depending on properties of v, the solution can be extended over to $[0, T]$. Time blow up reflected in exponential growth at ∞ in an appropriate dual variable.

Borel Summation Illustrated in a Simple Linear ODE

$$y' - y = \frac{1}{x^2}$$

Want solution $y \to 0$, as $x \to +\infty$

Dominant Balance (or formally plugging a series in $1/x$):

$$y \sim -\frac{1}{x^2} + \frac{2}{x^3} + \ ...\frac{(-1)^k k!}{x^{k+1}} + .. \equiv \tilde{y}(x)$$

Borel Transform:

$$\mathcal{B}[x^{-k}](p) = \frac{p^{k-1}}{\Gamma(k)} = \mathcal{L}^{-1}[x^{-k}](p) \ \text{ for } \ Re\, p > 0$$

$$\mathcal{B}\left[\sum_{k=1}^{\infty} a_k x^{-k}\right](p) = \sum_{k=1}^{\infty} \frac{a_k}{\Gamma(k)} p^{k-1}$$

Borel Summation for linear ODE -II

$$Y(p) \equiv \mathcal{B}[\tilde{y}](p) = \sum_{k=1}^{\infty} (-1)^k p^k = -\frac{p}{1+p}$$

$$y(x) \equiv \int_0^{\infty} e^{-px} Y(p) dp = \mathcal{L}\mathcal{B}[\tilde{y}]$$

is the linear ODE solution we seek. Borel Sum defined as $\mathcal{L}\mathcal{B}$.

Note once solution is found, it is not restricted to large x.

Necessary properties for Borel Sum to exist:

1. **The Borel Transform $\mathcal{B}[\tilde{y}_0](p)$ analytic for $p \geq 0$,**
2. **$e^{-\alpha p}|\mathcal{B}[\tilde{y}_0](p)|$ bounded so that Laplace Transform exists.**

Remark: Difficult to check directly for non-trivial problems

Borel sum of nonlinear ODE solution

Instead, directly apply $\mathcal{L}^{-1}$ to equation; for instance

$$y' - y = \frac{1}{x^2} + y^2; \quad \text{with} \lim_{x\to\infty} y = 0$$

Inverse Laplace transforming, with $Y(p) = [\mathcal{L}^{-1}y](p)$:

$$-pY(p) - Y(p) = p + Y * Y \quad \text{implying } Y(p) = -\frac{1}{1+p} - \frac{Y * Y}{1+p} \tag{1}$$

For functions Y analytic for $p \geq 0$ and $e^{-\alpha p}Y(p)$ bounded, it can be shown that (1) has unique solution for sufficiently large α.

Implies ODE solution $y(x) = \int_0^\infty Y(p)e^{-px}dp$ for $Re\ x > \alpha$

The above is a special case of results available for generic nonlinear ODEs (Costin, 1998)

Eg: Illustrative IVP: 1-D Heat Equation

$$v_t = v_{xx} \ , \ v(x,0) = v_0(x) \ \ , \ \ v(x,t) = v_0 + t v_1 + ..$$

Obtain recurrence relation

$$(k+1) v_{k+1} = v_k'' \ , \qquad \text{implies} \ \ v_k = \frac{v_0^{(2k)}}{k!}$$

Unless v_0 entire, series $\sum_k t^k v_k$ factorially divergent.

Borel transform in $\tau = 1/t$: $V(x,p) = \mathcal{B}[v(x, 1/\tau))](p)$, $V(x,p) = p^{-1/2} W(x, 2\sqrt{p})$, then $W_{qq} - W_{xx} = 0$

Obtain $v(x,t) = \int_{\mathbb{R}} v_0(y) (4\pi t)^{-1/2} \exp[-(x-y)^2/(4t)] dy$,

i.e. **Borel sum of formal series leads to usual heat solution.**

We seek applications of these simple ideas to more complicated PDEs, including 3-D Navier-Stokes

Background

Borel Summability for linear PDEs studied before (Balser, Miyake, Lutz, Schaefke, ..)

Sectorial existence for a class of nonlinear PDEs (Costin & T.)

Complex singularity formation for a nonlinear PDE (Costin & T.)

Navier Stokes is a nonlinear PDE governing fluid velocity $\mathbf{v}(\mathbf{x}, t)$:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \nu \Delta v \ + \ \mathbf{f}$$

$$\nabla \cdot \mathbf{v} = \mathbf{0} \ , \quad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$$

Using PDE techniques, Leray (1930s) proved local existence, uniqueness for classical solutions and global existence for weak solutions. Global existence of classical solutions known in 2-D, not in 3-D. Literature extensive ((Constantin, Temam, Foias,...).

Background II

Global existence of classical solution or lack of it has fundamental implications to fluid turbulence.

Blow up of classical solution with finite energy $\|v_0\|_{L^2(\mathbb{R}^3)}$ implies $\|\nabla \times v(.,t)\|_\infty$ and $\|v(.,t)\|_{L^3(\mathbb{R}^3)}$ blow up (Beale *et al*, Sverak).

This becomes incompatible with the modeling assumptions in deriving Navier-Stokes. Hence other parameters not included in Navier-Stokes would become important in turbulent flow.

For the usual PDE techniques, key to global existence question is believed to be *a priori* energy bounds involving ∇v (Tao). None is available thus far.

This motivates alternate formulation of initial value problems for nonlinear PDEs that are not dependent on energy bounds at all. Borel methods and its generalization allows such a formulation.

Illustration: Borel Transform for Burger's equation

Substitute $v = v_0(x) + u(x,t)$ into $v_t + v v_x = v_{xx}$ to obtain

$$u_t - u_{xx} = -v_0 u_x - u v_{0,x} - u u_x + v_1(x)$$

$$\text{where } \; v_1(x) = v_0'' - v_0 v_{0,x} \;\; , \quad \text{and} \;\; , u(x,0) = 0$$

Inverse Laplace Transform in $1/t$ and Fourier-Transform in x:

$$p\hat{U}_{pp} + 2U_p + k^2 \hat{U} = \hat{v}_1 - ik\hat{v}_0 \hat{*} \hat{U} - ik\hat{U} \overset{*}{*} \hat{U} \equiv \hat{G}(k,p) + \hat{v}_1 ,$$

$\hat{*}$ is Fourier convolution, $\overset{*}{*}$ Fourier-Laplace convolution. Hence

$$\hat{U}(k,p) = \int_0^p \mathcal{K}(p,p';k)\hat{G}(k,p')dp' + \hat{U}^{(0)}(k,p) \equiv \mathcal{N}\left[\hat{U}\right](k,p)$$

$$\mathcal{K}(p,p';k) = \frac{ik\pi}{z}\left\{z' Y_1(z') J_1(z) - z' Y_1(z) J_1(z')\right\}$$

$$z = 2|k|\sqrt{p} \; , \; z' = 2|k|\sqrt{p'} \; , \; \hat{U}^{(0)}(k,p) = 2\frac{J_1(z)}{z}\hat{v}_1(k)$$

Solution to integral equation $\hat{U} = \mathcal{N}[\hat{U}]$

$$\text{We find } |\mathcal{K}(p, p'; k)| \leq \frac{C}{\sqrt{p}} \;, \quad C \text{ a constant}$$

$$\|\hat{F}(., p) \hat{*} \hat{G}(., p)\|_{L^1(\mathbb{R})} \leq C \|\hat{F}(., p)\|_{L^1(\mathbb{R})} \|\hat{G}(., p)\|_{L^1(\mathbb{R})}$$

Define norm $\|.\|^{(\alpha)}$ for functions $F(p, k)$

$$\|F\|^{(\alpha)} = \int_0^\infty e^{-\alpha p} \|F(., p)\|_{L^1(\mathbb{R})} \, dp$$

$$\text{easily follows } \|F \overset{*}{*} G\|^{(\alpha)} \leq C \|F\|^{(\alpha)} \|G\|^{(\alpha)}$$

$\mathcal{N}$ seen to be contractive for large α implies Burgers solution for for $\mathrm{Re}\, \frac{1}{t} > \alpha$ in the form $v(x, t) = v_0(x) + \int_0^\infty e^{-p/t} U(x, p) dp$

Global classical PDE solution implied if $\|\hat{U}(., p)\|_{L^1(\mathbb{R}^3)}$ bounded.

Borel summability for analytic v_0 requires analyticity of $U(., p)$ for $p \in 0 \cup \mathbb{R}^+$; proof a bit more delicate.

Incompressible 3-D Navier-Stokes in Fourier-Space

Consider 3-D N-S in infinite geometry or periodic box. Similar results expected for finite domain with no-slip BC using eigenfunctions of Stokes operator as basis. In Fourier-Space

$$\hat{v}_t + \nu |k|^2 \hat{v} = -i k_j P_k \left[\hat{v}_j \hat{*} \hat{v} \right] + \hat{f}(k)$$

$$P_k = \left(I - \frac{k(k\cdot)}{|k|^2} \right) \quad , \quad \hat{v}(k,0) = \hat{v}_0(k)$$

where P_k is the Hodge projection in Fourier space, $\hat{f}(k)$ is the Fourier-Transform of forcing $f(x)$, assumed divergence free and t-independent. Subscript j denotes the j-th component of a vector. $k \in \mathbb{R}^3$ or $\mathbb{Z}^3$. Einstein convention for repeated index followed. $\hat{*}$ denotes Fourier convolution.

Integral equation for Navier Stokes in Borel plane

Substitute $\hat{v} = \hat{v}_0 + \hat{u}(k,t)$, into Navier-Stokes, inverse-Laplace Transform in $1/t$ and inverting as for Burger's equation obtain integral equation:

$$\mathrm{U}(\mathrm{k},p) = \int_0^p \mathcal{K}(p,p')\hat{\mathrm{R}}(\mathrm{k},p')dp' + \mathrm{U}^{(0)}(\mathrm{k},p),$$

$$\hat{\mathrm{R}}(k,p) = -ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{\mathrm{U}} + \hat{U}_j \hat{*} \hat{\mathrm{v}}_0 + \hat{U}_j \overset{*}{*} \hat{\mathrm{U}}\right]$$

$$\mathrm{U}^{(0)}(k,p) = 2\frac{J_1(z)}{z}\hat{\mathrm{v}}_1(\mathrm{k}), \quad \text{where}$$

$$\hat{\mathrm{v}}_1(k) = -|\mathrm{k}|^2\hat{\mathrm{v}}_0 - ik_j P_k\left[\hat{v}_{0,j} \hat{*} \hat{\mathrm{v}}_0\right] + \hat{\mathrm{f}}(\mathrm{k})$$

Some Results for Navier-Stokes (NS) in $\mathbb{R}^3$

Define $\|.\|_{\mu,\beta}$, for $\mu > 3$, $\beta \ge 0$:

$$\|v_0\|_{\mu,\beta} = \sup_{k\in\mathbb{R}^3} e^{\beta|k|}(1+|k|)^{\mu}|\hat{v}_0(k)|$$

Theorem 1: For $\beta > 0$, the NS solution v is Borel summable in $1/t$, i.e. there exists $U(x,p)$, analytic in a neighborhood of $0 \cup \mathbb{R}^+$, exponentially bounded, and analytic in x for $|\,\mathrm{Im}\ x| < \beta$ so that $v(x,t) = v_0(x) + \int_0^\infty U(x,p)e^{-p/t}dp$. When $t \to 0$, $v(x,t) \sim v_0(x) + \sum_{m=1}^\infty t^m v_m(x)$, where $|v_m(x)| \le m! A_0 B_0^m$, with A_0, B_0 generally dependent on v_0, f.

Remark: Same results valid for $\mathrm{x} \in \mathbb{T}^3$.

Theorem 2: If v_0 and f have a finite number of Fourier modes, then B_0 is independent of v_0 and f.

Further Results on NS in $\mathbb{T}^3$

Define $\|.\|^{(\alpha)}$ so that

$$\|\hat{V}\|^{(\alpha)} = \int_0^\infty e^{-\alpha p} \|\hat{V}(.,p)\|_{l^1(\mathbb{Z}^3)} dp$$

Theorem 3: If $\|\hat{v}_0\|_{l^1(\mathbb{Z}^3)}$, $\|\hat{f}\|_{l^1(\mathbb{Z}^3)} < \infty$ then there exists some $\alpha > 0$ so that integral equation $\hat{U} = \mathcal{N}\left[\hat{U}\right]$ has a unique solution for $p \in \mathbb{R}^+$ in the space of functions $\left\{\hat{U} : \|\hat{U}\|^{(\alpha)} < \infty\right\}$. Further, $\hat{v}(k,t) = \hat{v}_0(k) + \int_0^\infty \hat{U}(k,p) e^{-p/t} dp$ satisfies 3-D Navier-Stokes in Fourier-Space; corresponding $v(x,t)$ is a classical NS solution for $t \in (0, \alpha^{-1})$.

Remark 1: Classical PDE methods known to give similar results. However, in the present formulation, global PDE existence is a question of asymptotics of known solution to integral equation as $p \to \infty$. Sub-exponential growth implies global existence.

More Remarks on Theorem 3 for 3-D Navier-Stokes

Remark 2: Errors in Numerical solutions rigorously controlled. Discretization in p and Galerkin approximation in k results in:

$$\hat{U}_\delta(k, m\delta) = \delta \sum_{m'=0}^{m} \mathcal{K}_{m,m'} \mathcal{P}_N \mathcal{H}_\delta(k, m'\delta) + \hat{U}^{(0)}(k, m\delta)$$

$$\equiv \mathcal{N}_\delta \left[\hat{U}_\delta\right] \quad \text{for} \quad k_j = -N, ...N, \quad j = 1, 2, 3$$

$\mathcal{P}_N$ is the Galerkin Projection into N-Fourier modes. $\mathcal{N}_\delta$ has properties similar to $\mathcal{N}$. The continuous solution $\hat{U}$ satisfies $\hat{U} = \mathcal{N}_\delta\left[\hat{U}\right] + E$, where E is the truncation error. Thus, $\hat{U} - \hat{U}_\delta$ can be estimated using same tools as in Theorem 1.

Note: Similar control over discretized solutions to PDEs not available since truncation errors involve derivatives of PDE solution which are not known to exist beyond a short-time.

Extending Navier-Stokes interval of existence

Suppose $\hat{U}(.,p)$ is known over $[0,p_0]$ through Taylor series in p or otherwise, and computed $\|\hat{U}(.,p)\|_{l^1}$ is observed to decrease towards the end of this interval. Prior discussions show that any error in this computation can be rigorously controlled.

Results in the following page show that a more optimal Borel exponent $\alpha \le \alpha_0$ may be estimated using the known solution in $[0,p_0]$, where α_0 is the initial α estimate in Theorem 1. This implies a longer interval $\left[0,\alpha^{-1}\right)$ for NS solution.

A longer existence time for NS is relevant to the global existence question for $f=0$, since it is known that there exists T_c so that any weak Leray solution becomes classical for $t > T_c$

Extending Navier-Stokes interval of existence -II

For $\alpha_0 \geq 0$, define

$$\epsilon = \nu^{-1/2} p_0^{-1/2}\ , \ a = \|\hat{v}_0\|_{l^1}\ , \ c = \int_{p_0}^{\infty} \|\hat{U}^{(0)}(.,p)\|_{l^1} e^{-\alpha_0 p} dp$$

$$\epsilon_1 = \nu^{-1/2} p_0^{-1/2} \left(2 \int_0^{p_0} e^{-\alpha_0 s} \|\hat{U}(.,s)\|_{l^1} ds + \|\hat{v}_0\|_{l^1} \right)$$

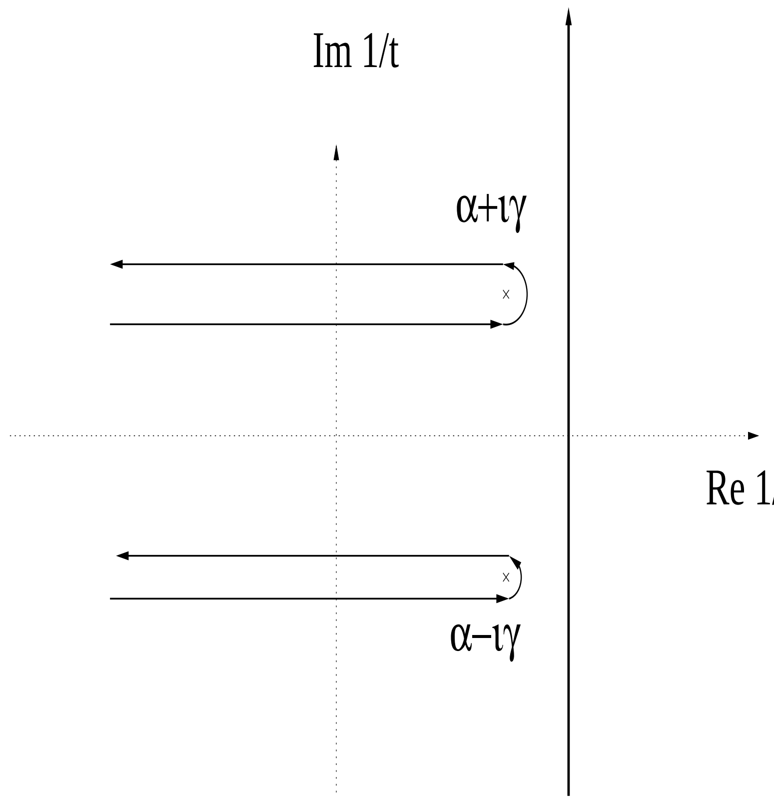
$$b = \frac{e^{-\alpha_0 p_0}}{\sqrt{\nu p_0} \alpha} \int_0^{p_0} \|\hat{U} \hat{*} \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{l^1} ds$$

Theorem 4: A smooth solution to 3-D Navier-Stokes equation exists on the interval $[0, \alpha^{-1})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

Relation of optimal α to NS time singularities

$$\hat{U}(k,p) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{p/t}\left[\hat{v}(k,t) - \hat{v}_0(k)\right] d\left[\frac{1}{t}\right]$$



Rightmost singularity(ies) of NS solution $\hat{v}(k,t)$ in the $1/t$ plane determines optimal α. γ gives dominant oscillation frequency.

Generalized Laplace-transform representation

Since the Borel domain growth rate α relates to complex right-half $\frac{1}{t}$ NS singularities, the following generalized Laplace Transform representation for $n > 1$ is sought:

$$\hat{v}(k,t) = \hat{v}_0(k) + \int_0^\infty e^{-q/t^n} \hat{U}(k,q) dq$$

In order that $\hat{U}(.,q)$ has no growth for large q, unless there is a NS singularity for $t \in \mathbb{R}^+$, need to know *a priori* **that there is a singularity free sector in the right-half t-plane. This is proved to be true for $f = 0$ and we have the following result:**

Theorem 5: For $f = 0$, if NS has a global classical solution, then for all sufficiently large n, $U(x,q) = O(e^{-C_n q^{1/(n+1)}})$ as $q \to +\infty$, for some $C_n > 0$.

Numerical Solutions to integral equation

We choose the Kida initial conditions and forcing

$$\mathbf{v}_0(\mathbf{x}) = (v_1(x_1, x_2, x_3, 0), v_2(x_1, x_2, x_3, 0), v_3(x_1, x_2, x_3, 0))$$

$$v_1(x_1, x_2, x_3, 0) = v_2(x_3, x_1, x_2, 0) = v_3(x_2, x_3, x_1, 0)$$

$$v_1(x_1, x_2, x_3, 0) = \sin x_3 \left(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3\right)$$

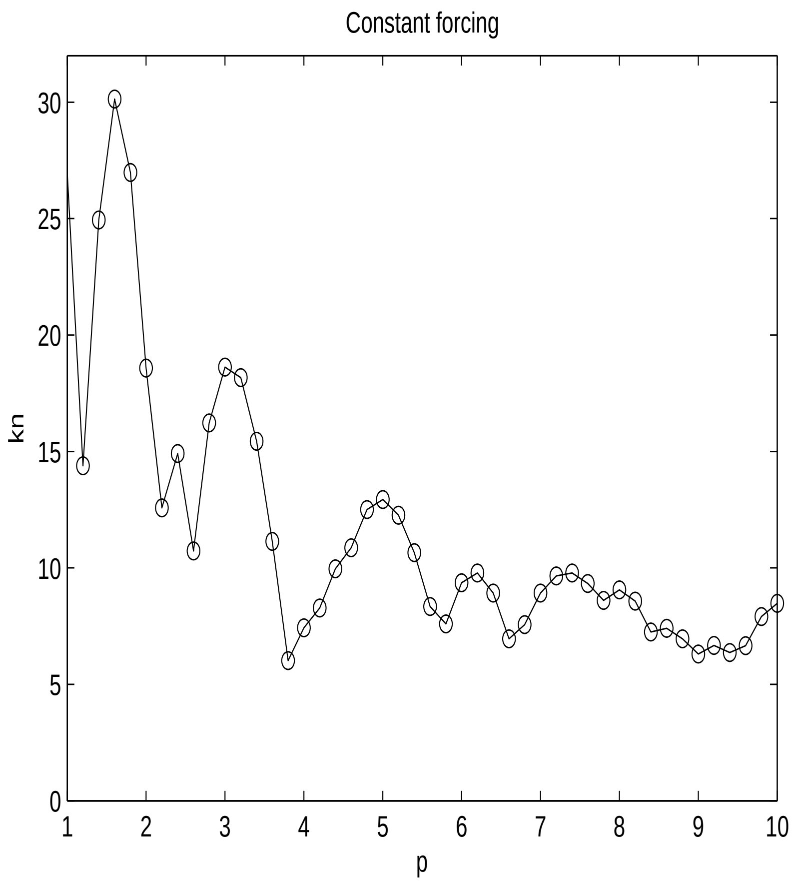
$$f_1(x_1, x_2, x_3) = \frac{1}{5} v_1(x_1, x_2, x_3, 0)$$

High Degree of Symmetry makes computationally less expensive

Corresponding Euler problem believed to blow up in finite time; so good candidate to study viscous effects

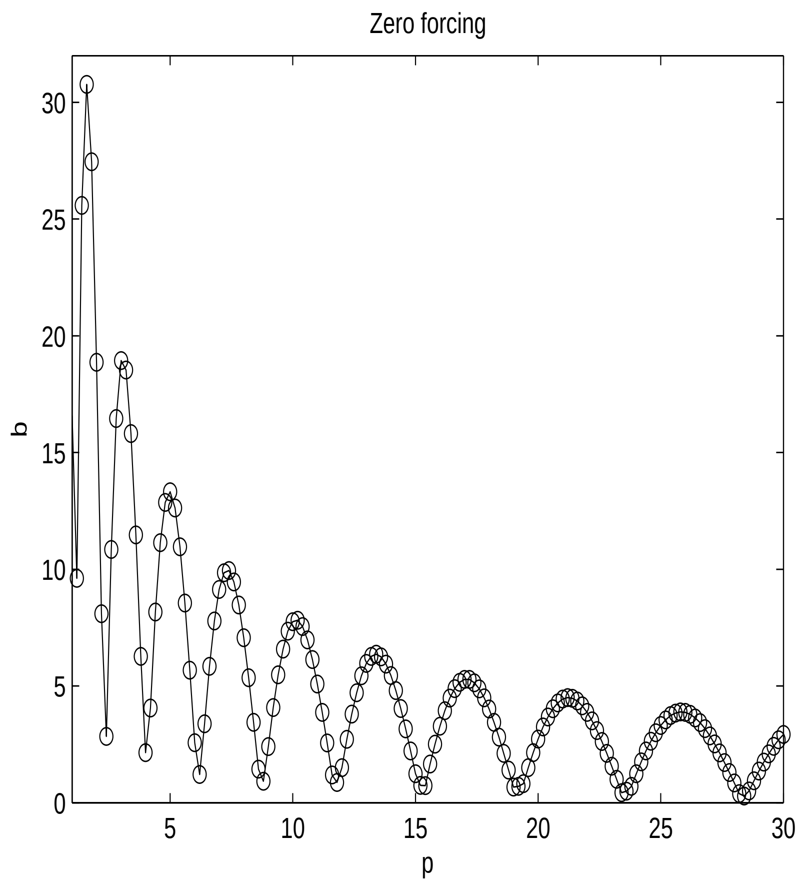
In the plots, "constant forcing" corresponds to $f = (f_1, f_2, f_3)$ as above, while zero forcing refers to $f = 0$. Recall sub-exponential growth in p corresponds to global N-S solution.

Numerical solution to integral equation-plot-1



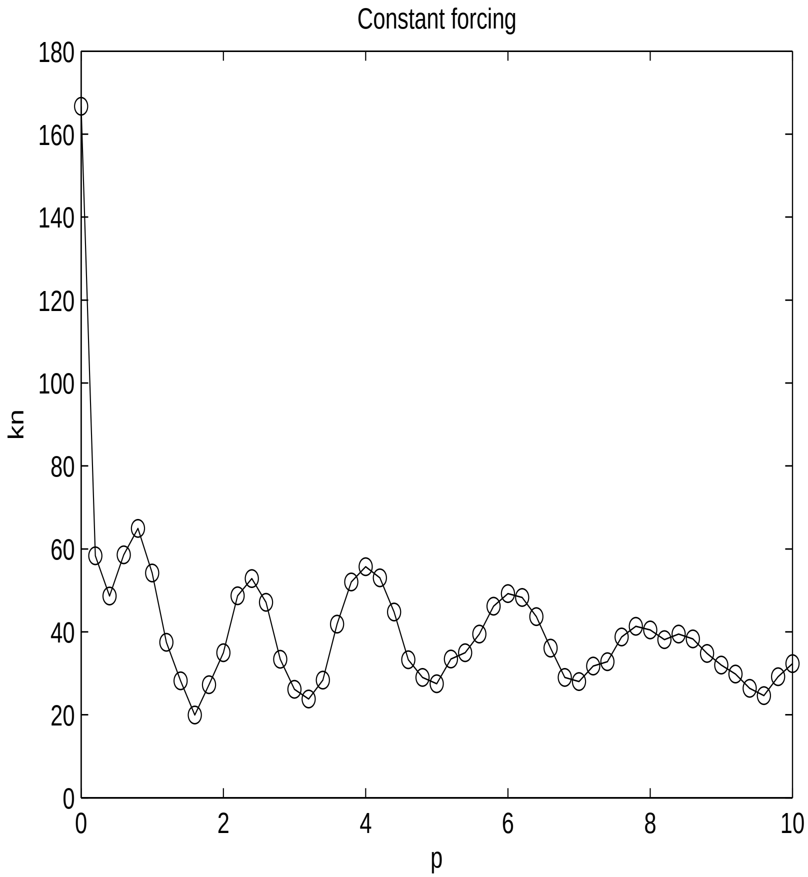
$\|\hat{U}(.,p)\|_{l^1}$ **vs.** p **for** $\nu = 1$**, constant forcing.**

Numerical solution to integral equation-plot-2



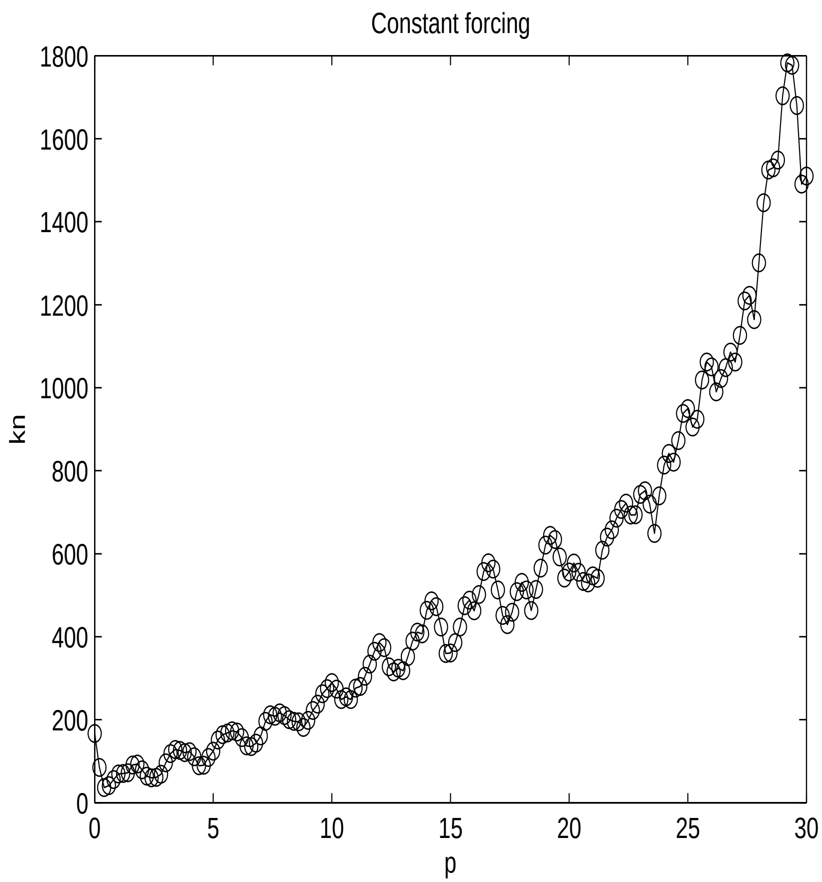
$\|\hat{U}(.,p)\|_{l^1}$ **vs.** p **for** $\nu = 1$**, no forcing**

Numerical solution to integral equation-plot-3



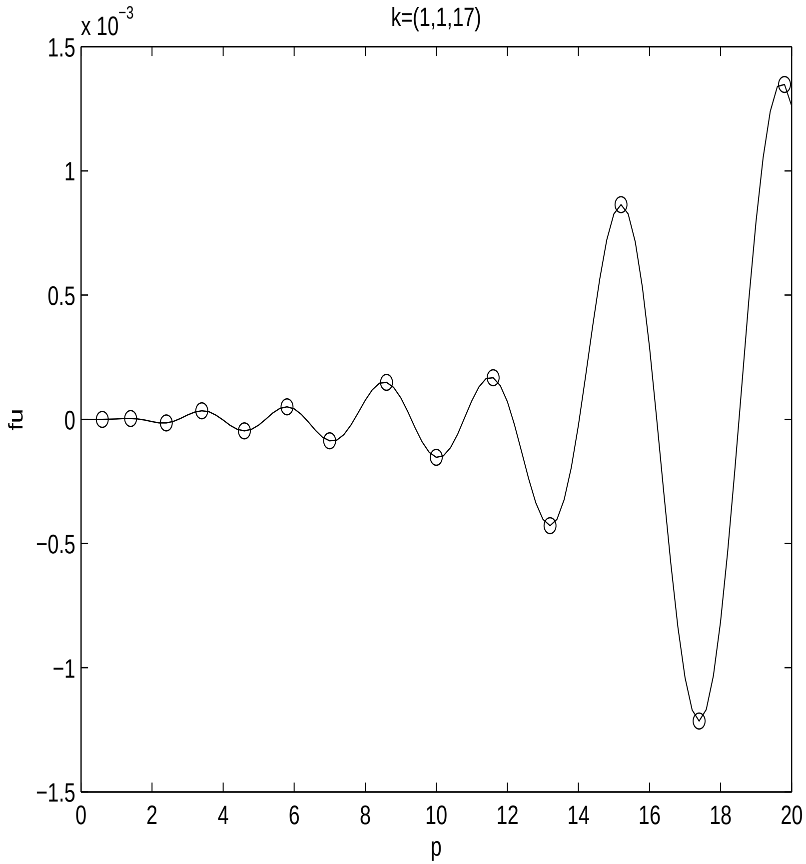
$\|\hat{U}(., p)\|_{l^1}$ **vs.** p **for** $\nu = 0.16$**, constant forcing**

Numerical solution to integral equation-plot-4



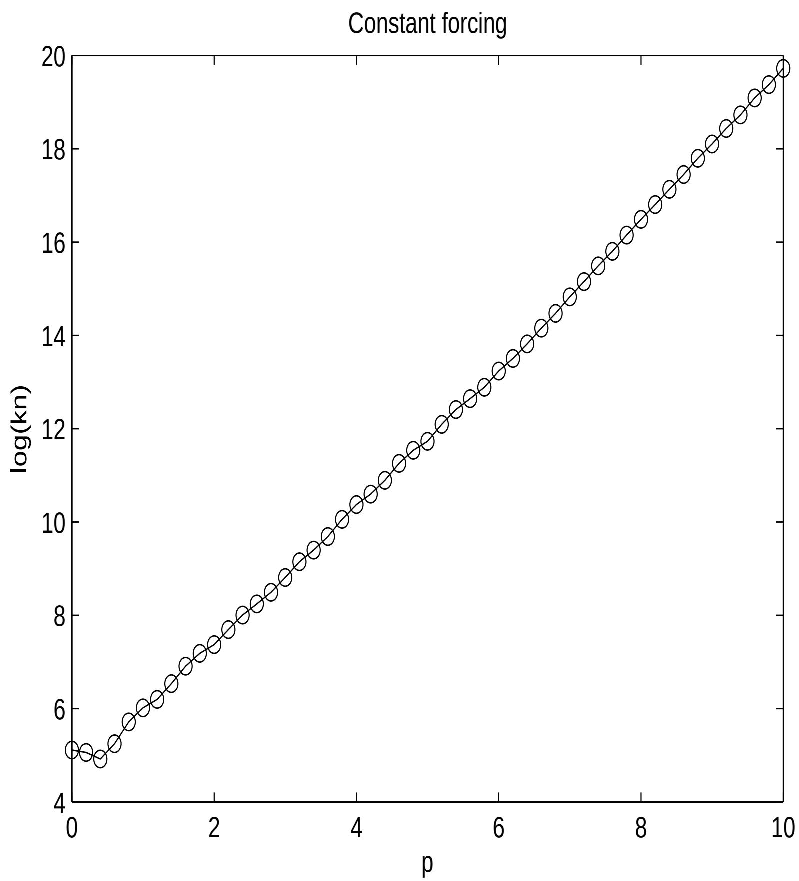
$\|\hat{U}(.,p)\|_{l^1}$ **vs.** p **for** $\nu = 0.1$**, constant forcing**

Numerical solution to integral equation-plot-5



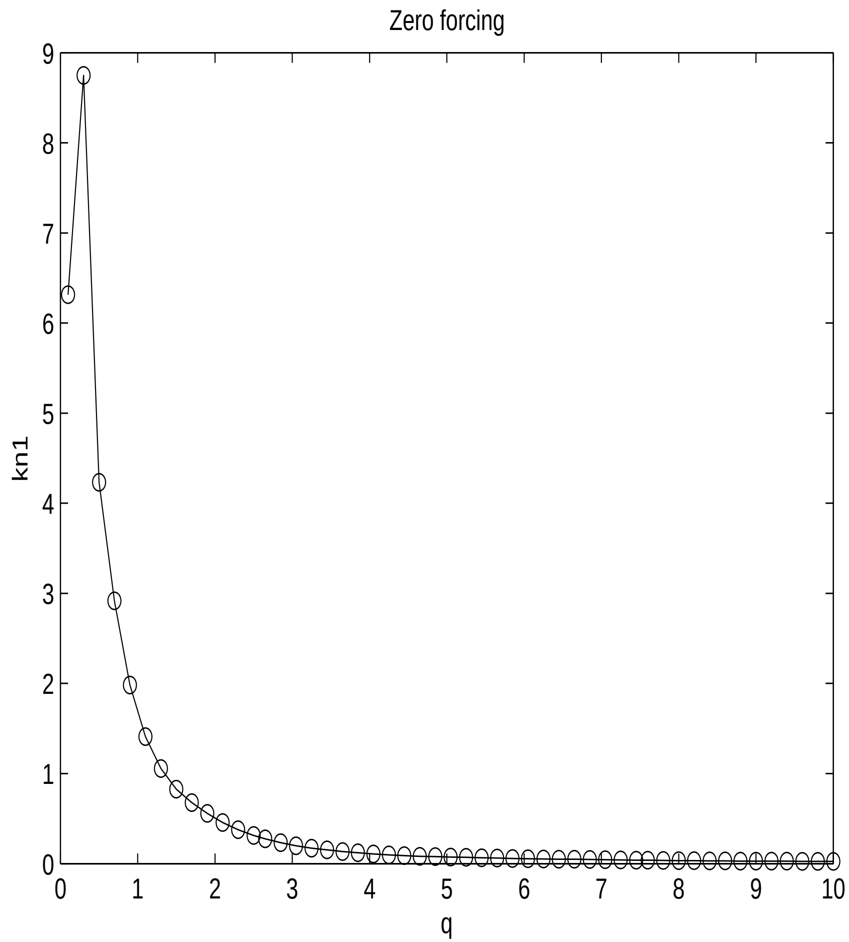
$\hat{U}(k, p)$ **vs.** p **for** $k = (1, 1, 17)$**,** $\nu = 0.1$**, no forcing.**

Numerical solution to integral equation-plot-6



$\log \|\hat{U}(.,p)\|_{l^1}$ **vs.** $\log p$ **for** $\nu = 0.001$**, constant forcing**

$\|\hat{U}(.,q)\|_{l^1}$ vs. q, $n = 2$, $\nu = 0.1$



Kida I.C. $v_1^{(0)} = \sin x_1 (\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3)$

Other components from cyclic relation:

$v_1^{(0)}(x_1, x_2, x_3) = v_1^{(0)}(x_3, x_1, x_2) = v_3^{(0)}(x_2, x_3, x_1)$

Extending Navier-Stokes interval of existence

For $\alpha_0 \geq 0$, define

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} \quad , \quad c = \int_{q_0}^{\infty} \|\hat{U}^{(0)}(.,q)\|_{l^1} e^{-\alpha_0 q} dq$$

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} \left(2 \int_0^{q_0} e^{-\alpha_0 s} \|\hat{U}(.,s)\|_{l^1} ds + \|\hat{v}_0\|_{l^1} \right)$$

$$b = \frac{e^{-\alpha_0 q_0}}{\sqrt{\nu} q_0^{1-1/(2n)} \alpha} \int_0^{q_0} \|\hat{U} \hat{*} \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{l^1} ds$$

Theorem 6: A smooth solution to 3-D Navier-Stokes equation exists in the $\|.\|_{l^1}$ space on the interval $[0, \alpha^{-1/n})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

Remark: If q_0 is chosen large enough, ϵ, ϵ_1 is small when computed solution in $[0, q_0]$ decays with q. Then α can be chosen rather small.

Other problems where approach is applicable

- **Navier-Stokes with temperature field (Boussinesq approximation)**

- **Fourth order Parabolic equations of the type:**

$$u_t + \Delta^2 u = N[u, Du, D^2 u, D^3 u]$$

- **Magneto-hydrodynamic equation with certain approximations.**

- **For some PDE problems with finite-time blow-up, blow-up time related to exponent α of exponential growth of Integral equation as $n \to \infty$.**

Conclusions

We have shown how Borel summation methods provides an alternate existence theory for PDE Initial value problems like N-S.

With this integral equation (IE) approach, the PDE global existence is implied if known solution to IE has subexponential growth at ∞.

The solution to integral equation in a finite interval can be computed numerically with rigorously controlled errors.

Integral equation in a suitable accelerated variable q will decay exponentially for unforced N-S equation, unless there is a real time singularity of PDE solution.

The computation over a finite $[0, q_0]$ interval gives a refined bound on exponent α at ∞, and hence a longer existence time $[0, \alpha^{-1/n})$ to 3-D Navier-Stokes.

Approach is applicable to a wide class of other PDE initial value problems.