Regularity, singularity and well-posedness of some mathematical models in physics

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Background

Mathematical models involves simplifying assumptions where "small" terms are ignored. However, for physical relevance, any term dropped cannot have a "singular" effect on the solution.

For instance, it is reasonable to ignore the gravitational pull of distant stars in planetary orbit computation, except on very large time scales.

However, for fluid motion past a solid body, viscous effects cannot be ignored regardless of viscosity size.

Whether or not one can ignore a term that seems small depends on whether it is a regular or singular perturbation. A perturbation is regular if every $\epsilon = 0$ solution is obtained as $\lim_{\epsilon \to 0}$ of some non zero ϵ solution and vice versa.

Prediction of singular effects or lack of it not always obvious

Simple illustration of regular and singular perturbation

$$\text{Want Solution } X \text{ to } AX = B, \quad \text{where } A = A(\epsilon), \quad \epsilon \text{ small}$$

We compare two cases (a) and (b) as follows:

$$\text{Case (a)}: \; A(\epsilon) = \begin{pmatrix} 1 & 1 \\ \epsilon & 2 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{Case (b)}: \; A(\epsilon) = \begin{pmatrix} 1 & 1 \\ 2 & 2+\epsilon \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{Solutions}: \; X = \begin{pmatrix} \frac{2}{2-\epsilon} \\ -\frac{\epsilon}{2-\epsilon} \end{pmatrix} \quad , \quad X = \begin{pmatrix} \frac{2+\epsilon}{\epsilon} \\ -\frac{2}{\epsilon} \end{pmatrix},$$

Regular perturbation $X = X^0 + \epsilon X^1 + ..$ in case (a), but not (b)

ODE examples of perturbations

For $x \in (0, L)$, where f and g are given regular functions,

Find $y(x)$ satisfying $y'' + f(x)y' + \epsilon g(x)y = 0\,,\ y(0) = 1,\ y'(0) = 0.$

Find $y(x)$ satisfying $\epsilon y'' + f(x)y' + g(x)y = 0\,,\ y(0) = 1\,,\ y'(0) = 0.$

Not difficult to prove that ϵ term in Case (a) is a regular perturbation, while it is a singular perturbation in Case (b).

Distant Star effects on planetary orbits is similar to the ϵ term in first case, while viscous effects in fluid flow problem is like the ϵ term in the second case above

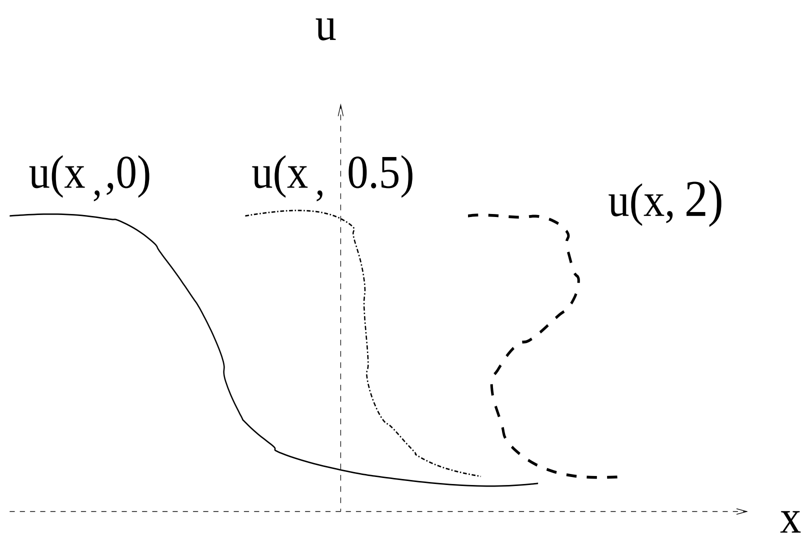
A formal *ansatz* $y \sim y_0 + \epsilon y_1 \ + \ ..$ **will be consistent in the first case but inconsistent in the second. Typically, users of perturbation theory take this as evidence for the type of perturbation. This type of evidence is not always reliable.**

Example of Pertubation in PDE evolution problems

Find $u(x,t)$ for $x \in (-\infty, \infty)$, and $t > 0$ satisfying:

$$u_t + u u_x = \epsilon u_{xx} \quad , \quad u(x,0) = u_0(x)$$

When $\epsilon = 0$, $u = u^{(0)}$ evolution typically looks like:



$\partial_x u^{(0)}$ blows up first at some time $t = t_s$. As $\epsilon \to 0^+$, actual solution $u \to u^{(0)}$ for $t < t_s$. For $t \geq t_s$, $\lim_{\epsilon \to 0} u \neq u^{(0)}$, *i.e.* ϵ term is a singular perturbation. Here, singular perturbation warned by singular behavior of $\partial_x u^{(0)}$.Not always the case!!

Mathematically Precise Notions

Any mathematical model can be described abstractly by

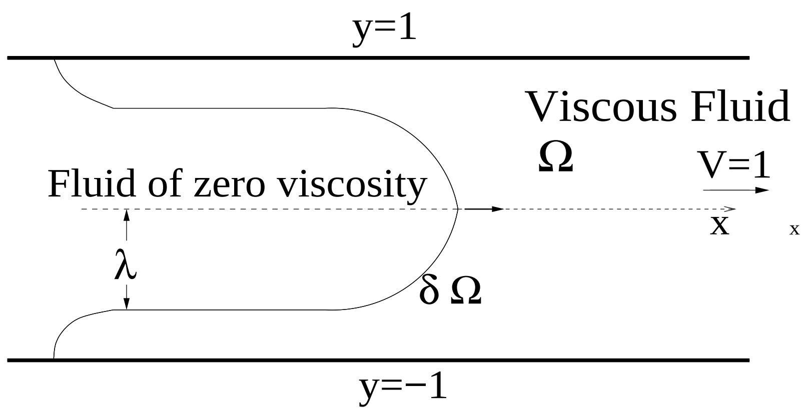
$$\mathcal{N}\left[\mathbf{u};\mathbf{u}_0,\epsilon\right]=0,$$

where operator $\mathcal{N}$ can describe arbitrary differential, integral or algebraic operator. $\mathbf{u}_0$ describes initial and/or boundary conditions and ϵ describes parameters.

Definition: The above problem is *well posed* if there is a unique solution u that depends continuously on $\mathbf{u}_0$ and parameters ϵ.

Ill-posed problems are physically irrelevant since we cannot measure $\mathbf{u}_0$ and ϵ to infinite precision. If solution changes discontinously, say at $\epsilon = 0$, then the $\epsilon = 0$ simplified model is not a good model since singular perturbation effect has been ignored. In the evolution problem for $\epsilon = 0$, if u^0 blows up in finite time, then at least for $t \geq t_s$, cannot ignore the ϵ term.

Mathematical Model of Viscous Fingering Problem



$$\Delta\phi = 0 \quad , \quad (x, y) \in \Omega \ \text{(exterior of finger)}$$

On $\partial\Omega$: $v_n = \frac{\partial\phi}{\partial n}$, and $\phi = \epsilon\kappa$,where κ, ϵ denote curvature and surface tension

Boundary condition at ∞: $\phi \sim x + O(1) \ \text{as} \ x \to \infty$

At the walls: $\frac{\partial\phi}{\partial y}(x, \pm 1) = 0$

For $\epsilon = 0$, Saffman & Taylor ('58) found steady finger solutions with width $\lambda \in (0, 1)$; experiment yields $\lambda \approx \frac{1}{2}$. Perturbation in powers of ϵ consistent! Yet, we now know the model is ill-posed

Perils in relying simply on consistency check

Consider the solution $\phi(x, y)$ to

$$\Delta \phi = 0 \ \text{ for } y > 0$$

On $y = 0$, require Boundary Condition

$$\epsilon \phi_{xxx}(x, 0) + (1 - x^2 + a)\phi_x(x, 0) - 2x\phi_y(x, 0) = 1,$$

where $a \in (-1, \infty)$ is real. Also require that as $x^2 + y^2 \to \infty$, $(x^2 + y^2) \, |\nabla \phi|$ bounded.

Can show $W(x + iy) = \phi_x(x, y) - i\phi_y(x, y)$ satisfes

$$\epsilon W'' + \left[-(z + i)^2 + a \right] W = 1$$

For $\epsilon = 0$, $W = W_0 \equiv \frac{1}{-(z+i)^2 + a}$. *Ansatz* $W = W_0 + \epsilon W_1 + ..$ consistent. Suggests no restriction on a. Yet, we will discover this conclusion to be incorrect !

Perils in relying on simply consistency –II

With scaling of dependent and independent variable, obtain:

$$z + i = i 2^{-1/2} \epsilon^{1/2} Z \ ; \ W = 2^{-1} \epsilon^{-1} G(Z) \ ; a = 2 \epsilon \alpha$$

$$G'' - (\frac{1}{4} Z^2 + \alpha) G = -1$$

Using parabolic cylinder functions, the above problem has an explicit solution. Requiring $G \to 0$ as $Z \to \infty$ for $\arg Z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is possible if and only for integer $n \geq 0$

$$\alpha = (2n + \frac{3}{2}) \ , \ i.e. \ a = 2\epsilon\alpha = 2\epsilon \left(2n + \frac{3}{2}\right)$$

$\lim_{\epsilon \to 0+}$ solution not equal $\epsilon = 0$ solution, unless a is as above. Discontinuity of solution set at $\epsilon = 0$. So ϵ term cannot be discarded, despite consistency of regular perturbation series.

Surprises when close to an ill-posed system

Suppose $\epsilon_1 << \epsilon$ in the following variation of the toy problem:

$$\epsilon W'' + \left[-(z+i)^2 + a - \frac{\epsilon_1}{(z+i)^2} \right] W = 1 \quad \text{for} \quad y = \text{Im}\, z \geq 0$$

Question: Should we ignore ϵ_1 term ? Appears reasonable since a scales as ϵ without ϵ_1 term and $\frac{\epsilon_1}{(z+i)^2} << a$ for $y = \text{Im}\, z \geq 0$.

This reasoning is incorrect.

Explanation: what matters is the size of ϵ_1-term in an $\epsilon^{1/2}$ neighborhood of $z = -i$. It is $O(\epsilon)$ when $\epsilon_1 = O(\epsilon^2)$.

Similar situation arises for small surface tension viscous fingering Saffman-Taylor problem. Combescot et al (1986), Shraiman (1986), T. (1986, 1987), Xie & T (2003)

The toy problem also illustrates that disparate length (and time) scales can interact in nearly ill-posed problem.

Regularity study for time-evolving problems

Recall for 1-D Burger's equation, any small viscous term become important if the inviscid Burger's equation develops singularity in u_x.

By analogy, if we study any time evolution equation, it is important to know if the problem has smooth solution for all time or if the solution becomes singular like inviscid Burger's solution. If it becomes singular, one seeks to determine what regularization ignored in the model should be included to describe the physical situation. This is true whether or not we model bacterial growth, thin fluid film, or evolution of stars. One of this outstanding problem is the master equation for fluid dynamics: 3D Navier-Stokes equation

3-D Navier-Stokes (NS) problem

$$v_t + (v \cdot \nabla) v = -\nabla p + \nu \Delta v + f \quad ; \quad \nabla \cdot v = 0,$$

where $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ is the fluid velocity and $p \in \mathbb{R}$ pressure at $x = (x_1, x_2, x_3) \in \Omega$ at time $t \geq 0$. Further, the operator $(v \cdot \nabla) = \sum_{j=1}^{3} v_j \partial_{x_j}$, ν = nondimensional visocity (inverse Reynolds number)

The problem supplemented by initial and boundary conditions:

$$v(x, 0) = v_0(x) \;\; (IC) \,, \; v = 0 \text{ on } \partial\Omega \text{ for stationary solid boundary}$$

We take $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3[0, 2\pi]$; no-slip boundary condition avoided, but assume in the former case $\|v_0\|_{L^2(\mathbb{R}^3)} < \infty$.

Millenium problem: Given smooth v_0 and f, prove or disprove that there exists smooth 3-D NS solution v for all $t > 0$. Note: global solution known in 2-D.

NS - a fluid flow model; importance of blow-up

$$v_t + (v \cdot \nabla)v = -\nabla p + \nu \Delta v + f \quad ; \quad \nabla \cdot v = 0,$$

Navier-Stokes equation models incompressible fluid flow.

$v_t + (v \cdot \nabla)v \equiv \frac{Dv}{Dt}$ represents fluid particle acceleration. The right side (force/mass) can be written: $\nabla \cdot \mathrm{T} + f$, where T: a tensor of rank 2, called stress with

$$\mathbf{T}_{jl} = -p\delta_{j,l} + \frac{\nu}{2}\left[\frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j}\right]$$

The second term on the right is viscous stress approximated to linear order in ∇v. Invalid for large $||\nabla v||$ or for non-Newtonian fluid (toothpaste, blood)

Incompressibility not valid if v comparable to sound velocity

Whether or not fluid turbulence is describable by Navier-Stokes depends on its global regularity.

Definition of Spaces of Functions

$H^m(\mathbb{R}^3)$: closure of C_0^∞ functions under the norm

$$\|\phi\|_{H^m} = \left\{ \sum_{0 \le l_1+l_2+l_3 \le m} \| \frac{\partial^{l_1+l_2+l_3}\phi}{\partial x_1^{l_1} \partial x_2^{l_2} \partial x_3^{l_3}} \|_{L_2}^2 \right\}^{1/2}$$

Note $H^0 = L_2$ If ϕ is a vector or tensor, components are also involved in the summation.

$H^m(\mathbb{T}^3[0, 2\pi])$: Closure under the above norm of C^∞ periodic functions in $x = (x_1, x_2, x_3)$ with 2π period in each direction.

$L_p\left([0, T], H^m(\mathbb{R}^3)\right)$ will denote the closure of the space of smooth functions of (x, t) under the norm:

$$\|v\|_{L_{p,t} H_{m,x}} \equiv \| \|v(., t)\|_{H^m} \|_{L_p}$$

Basic Steps in a typical nonlinear PDE analysis

Construct an approximate equation for $v^{(\epsilon)}$ that formally reduces to the PDE as $\epsilon \rightarrow 0$ such that ODE theory guarantees solution $v^{(\epsilon)}$

Find *a priori* estimate on v that satisfies PDE and also obeyed by $v^{(\epsilon)}$

Use some compactness argument to pass to the limit $\epsilon \rightarrow 0$ to obtain local solution of PDE

If *a priori* bounds on appropriate norms are globally controlled, then global solution follows. One way to get to classical (strong) solutions is to have *a priori* bounds on $\|v(.,t)\|_{H^m}$ for any m large enough.

For weak solutions, starting point is an equation obtained through inner product (in L_2) with a test function.

Some basic observations about Navier Stokes

For $f = 0$, $\Omega = \mathbb{R}^3$, if $v(x,t)$ is a solution, so is
$v_\lambda(x,t) = \frac{1}{\lambda} v\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$.

A space-time norm $\|.\|$ is called sub-critical if for $\lambda > 1$, $\|v_\lambda\| = \lambda^{-q}\|v\|$ for some $q > 0$. If the above is true for $q < 0$, the norm is termed super-critical

Basic Energy Equality for $f = 0$:

$$\frac{1}{2}\|v(.,t)\|_{L_2}^2 + \nu \int_0^t \|\nabla v(.,t'\|_{L_2}^2 dt' = \frac{1}{2}\|v_0\|_{L_2}^2$$

Therefore, for following *super-critical* norms over time interval $[0,T]$:

$$\|v\|_{L_{\infty,t} L_{2,x}} \leq \|v_0\|_{L_2} \quad , \quad \|v\|_{L_{2,t} H_x^1} \leq C$$

These are the only two known globally controlled quantities

Results by Leray

Leray (1933a,b, 1934) made seminal contributions:

A solution guaranteed in the space

$L_\infty\left((0,T), L_2(\mathbb{R}^3)\right) \cap L_2\left((0,T), H^1(\mathbb{R}^3)\right)$ for any $T > 0$.

For regular f and v_0, unique smooth solution in $(0, T^*)$

For $t \in (0, T^*)$, weak and strong solution the same. Only small v_0, f or large viscosity gives $T^* = \infty$

$\int_0^T \|\nabla v(.,t)\|_\infty dt < \infty$ guarantees smooth solution on $(0, T]$.

Uniqueness of Leray's global weak solution for $t > T^*$ not known

Leray conjectured formation of singular 1-D line vortices where $\nabla \times v$ blows up at some time t_0.

Also conjectured blow up for $f = 0$ via similarity solution

$$v(x,t) = (t_0 - t)^{-1/2} V\left(\frac{x}{(t_0 - t)^{1/2}}\right)$$

Some known important results -II

Cafarelli-Kohn-Nirenberg (1982): 1-D Hausdorff measure of the singular space-time set for Leray's weak solution is 0.

Necas-Ruzicka-Sverak (1996): no Leray similarity solution for $v_0 \in L^3$. Tsai (2003): no Leray-type similarity solution with finite energy and finite dissipation.

Beale-Kato-Majda (1984): $\int_0^T \|\nabla \times v(.,t)\|_\infty dt < \infty$ guarantees smooth v over $[0,T]$

Other controlling norms by Prodi-Serrin-Ladyzhenzkaya and Escauriaza, Seregin & Sverak (2003): $\|.\|_{L_{p_t} L_{s,x}}$ for $\frac{3}{s} + \frac{2}{p} = 1$ for $s \in [3, \infty)$.

Constantin-Fefferman (1994): If $\frac{\nabla \times v}{|\nabla \times v|}$ is uniformly Holder continuous in x in a region where $|\nabla \times v| > c$ for a sufficiently large c for $t \in (0,T]$, then smooth N-S solution exists over $(0,T]$

Difficulty with Navier-Stokes in the usual PDE analysis

Nonlinearity strong unless ν is large enough for given v_0 and f. Rules out perturbation about linear problem.

$\nu = 0$ approximation (3-D Euler equation) no simpler. Rules out any perturbative treatment.

The norms that are controlled globally are all super-critical: does not give sufficient control over small scales.

Other techniques include introduction of ϵ regularizations like hyperviscosity, compressibility, etc. and taking limit $\epsilon \to 0$

Maddingly-Sinai (2003): if $-\Delta$ is replaced by $(-\Delta)^\alpha$ in N-S equation, and $\alpha > \frac{5}{4}$ then global smooth solution exists.

Tao (2007) believes that no "soft" estimate can work including introduction of regularization. Believes global control on some critical or subcritical norm a must.

An alternate Borel based approach

Sobolev methods give no information about solution at $t = T^*$ when *a priori* Energy estimates breakdown.

A more constructive approach is to use ideas of Borel sum, with specific v_0, f and ν in mind.

Borel summation is a procedure that, under some conditions, generates an isomorphism between formal series and actual functions they represent (Ecalle, ..., O. Costin).

Formal expansion of N-S solution possible for small t: $v(x,t) = v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$.

Borel Sum of this series, which is sensible for analytic v_0 and f, leads to an actual solution to N-S (O. Costin & S. Tanveer, '06) in the form: $v(x,t) = v_0(x) + \int_0^{\infty} e^{-p/t} U(x,p) dp$. This form transcends assumptions on analyticity of v_0 and f or of t small

Borel Summation Illustrated in a Simple Linear ODE

$$y' - y = \frac{1}{x^2}$$

Want solution $y \to 0$, as $x \to +\infty$

Dominant Balance (or formally plugging a series in $1/x$):

$$y \sim -\frac{1}{x^2} + \frac{2}{x^3} + \ ...\frac{(-1)^k k!}{x^{k+1}} + .. \equiv \tilde{y}(x)$$

Borel Transform:

$$\mathcal{B}[x^{-k}](p) = \frac{p^{k-1}}{\Gamma(k)} = \mathcal{L}^{-1}[x^{-k}](p) \ \text{ for } \ Re\, p > 0$$

$$\mathcal{B}\left[\sum_{k=1}^{\infty} a_k x^{-k}\right](p) = \sum_{k=1}^{\infty} \frac{a_k}{\Gamma(k)} p^{k-1}$$

Borel Summation for linear ODE -II

$$Y(p) \equiv \mathcal{B}[\tilde{y}](p) = \sum_{k=1}^{\infty} (-1)^k p^k = -\frac{p}{1+p}$$

$$y(x) \equiv \int_0^{\infty} e^{-px} Y(p) dp = \mathcal{L}\mathcal{B}[\tilde{y}]$$

is the linear ODE solution we seek. Borel Sum defined as $\mathcal{L}\mathcal{B}$.

Note once solution is found, it is not restricted to large x.

Necessary properties for Borel Sum to exist:

1. **The Borel Transform $\mathcal{B}[\tilde{y}_0](p)$ analytic for $p \geq 0$,**
2. **$e^{-\alpha p}|\mathcal{B}[\tilde{y}_0](p)|$ bounded so that Laplace Transform exists.**

Remark: Difficult to check directly for non-trivial problems

Borel sum of nonlinear ODE solution

Instead, directly apply $\mathcal{L}^{-1}$ to equation; for instance

$$y' - y = \frac{1}{x^2} + y^2; \quad \text{with} \lim_{x\to\infty} y = 0$$

Inverse Laplace transforming, with $Y(p) = [\mathcal{L}^{-1}y](p)$:

$$-pY(p) - Y(p) = p + Y * Y \;\; \text{implying} \; Y(p) = -\frac{1}{1+p} - \frac{Y * Y}{1+p} \tag{1}$$

For functions Y analytic for $p \geq 0$ and $e^{-\alpha p}Y(p)$ bounded, it can be shown that (1) has unique solution for sufficiently large α.

Implies ODE solution $y(x) = \int_0^\infty Y(p)e^{-px}dp$ for $Re\, x > \alpha$

The above is a special case of results available for generic nonlinear ODEs (Costin, 1998)

Eg: Illustrative IVP: 1-D Heat Equation

$$v_t = v_{xx} \ , \ v(x,0) = v_0(x) \ \ , \ \ v(x,t) = v_0 + t v_1 + ..$$

Obtain recurrence relation

$$(k+1) v_{k+1} = v_k'' \ \ , \qquad \text{implies} \ \ v_k = \frac{v_0^{(2k)}}{k!}$$

Unless v_0 entire, series $\sum_k t^k v_k$ factorially divergent.

Borel transform in $\tau = 1/t$: $V(x,p) = \mathcal{B}[v(x, 1/\tau))](p)$,
$V(x,p) = p^{-1/2} W(x, 2\sqrt{p})$, then $W_{qq} - W_{xx} = 0$

Obtain $v(x,t) = \int_{\mathbb{R}} v_0(y) (4\pi t)^{-1/2} \exp[-(x-y)^2/(4t)] dy$,

i.e. **Borel sum of formal series leads to usual heat solution.**

We have applied these simple ideas to provide an alternate existence theory for 3-D Navier-Stokes (Costin & T., '08, Costin,

Generalized Laplace Representation and Results

Solution representation in the form

$$v(x,t) = v_0(x) + \int_0^\infty U(x,q) e^{-q/t^n} dq$$

Gives rise to an integral equation for $U(x,q)$ which was shown to have global smooth solution for $q \in \mathbb{R}^+$

If the solution $\hat{U}$ decays for large q, global NS existence follows. On the other hand, if global smooth NS solution exists, then for some large enough n, $\|\hat{U}(.,q)\|_{l^1}$ decreases exponentially in q.

Conclusions

Tried to show why theoretical questions of existence, uniqueness and well-posedness have a bearing on mathematical model and their physical predictions.

In particular, a model that is not well-posed is intrinsically deficient in predicting physical reality. Explained this in terms of the viscous fingering problem. How this 50 year old problem was not recognized to be ill-posed until relatively recently. Also showed how a system close to ill-posedness behaves in unpredictable manner.

Global regularity and singularity of smooth solutions in a time evolution model are also important physically. Explained how arbitrarily small viscosity effects the solution after the invisid equation becomes singular.

Explained why regularity questions in 3-D Navier Stokes problem is such an important problem to generate the attention of Clay