Mathematical Analysis of Floquet Problem

with applications to pipe/channel flows

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Background

Stability of oscillating pipe or channel flows important in transition to turbulence

Numerical methods become difficult and unreliable for large Reynolds number, most analytic calculations limited to small amplitude oscillations

Earlier investigations (Hall, ’78), Hall (’03), (Blennerhasset-Bassom, ’08) present a confusing picture; quantitative agreement with experiment (Eckmann-Grotberg, 1991) not good.

More generally, analysis methods available for stability of time oscillating states are quite limited- hence the motivation for this line of research.

2-D Linearized disturbance equation for parallel flow

$$2\partial_t[\partial_y^2 - \alpha^2]\psi - [\partial_y^2 - \alpha^2]^2\psi = -\frac{iU}{\epsilon}[\partial_y^2 - \alpha^2]\psi + \frac{iU_{yy}}{\epsilon}\psi,$$

where $0 < y < \beta$, $U(y,t)$ is the known time-periodic base flow, ψ: perturbed stream function, ϵ: reciprocal Reynolds number. Disturbance wavelength α fixed.

Initial condition: $\psi(y,0) = \psi_0(y)$ and no slip wall BC implies:

$$\psi(0,t) = \psi_y(0,t) = 0 = \psi(\beta,t) = \psi_y(\beta,t)$$

$U(y,t)$ depends on flow situation. For $\beta = \infty$ for walls oscillating along x-direction, $U(y,t) = e^{-y}\cos(t-y)$. Other expressions for time-oscillating pressure or for finite β. In pipe flows, equations more complicated, though similar mathematical structure

Relation of IVP with Floquet problem

In a general context, if

$$u_t = [\mathcal{A} + 2\cos t \;\; \mathcal{B}]\, u \;\; , u(x,0) = u_0,$$

where $\mathcal{A}$ and $\mathcal{B}$ are time-independent spatial operators.

If $\mathcal{A}^{-1}$ incorporates boundary or decay conditions at ∞, we can can write

$$\mathcal{A}^{-1} u_t = \left[\mathcal{I} + 2\cos t \;\; \mathcal{A}^{-1}\mathcal{B}\right] u$$

When *a priori* **exponential bounds in t exist, Laplace transform $U(.,p) = \int_0^t e^{-pt} u(.,t) dt$ justified and satisfies**

$$(\mathcal{I} - K)\, U(.,p) = u_0, \qquad (1)$$

where $K = p\mathcal{A}^{-1} - \mathcal{A}^{-1}\mathcal{B}S_+ - \mathcal{A}^{-1}\mathcal{B}S_-$, where shift operators defined by $[S_- U](.,p) = U(.,p-i)$, $[S_+ U](.,p) = U(.,p+i)$

Relation to Floquet problem- page II

If we define $p = \sigma + in$, $U(., \sigma + in) = U_n$,

$$[\mathcal{A} - in]\, U_n - \sigma U_n - \mathcal{B} U_{n-1} - \mathcal{B} U_{n+1} = u_0 \qquad (2)$$

When $\mathcal{R}_n \equiv [\mathcal{A} - in]^{-1}$ exists,

$$U_n = \sigma \mathcal{R}_n U_n + \mathcal{R}_n \mathcal{B} U_{n-1} + \mathcal{R}_n \mathcal{B} U_{n+1} + \mathcal{R}_n u_0$$

We may define operator $\mathcal{K}$ acting on $\mathrm{U} = \{U_n\}_{n\in\mathbb{Z}}$ such that

$$[\mathcal{K}\mathrm{U}]_n = \sigma \mathcal{R}_n U_n + \mathcal{R}_n \mathcal{B} U_{n-1} + \mathcal{R}_n \mathcal{B} U_{n+1}$$

$$\text{Then,} \quad [\mathcal{I} - \mathcal{K}]\, \mathrm{U} = \mathrm{U}^0$$

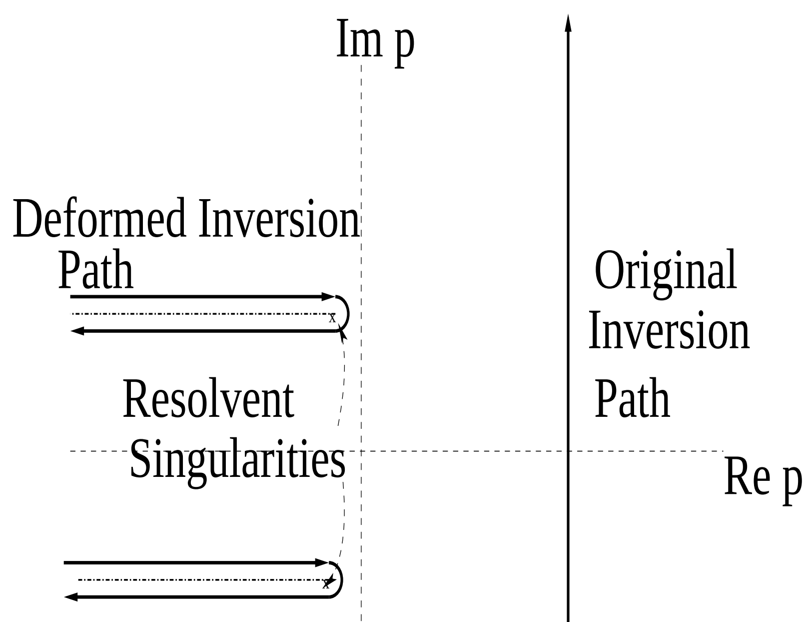
Fredholm applies when $\mathcal{K}$ is a compact operator on a Hilbert space, implying solvability iff only solution to Floquet problem $(\mathcal{I} - \mathcal{K})\mathrm{U} = 0$ is $\mathrm{U} = 0$.

Stability criteria

If Floquet problem has only zero solution for $\mathrm{Re}\,\sigma \geq 0$ in a Hilbert space where u_n decays sufficiently rapidly in n, then $u(x,t)$ decays since

$$u(x,t) = \int_{-i\infty}^{i\infty} e^{pt} U(x,p) dp$$

Since $U(.,p) = (I - \mathcal{K})^{-1} u_0 = \mathcal{R}_\sigma u_0$, the singularities of resolvent $\mathcal{R}_\sigma$ in σ determine the long-term behavior of $u(x,t)$



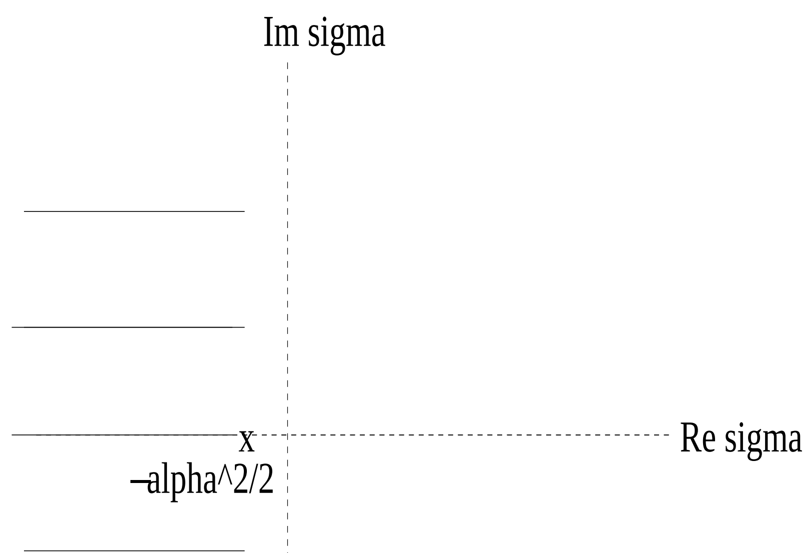
Floquet spectrum for oscillating plates

For the oscillating plate, the floquet problem becomes

$$\left(\partial_y^2 - \alpha^2 - 2\sigma - 2in\right)\Phi_n = \frac{iV}{2\epsilon}\left(1+\mathcal{I}\right)\Phi_{n+1} + \frac{iV^*}{2\epsilon}\left(1+\mathcal{I}\right)\Phi_{n+1},$$

$$\text{where } V(y) = a\left[e^{-(1+i)y} + e^{-(1+i)(\beta-y)}\right] \;\; , \psi = \mathcal{I}\phi$$

Theorem: For $0 < \beta < \infty$, the Floquet problem has only discrete spectrum. For $\beta = \infty$, discrete spectrum also, except for $\sigma \in \left\{-\frac{\alpha^2}{2} + i\mathbb{Z} + \mathbb{R}^-\right\}$



Floquet problem for oscillating plate for $\beta = \infty$

For $\beta = \infty$, $V(y) = \frac{1}{2}e^{-y(1+i)}$, with Floquet problem:

$$\left(\partial_y^2 - \alpha^2 - 2\sigma - 2in\right)\Phi_n = \frac{iV(y)}{\epsilon}\Phi_{n-1} + \frac{2V}{\epsilon}\mathcal{I}[\Phi_{n-1}]$$

$$+\frac{iV^*(y)}{\epsilon}\Phi_{n+1} - \frac{2V^*}{\epsilon}\mathcal{I}[\Phi_{n+1}].$$

Let $\gamma_n = \sqrt{\alpha^2 + 2\sigma + 2in}$. Hall ('74) assumed

$$\Phi_n = \sum_{j,k,n} A_{j,k,n} e^{-(\gamma_n + k + ij)y} + \sum_{j,k} B_{j,k} e^{-(\alpha + k + ij)y},$$

constrained by $\mathrm{Re}\,\gamma_n + k > 0$, $\alpha + k > 0$. The recurrence relations for $A_{j,k,n}$, $B_{j,k}$ were solved numerically. Concluded $\mathrm{Re}\,\sigma < 0$ for $\epsilon \geq 1/200$. Note: more and more terms are needed for accuracy as $\epsilon \to 0^+$.

Floquet problem for oscillating plate

Blennerhasset and Bassom ('08) concluded instability for $\epsilon \approx \frac{1}{700}$ based on numerics on the same recurrence relation. They suggest an inviscid instability mode

Based on U varying on relatively slow time scale, a quasi-steady calculation (Hall, '03) based on inviscid Rayleigh equation:

$$(U - c)\left[\partial_y^2 - \alpha^2\right]\psi - U_{yy}\psi = 0 \ \ , \text{with} \ \ c = 2i\epsilon\sigma$$

suggested stability.

Experiment (Merkli-Thomann, '75, Clemen-Minton, '77, Eckmann-Grotberg '91) suggests instability, though quantitative disagreement with theory about onset.

Effect of transients on possible nonlinear stability, like for non-oscillatory pipe and channel flow, is not known.

Further Laplace transform for $\beta = \infty$

Laplace Transform in y, which can be rigorously justified, gives

$$(s^2 - \lambda_n^2)\hat{\Phi}_n(s) = (s^2 - \lambda_n^2)\hat{\Phi}_n^{(0)}(s)$$

$$+\frac{i}{2\epsilon}\left(1 + \frac{2i}{[(s+1-i)^2 - \alpha^2]}\right)\hat{\Phi}_{n+1}(s+1-i)$$

$$+\frac{i}{2\epsilon}\left(1 - \frac{2i}{[(s+1+i)^2 - \alpha^2]}\right)\hat{\Phi}_{n-1}(s+1+i),$$

where

$$\hat{\Phi}_n^{(0)}(s) = \frac{\Phi_n'(0) + s\Phi_n(0)}{s^2 - \lambda_n^2},$$

$$\lambda_n^2 = \alpha^2 + 2\sigma + 2in$$

Contraction argument gives for large $\mathrm{Re}\, s$, unique solution

$\Phi(s) \sim \Phi^{(0)}(s)$

More on Floquet Problem for $\beta = \infty$

Convenient to introduce discretized variables

$$s_{k,j} = s+k-ij \ , \ \lambda_{n,k,j} = \lambda_n+k-ij \ , \ \Phi_{n,k,j}(s) = \Phi_n(s+k-ij)$$

Then, with

$$\beta^{(1)}_{n,k,j}(s) \equiv \frac{1}{s^2_{k,j} - \lambda^2_{n+j}} \left\{ 1 + \frac{2i}{\left[s^2_{k+1,j+1} - \alpha^2 \right]} \right\},$$

$$\beta^{(-1)}_{n,k,j}(s) \equiv \frac{1}{s^2_{k,j} - \lambda^2_{n+j}} \left\{ 1 - \frac{2i}{\left[s^2_{k+1,j-1} - \alpha^2 \right]} \right\},$$

$$\Phi_{n,k,j}(s) = \Phi^{(0)}_{n,k,j}(s) + \beta^{(1)}_{n,k,j}(s)\Phi_{n+1,k+1,j+1}(s)$$

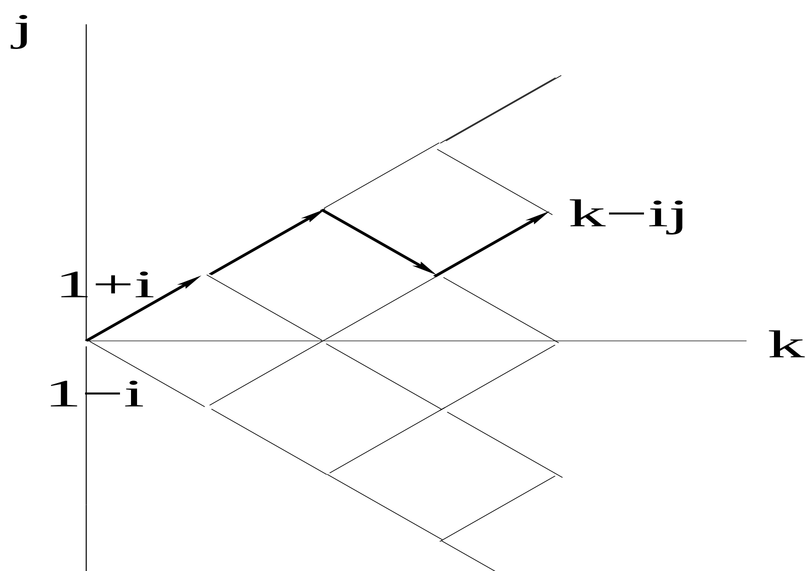
$$+\beta^{(-1)}_{n,k,j}(s)\Phi_{n-1,k+1,j-1}(s)$$

Associated Homogeneous Equation and Solution

$$G_{k,j}^{(n)} = \beta_{n+j,k,j}^{(1)} G_{k-1,j-1}^{(n)} + \beta_{n+j,k,j}^{(-1)} G_{k-1,j+1}^{(n)} \ , \ \text{with} \ G_{0,0}^{(n)} = 1$$

Introduce $\tau = \{a_1, a_2, .., a_k\} \in \{-1, 1\}^k$ **with** $j_k \equiv a_1 + a_2 \ + \ .. + a_k$**. Then for** $|j| \leq k$**,**

$$G_{k,j}^{(n)}(s) = \sum_{\tau, j_k = j} \prod_{l=1}^{k} \beta_{n+j_{l-1}, l-1, j_{l-1}}^{(a_l)}(s)$$



$$G_{k,j}^{(n)}(s) = \sum_{\tau, j_k = j} \prod_{l=1}^{k} \frac{1}{\left(s + l - 1 + i j_{l-1})^2 - \lambda^2_{n+j_{l-1}} \right)}$$

$$\times \left[1 + \frac{2 i a_l}{(s + l + i j_l)^2 - \alpha^2)}, \right]$$

where

$$j_{l-1} = a_1 + a_2 + ..a_{l-1} \quad , \quad j_0 = 0 \quad , \quad \{a_1, a_2, ...a_k\} \in \{-1, 1\}^k$$

$$\lambda_n = \sqrt{\alpha^2 + 2\sigma + 2in}$$

Solution in terms of $\{\Phi_n(0), \Phi_n'(0)\}_{n\in\mathbb{Z}}$

It can be proved that

$$\hat{\Phi}_n(s) = \sum_{k=0}^{\infty} \left(\frac{i}{2\epsilon}\right)^k \sum_{j=-k,2}^{k} G_{k,j}^{(n)}(s)\Phi_{n+j,k,j}^{(0)}(s)$$

Requiring solution to be pole free at $s = \lambda_n$, $s = \alpha$ gives

$$\sum_{j\in\mathbb{Z}} a_{n,n+j}\Phi_n(0) + \sum_{j\in\mathbb{Z}} b_{n,n+j}\Phi_n'(0) = 0, \text{ for } n \in \mathbb{Z}$$

$$\sum_{j\in\mathbb{Z}} c_{n,n+j}\Phi_n(0) + \sum_{j\in\mathbb{Z}} d_{n,n+j}\Phi_n'(0) = 0, \text{ for } n \in \mathbb{Z},$$

$$\text{where } a_{n,n+j} = \sum_{k=|j|}^{\infty} \left(\frac{i}{2\epsilon}\right)^k \frac{\alpha_{k,j} G_{k,j}^{(n)}(\alpha)}{\alpha_{k,j}^2 - \lambda_{n+j}^2},$$

Similarly expressions for $b_{n,n+j}$, $c's$, $d's$. Note $\left|G_{k,j}^{(n)}\right| \leq \frac{C}{k!}$.

Asymptotics for $G_{k,j}^{(n)}$ for $|j| << k$

for $|n| << k$, $k >> 1$, $\sigma << \frac{1}{\epsilon}$

We note that

$$\beta^{(a_l)}_{n+j_{l-1},l-1,j_{l-1}}(s) = \frac{1}{(s+l-1)^2-\lambda_n^2}\left[1-\frac{2ilj_{l-1}+j_{l-1}^2+2ij_{l-1}}{(s+l-1)^2-\lambda_n^2}\right]^{-}$$

For $l >> 1$, if $j_{l-1} << l$, then we have

$$\beta^{(a_l)}_{n+j_{l-1},l-1,j_{l-1}}(s) = \frac{1}{(s+l-1)^2-\lambda_n^2}+\frac{2ilj_{l-1}}{[(s+l-1)^2-\lambda_n^2]^2}+O(\frac{j_l^2}{l^2})$$

$$G_{k,j}^{(n)}(s) \sim \frac{A(n)\Gamma(s-\lambda_n)\Gamma(s+\lambda_n)}{\Gamma(s+k-\lambda_n)\Gamma(s+k+\lambda_n)}\frac{k!}{\left(\frac{k-j}{2}\right)!\left(\frac{k+j}{2}\right)!}\left[1+A_1\frac{j}{k}+..\right]$$

Computational details in $G^{(n)}_{k,j}(s)$

To get results for $G^{(n)}_{k,j}$ as quoted, we need

$$S_{k,j;m} \equiv \sum_{l=1}^{k} \sum_{\tau, j_k = j} g(l) j_{l-1}^m.$$

$$\text{Note that} \quad S_{k,j;m} = \sum_{l=1}^{k} f(l) \partial_\beta^m |_{\beta=0} \quad T_{l,k,j}(\beta),$$

$$T_{l,k,j}(\beta) = \sum_{\tau, j_k = j} e^{\beta j_{l-1}}$$

$$\zeta(z;\beta) \equiv \sum_{j=-k}^{k} T_{l,k,j} z^j = \sum_{\tau} e^{a_1(\beta + \log z)} .. e^{a_{l-1}(\beta + \log z)} e^{a_l \log z} ... e^{a_k \log z}$$

$$\zeta = \left(z e^{\beta} + \frac{1}{z} e^{-\beta} \right)^{l-1} \left(z + \frac{1}{z} \right)^{k-l+1}$$

Floquet Spectrum in the closed right-half plane

Use of Gamma function asymptotics and Euler-McLaurin summation converts the system of equation into a set of integral equations for which there is no nonzero solution for $\mathrm{Re}\,\sigma \geq 0$ for $|\sigma| \leq \frac{c}{\epsilon}$ for some small c.

Theorem: For $\beta = \infty$, the Floquet problem for oscillating plate has no spectrum in the region $\mathbf{Re}\,\boldsymbol{\sigma} \geq \mathbf{0}$ for $|\boldsymbol{\sigma}| \leq \frac{c}{\epsilon}$ for some small $\boldsymbol{c}$.

For $\sigma = O\left(\frac{1}{\epsilon}\right)$ a different asymptotic analysis is needed.

Further, for finite β, we use a Neumann series based on Volterra kind of integral equation, instead of explicit Laplace transform in y, though analysis is more complicated.

Other non-perturbative Floquet problems require somewhat different techniques, as exemplified in the following for the 3-D Schroedinger equation with time-periodic potential.

Floquet problem in ionization of hydrogen atom

Reference: O. Costin, J. Lebowitz, S.T, Comm. Math. Phys, 2010

$$\left(-\Delta - \frac{b}{r} - i\sigma + n\omega\right)\Phi_n = -i\Omega(|x|)\left[\Phi_{n+1} - \Phi_{n-1}\right]$$

$$\text{reduces to } \left[\frac{d^2}{dr^2} + \frac{b}{r} - \frac{l(l+1)}{r^2} + i\sigma - n\omega\right] w_n = -i\Omega[w_{n+1} - w_{n-1}]$$

$\Omega(r)$ assumed smooth and nonzero in support $r \le 1$. Also, can prove $i\sigma \in \mathbb{R}$

Can prove $w_n = 0$ for $r > 1$ for $n < 0$ as otherwise $\Phi_n = \frac{w_n(r)}{r} Y_{l,m}(\theta,\phi) \notin L^2(\mathbb{R}^3)$, implying $w_n(1)$, $w_n'(1) = 0$ for $n < 0$.

Floquet problem asymptotics for Hydrogen atom

Define n_0 as the smallest positive integer for which either $w_{n_0}(1)$ or $w'_{n_0}(1)$ nonzero for assumed nonzero solution. Take the case $w_{n_0}(1) \neq 0$, taken 1 w.l.o.g. Find $\frac{\partial^j}{\partial \xi^j} w_{n_0-k}(1) = i^k \delta_{j,2k}$, where $\xi = \int_r^1 \sqrt{\Omega(s)} ds$. For ξ small, $w_{n_0-k} \sim \frac{i^k \xi^{2k}}{(2k)!}$.

Above suggests that for $r = O(1)$, for $k >> 1$,

$$w_{n_0-k} \sim \frac{i^k \xi^{2k}}{(2k)!} f(r),$$

Requiring $O(k^2), O(k)$ terms to vanish in the residual

$$R_k \equiv \frac{\mathcal{L}_k w_{n_0-k} - i\Omega \left[w_{n_0-k+1} - w_{n_0-k+1} \right]}{g_{n_0-k}(r)},$$

$$\text{gives } f(r) = \Omega^{-1/4}(r) \Omega^{1/4}(0) \exp \left[\frac{1}{4} \int_1^r ds \frac{\omega \xi(s)}{\sqrt{\Omega(s)}} \right]$$

Hydrogen Floquet Problem asymptotics

The asymptotics $w_{n_0-k} \sim \frac{i^k \xi^{2k}}{(2k)!} f(r)$ invalid when $kr = O(1)$. We demand substitution of

$$w_{n_0-k} = \frac{i^k \xi^{2k}}{(2k)!} f(r) \frac{H(k\alpha r)}{H(k\alpha)}$$

result in residuals of $O(1)$ uniformly in $r \in (0, 1]$. Obtain to the leading order in k,

$$H(\zeta) \sim \sqrt{\frac{2}{\pi}} e^{\zeta} \zeta^{1/2} K_{l+1/2}(\zeta) \text{ where } K_{l+1/2} \text{ is a Bessel function.}$$

Any assumed nonzero solution is singular at $r = 0$. Therefore, Floquet problem has no acceptable solution for $\mathrm{Re}\, \sigma \geq 0$, implying hydrogen atom ionizes for assumed time-periodic compact potential of arbitrary size. (Proofs appear in the paper cited).

Conclusions

The Floquet spectral problem arises naturally in the linearized time-evolution equation for disturbance on a time-periodic solution. May be rigorously and constructively analyzed in a number of situations, including oscillating channel and pipe flows, 3-D Schroedinger equations, etc.

For Stokes layer problem $\beta = \infty$ problem, an intriguing connection revealed with calculation of expected value in some stochastic process. A continuum limit is identified as $\epsilon \to 0$ that reduces an infinite discrete system of linear equation into a system of integral equations for which the only solution is 0 for $\mathrm{Re}\,\sigma \geq 0$ when $\sigma << \frac{1}{\epsilon}$. Analysis for $\sigma = O(\frac{1}{\epsilon})$ is in progress

In some problems like the 3-D Schroedinger equation with a time-periodic compact potential added to Coulomb potential, the infinite set of differential-difference equations may be analyzed through rigorous WKB analysis.

An integral reformulation of 2-D channel IVP

If we introduce $\phi = (\partial_y^2 - \alpha^2)\psi$, then equation may be written as:

$$2\partial_t \phi - \left(\partial_y^2 - \alpha^2\right)\phi = -\frac{iU}{2\epsilon}\phi + \frac{iU_{yy}}{2\epsilon}\mathcal{I}[\phi],$$

where operator $\mathcal{I} : L^2(0,\beta) \to H^2(0,\beta)$ is defined by

$$\mathcal{I}[\phi](y) = \frac{\sinh(\alpha y)}{\alpha \sinh(\alpha\beta)} \int_\beta^y \sinh[\alpha(\beta - y')]\phi(y')dy'$$

$$-\frac{\sinh(\alpha(\beta - y))}{\alpha \sinh[\alpha\beta]} \int_0^y \sinh(\alpha y')\phi(y')dy',$$

which incorporates $\mathcal{I}[\phi](0) = 0 = \mathcal{I}[\phi](\beta)$. For $\beta = \infty$,

$$\mathcal{I}[\phi](y) = \frac{e^{-\alpha y}}{\alpha} \int_\infty^y \sinh(\alpha y')\phi(y')dy' - \frac{\sinh(\alpha y)}{\alpha} \int_0^y e^{-\alpha y'}\phi(y')dy',$$

Integral reformulation-II

An operator $\mathcal{R}$ similar to similar to $\mathcal{I}$ can be defined as an inversion of $\left(\partial_y^2 - \alpha^2\right)$ such that for $\chi \in L^2(0, \beta)$, $\frac{d}{dy}\mathcal{I}\left[\mathcal{R}[\chi]\right]$ is zero at $y = 0$ and $y = \beta$. When $\beta = \infty$, replace by decay.

Evolution for ϕ may be written as:

$$\phi - \partial_t \mathcal{R}[\phi] = \frac{i}{2\epsilon}\mathcal{R}\left[U\phi\right] - \frac{i}{2\epsilon}\mathcal{R}\left[U_{yy}\mathcal{I}[\phi]\right]$$

Integration in time over $(0, t)$ results in an integral reformulation for rigorous justification of Laplace transform in t, and determining how Floquet spectrum relates to initial value problem.

Space integration of ψ equation gives $O(\frac{1}{\epsilon})$ growth rate, since

$$\frac{d}{dt}\left\{\|\psi_y\|^2 + \alpha^2\|\psi\|^2\right\} + \|\psi_{yy}\|^2 \leq \frac{|U_y|_\infty}{2\epsilon\alpha}\left\{\|\psi_y\|^2 + \alpha^2\|\psi\|^2\right\}$$