A new approach to Regularity and Singularity Questions in some PDEs including 3-D Navier-Stokes

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Regularity and Singularity in PDEs-background

- PDEs modeling physical phenomena typically include some effects while ignoring others.

- Existence and uniqueness questions of smooth solutions fundamental to relevance of a PDE model, as is blow-up.

- Global existence of evolutionary PDE solutions typically rely on "Energy" methods. Control over sufficiently higher order Sobolev norm often necessary.

- Numerical discretization not rigorously controllable, generally. Further, numerical resolution becomes an issue in higher dimensions.
Navier-Stokes existence—background

- Global Existence of smooth 3-D Navier-Stokes solution is an important open problem.

- Deviation from linear stress-strain relation or incompressibility is potentially important if N-S solutions are singular

- Globally smooth solutions known only when Reynolds number small

- Generally, smooth solutions for smooth data on $[0, T]$ known to exist, for $T$ scaling inversely with initial data/forcing.

- Global weak solutions known since Leray, but not known whether they are unique. For unforced problem in $\mathbb{T}^3$, such a solution becomes smooth again for $t > T_c$, $T_c$ depends on IC
Borel Summation—background and main idea

- Borel summation generates an isomorphism between formal series and actual functions with respect to all usual algebraic operations (Ecalle, Costin,..). Borel summation used in exponential asymptotics (Dingle, Berry,..).

- Borel sum can involve large or small variable(s)/ parameter(s).

- Formal expansion for $t << 1$: $v(x, t) = v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$ obtained algorithmically by plugging into $v_t = \mathcal{N}[v]$, where $\mathcal{N}$ being some differential operator. Series usually divergent

- Borel Sum of this series gives actual solution, which transcends restriction $t << 1$

- For NS or Burger’s equation, Borel sum given by:

$$v(x, t) = v_0(x) + \int_0^{\infty} U(x, p)e^{-p/t} dp$$

$U$ satisfies Integral equation obtained by inverse LT of PDE.
Borel Summation Illustrated in a Simple Linear ODE

\[ y' - y = \frac{1}{x^2} \]

Want solution \( y \to 0 \), as \( x \to +\infty \)

Dominant Balance (or formally plugging a series in \( 1/x \)):

\[
y \sim -\frac{1}{x^2} + \frac{2}{x^3} + \cdots \frac{(-1)^k k!}{x^{k+1}} + \cdots \equiv \tilde{y}(x)
\]

Borel Transform:

\[
\mathcal{B}[x^{-k}](p) = \frac{p^{k-1}}{\Gamma(k)} = \mathcal{L}^{-1}[x^{-k}](p) \quad \text{for} \quad \text{Re } p > 0
\]

\[
\mathcal{B} \left[ \sum_{k=1}^{\infty} a_k x^{-k} \right] (p) = \sum_{k=1}^{\infty} \frac{a_k}{\Gamma(k)} p^{k-1}
\]
Borel Summation for linear ODE -II

\[ Y(p) \equiv B[\tilde{y}](p) = \sum_{k=1}^{\infty} (-1)^k p^k = -\frac{p}{1 + p} \]

\[ y(x) \equiv \int_0^{\infty} e^{-px} Y(p) dp = \mathcal{LB}[\tilde{y}] \]

is the linear ODE solution we seek. Borel Sum defined as \( \mathcal{LB} \).
Note once solution is found, it is not restricted to large \( x \).

Necessary properties for Borel Sum to exist:

1. The Borel Transform \( \mathcal{B}[\tilde{y}_0](p) \) analytic for \( p \geq 0 \),
2. \( e^{-\alpha p} |\mathcal{B}[\tilde{y}_0](p)| \) bounded so that Laplace Transform exists.

Remark: Difficult to check directly for non-trivial problems
Borel sum of nonlinear ODE solution

Instead, directly apply $\mathcal{L}^{-1}$ to equation; for instance

$$y' - y = \frac{1}{x^2} + y^2; \quad \text{with} \quad \lim_{x \to \infty} y = 0$$

Inverse Laplace transforming, with $Y(p) = [\mathcal{L}^{-1} y](p)$:

$$-pY(p) - Y(p) = p + Y \ast Y \quad \text{implying} \quad Y(p) = -\frac{1}{1 + p} - \frac{Y \ast Y}{1 + p}$$

For functions $Y$ analytic for $p \geq 0$ and $e^{-\alpha p}Y(p)$ integrable, it can be shown above has unique solution for sufficiently large $\alpha$.

Implies ODE solution $y(x) = \int_0^\infty Y(p)e^{-px}dp$ for $Re \ x > \alpha$

The above is a special case of nonlinear ODEs (Costin, 1998). Generalized to sectorial PDE solutions (Costin & T., ’07)
Borel sum of nonlinear ODE solution-II

Define $\chi_j(p)$ characteristic function, equalling 1 for $p \in [j, (j + 1))$ and zero otherwise.

Define $Y_j(p) = Y(p)\chi_j(p)$. Then from property of Laplace convolution $*$ for $p \in [j, j + 1)$: $Y * Y = \sum_{l=0}^{j} Y_l * Y_{j-l}$

Therefore, integral equation for $p \in [j, j + 1)$ becomes:

$$Y_j + \frac{2Y_0 * Y_j}{1 + p} = -\frac{p}{1 + p} - \frac{1}{1 + p} \sum_{l=1}^{j-1} Y_l * Y_{j-l}$$

Nonlinear ODE problem transformed to a sequence of linear problems beyond $[0, 1)$ interval. If a convergent series or other representation is available in $[0, 1)$, the rest involves a sequence of linear problem. This feature generalizes to nonlinear PDEs as well.
Integral Equation corresponding to Burger’s equation

Plug in $v = v_0(x) + u(x, t)$ into 1-D Burger’s to obtain

$$u_t - u_{xx} = -v_0 u_x - uv_{0,x} - uu_x + v_1(x), \quad v_1(x) = v''_0 - v_0v_{0,x}$$

with $u(x, 0) = 0$

Inverse Laplace Transform in $1/t$ and Fourier-Transform in $x$:

$$p\hat{U}_{pp} + 2U_p + k^2\hat{U} = -ik\hat{v}_0\ast\hat{U} - ik\hat{U}\ast\hat{U} \equiv \hat{G}(k, p)$$

Inverting left side using $\hat{U}(k, 0) = 0$ gives:

$$\hat{U}(k, p) = \int_0^p \mathcal{K}(p, p'; k)\hat{G}(k, p')dp' + \hat{U}^{(0)}(k, p) \equiv \mathcal{N}\left[\hat{U}\right](k, p)$$

$$\mathcal{K}(p, p'; k) = \frac{ik\pi}{z} \left\{ z'Y_1(z')J_1(z) - zY_1(z)J_1(z') \right\}$$

$$z = 2|k|\sqrt{p}, \quad z' = 2|k|\sqrt{p'}, \quad \hat{U}^{(0)}(k, p) = 2\frac{J_1(z)}{z}\hat{v}_1(k)$$
Solution to integral equation $\hat{U} = \mathcal{N} [\hat{U}]$

\[ |\mathcal{K}(p, p'; k)| \leq \frac{C}{\sqrt{p}}, \quad C \text{ a constant} \]

\[ \|\hat{F}(., p) \ast \hat{G}(., p)\|_{L^1(\mathbb{R}^3)} \leq C \|\hat{F}(., p)\|_{L^1(\mathbb{R}^3)} \|\hat{G}(., p)\|_{L^1(\mathbb{R}^3)} \]

Define for functions of $F(p, k)$ the norm:

\[ \|F\|^{(\alpha)} = \int_{0}^{\infty} e^{-\alpha p} \|F(., p)\|_{L^1(\mathbb{R}^3)} \, dp \], then can show

\[ \|F^*G\|^{(\alpha)} \leq C \|F\|^{(\alpha)} \|G\|^{(\alpha)} \]

Using above, can show $\mathcal{N}$ contractive for large $\alpha$; implies integral equation has unique solution and so Burger PDE has continuous solution for $\text{Re} \frac{1}{t} > \alpha$ as $v(x, t) = v_0(x) + \int_{0}^{\infty} e^{-p/t}U(x, p) \, dp$

Global PDE solution if $\|\hat{U}(., p)\|_{L^1(\mathbb{R}^3)}$ does not grow as $p \to \infty$
Incompressible 3-D Navier-Stokes in Fourier-Space

Consider 3-D N-S in infinite geometry or periodic box. Similar results expected for finite domain with no-slip BC using eigenfunctions of Stokes operator as basis. In Fourier-Space

\[ \hat{v}_t + \nu |k|^2 \hat{v} = -i k_j P_k [\hat{v}_j \hat{v}^*] + \hat{f}(k) \]

\[ P_k = \left( I - \frac{k(k\cdot)}{|k|^2} \right) \quad , \quad \hat{v}(k, 0) = \hat{v}_0(k) \]

where \( P_k \) is the Hodge projection in Fourier space, \( \hat{f}(k) \) is the Fourier-Transform of forcing \( f(x) \), assumed divergence free and \( t \)-independent. Subscript \( j \) denotes the \( j \)-th component of a vector. \( k \in \mathbb{R}^3 \) or \( \mathbb{Z}^3 \). Einstein convention for repeated index followed. \( \ast \) denotes Fourier convolution.

Decompose \( \hat{v} = \hat{v}_0 + \hat{u}(k, t) \), inverse-Laplace Transform in \( 1/t \) and invert the differential operator on the left side.
Integral equation associated with Navier-Stokes

We obtain:

\[
\hat{U}(k, p) = \int_0^p \mathcal{K}_j(p, p'; k) \hat{H}_j(k, p') dp' + \hat{U}^{(0)}(k, p) \equiv \mathcal{N} \left[ \hat{U} \right] (k, p)
\]

(1)

\[
\mathcal{K}_j(p, p'; k) = \frac{ik_j \pi}{z} \{ z' Y_1(z') J_1(z) - z Y_1(z) J_1(z') \}
\]

\[
z = 2|k|\sqrt{\nu p}, \quad z' = 2|k|\sqrt{\nu p'}, \quad \hat{H}_j = P_k \left\{ \hat{v}_{0,j} \hat{U} + \hat{U}_j \hat{v}_0 + \hat{U}_j \hat{U} \right\}
\]

\[
\hat{U}^{(0)}(k, p) = 2 \frac{J_1(z)}{z} \hat{v}_1(k), \quad P_k = \left( I - \frac{k(k \cdot)}{|k|^2} \right)
\]

\[
\hat{v}_1(k) = (-\nu |k|^2 \hat{v}_0 - ik_j \mathcal{P}_k [\hat{v}_{0,j} \hat{v}_0]) + \hat{f}(k),
\]

\*, denotes Fourier Convolution, \* denotes Laplace convolution, while \* denotes Fourier followed by Laplace convolution. \( J_1 \) and \( Y_1 \) are the usual Bessel functions.
Results for Integral equation and Navier-Stokes-1

**Theorem:** If \( \| \hat{v}_0 \|_{l^1(\mathbb{Z}^3)}, \| \hat{f} \|_{l^1(\mathbb{Z}^3)} < \infty \) then there exists some \( \alpha \) so that integral equation \( \hat{U} = \mathcal{N} \left[ \hat{U} \right] \) has a unique solution for \( p \in \mathbb{R}^+ \) in the space of functions \( \left\{ \hat{U} : \| \hat{U} \|^{(\alpha)} < \infty \right\} \). Further, \( \hat{v}(k, t) = \hat{v}_0(k) + \int_0^\infty \hat{U}(k, p)e^{-p/t}dp \) solves 3-D Navier-Stokes in Fourier-Space; the corresponding \( v(x, t) \) is a classical Navier-Stokes solution for \( t \in (0, \alpha^{-1}) \).

**Remark 1:** Local existence results in Theorem 1 already known through classical methods. In the present formulation, global PDE existence is a question of asymptotics of known solution to integral equation in the sense that a sub-exponential growth of \( \hat{U} \) as \( p \to \infty \) implies global existence of PDE solution.
Remark 2: Errors in Numerical solutions rigorously controlled. Discretization in $p$ and Galerkin approximation in $k$ results in:

$$\hat{U}_\delta(k, m\delta) = \delta \sum_{m'=0}^{m} K_{m,m'} P_N H_\delta(k, m'\delta) + \hat{U}^{(0)}(k, m\delta)$$

$$\equiv N_\delta \left[ \hat{U}_\delta \right] \quad \text{for} \quad k_j = -N, \ldots, N, \quad j = 1, 2, 3$$

$P_N$ is the Galerkin Projection into $N$-Fourier modes. $N_\delta$ has properties similar to $N$. The continuous solution $\hat{U}$ satisfies

$$\hat{U} = N_\delta \left[ \hat{U} \right] + E,$$

where $E$ is the truncation error. Thus, $\hat{U} - \hat{U}_\delta$ can be estimated using same tools as in Theorem 1.

Note: Similar control over discretized solutions to PDEs not available since truncation errors involve derivatives of PDE solution which are not known to exist beyond a short-time.
Numerical Solutions to integral equation

We choose the Kida initial conditions and forcing

\[ v_0(x) = (v_1(x_1, x_2, x_3, 0), v_2(x_1, x_2, x_3, 0), v_3(x_1, x_2, x_3, 0)) \]

\[ v_1(x_1, x_2, x_3, 0) = v_2(x_3, x_1, x_2, 0) = v_3(x_2, x_3, x_1, 0) \]

\[ v_1(x_1, x_2, x_3, 0) = \sin x_3 (\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3) \]

\[ f_1(x_1, x_2, x_3) = \frac{1}{5} v_1(x_1, x_2, x_3, 0) \]

High Degree of Symmetry makes computationally less expensive

Corresponding Euler problem believed to blow up in finite time; so good candidate to study viscous effects

In the plots, "constant forcing" corresponds to \( f = (f_1, f_2, f_3) \) as above, while zero forcing refers to \( f = 0 \). Recall sub-exponential growth in \( p \) corresponds to global N-S solution.
Numerical solution to integral equation-plot-1

\[ \|\hat{U}(\cdot, p)\|_{l^1} \text{ vs. } p \text{ for } \nu = 1, \text{ constant forcing.} \]
\[ \| \hat{U}(\cdot, p) \|_{l^1} \text{ vs. } p \text{ for } \nu = 1, \text{ no forcing} \]
Numerical solution to integral equation-plot-3

\[ \| \hat{U}(\cdot, p) \|_{L^1} \text{ vs. } p \text{ for } \nu = 0.16, \text{ constant forcing} \]
Numerical solution to integral equation-plot-4

\[ \| \hat{U}(., p) \|_{L^1} \text{ vs. } p \text{ for } \nu = 0.1, \text{ constant forcing} \]
$\hat{U}(k, p)$ vs. $p$ for $k = (1, 1, 17)$, $\nu = 0.1$, no forcing.
Numerical solution to integral equation-plot-6

\[ \log \| \hat{U}(., p) \|_{L^1} \text{ vs. } \log p \text{ for } \nu = 0.001, \text{ constant forcing} \]
Issues raised by numerical computations

Numerical solutions to integral equation available on finite interval \([0, p_0]\), yet N-S solution requires \([0, \infty)\) interval since
\[
\hat{v}(k, t) = \hat{v}_0 + \int_0^\infty e^{-p/t} \hat{U}(k, p) dp
\]

Actually, the integral over \(\int_0^{p_0}\) gives an approximate N-S solution, with errors that can be bounded for a time interval \([0, T]\), if computed solution to integral equation eventually decreases with \(p\) on a sufficiently large interval \([0, p_0]\).

Further, a non-increasing \(\hat{U}\) over a sufficiently large interval \([0, p_0]\) gives smaller bounds on growth rate \(\alpha\) as \(p \to \infty\).

Therefore, in such cases smooth NS solution exists over a long interval \([0, \alpha^{-1})\).

Recall for unforced problem in \(T^3\), even weak solution to NS becomes smooth for \(t > T_c\), with \(T_c\) estimated from initial data. Hence global existence follows under some conditions.
Extending Navier-Stokes interval of existence

For \( \alpha_0 \geq 0 \), define

\[
\epsilon = \nu^{-1/2} p_0^{-1/2}, \quad a = \|\hat{v}_0\|_{l^1}, \quad c = \int_{p_0}^{\infty} \|\hat{U}^{(0)}(., p)\|_{l^1} e^{-\alpha_0 p} dp
\]

\[
\epsilon_1 = \nu^{-1/2} p_0^{-1/2} \left( 2 \int_{0}^{p_0} e^{-\alpha_0 s} \|\hat{U}(., s)\|_{l^1} ds + \|\hat{v}_0\|_{l^1} \right)
\]

\[
b = \frac{e^{-\alpha_0 p_0}}{\sqrt{\nu p_0 \alpha}} \int_{0}^{p_0} \|\hat{U}^* \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{l^1} ds
\]

**Theorem 3:** A smooth solution to 3-D Navier-Stokes equation exists on the interval \([0, \alpha^{-1})\), when \( \alpha \geq \alpha_0 \) is chosen to satisfy

\[
\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon} - \epsilon_1^2
\]

**Remark:** If \( p_0 \) is chosen large enough, \( \epsilon, \epsilon_1 \) is small when computed solution in \([0, p_0]\) decays with \( q \). Then \( \alpha \) can be chosen rather small.
Relation of Optimal $\alpha$ to Navier-Stokes singularities

$$\hat{U}(k, p) = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} e^{p/t} [\hat{v}(k, t) - \hat{v}_0(k)] d \left[ \frac{1}{t} \right]$$

Rightmost singularity(ies) of NS solution $\hat{v}(k, t)$ in the $1/t$ plane determines optimal $\alpha$. $\gamma$ gives dominant oscillation frequency.
Laplace-transform and accelerated representation

To get rid of the effect of complex singularity, it is prudent to seek a more general Laplace-transform involves

\[ \hat{v}(k, t) = \hat{v}_0(k) + \int_0^\infty e^{-q/t^n} \hat{U}(k, q) dq \]

We have proved that for the unforced problem, if there are complex singularities \( t_s \) in the right-half plane, but not on the real axis, then a nonzero lower bound for \( |\arg t_s| \) exists. Then, for sufficiently large \( n \), no singularities in the \( \tau = t^{-n} \) plane in the right-half plane. Hence, \( \hat{U}(k, q) \) will not grow with \( q \)

\( \hat{U}(k, q) \) satisfies an integral equation similar to the one satisfied by \( \hat{U}(k, p) \) and Theorems similar to Theorem 1 follow. In the context of ODEs, change of variable \( p \rightarrow q \) is called acceleration (Ecalle)
$\| \hat{U}(., q) \|_{l^1}$ vs. $q$, $n = 2$, $\nu = 0.1$

Kida I.C. $v_1^{(0)} = \sin x_1 (\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3)$

Other components from cyclic relation:

$v_1^{(0)}(x_1, x_2, x_3) = v_1^{(0)}(x_3, x_1, x_2) = v_3^{(0)}(x_2, x_3, x_1)$
Extending Navier-Stokes interval of existence

For $\alpha_0 \geq 0$, define

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} , \quad c = \int_{q_0}^{\infty} \|\hat{U}(0) (\cdot, q)\|_{l^1} e^{-\alpha_0 q} dq$$

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} \left( 2 \int_0^{q_0} e^{-\alpha_0 s} \|\hat{U} (\cdot, s)\|_{l^1} ds + \|\hat{v}_0\|_{l^1} \right)$$

$$b = \frac{e^{-\alpha_0 q_0}}{\sqrt{\nu q_0^{1-1/(2n)}}} \int_0^{q_0} \|\hat{U}^* \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{l^1} ds$$

Theorem 4: A smooth solution to 3-D Navier-Stokes equation exists in the $\|\cdot\|_{l^1}$ space on the interval $[0, \alpha^{-1/n})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

Remark: If $q_0$ is chosen large enough, $\epsilon$, $\epsilon_1$ is small when computed solution in $[0, q_0]$ decays with $q$. Then $\alpha$ can be chosen rather small.
Example problems where approach is applicable

- Navier-Stokes with temperature field (Boussinesq approximation)
- Fourth order Parabolic equations of the type:
  \[ u_t + \Delta^2 u = N[u, Du, D^2u, D^3u] \]
- KDV and related equations.
- Magneto-hydrodynamic equation with certain approximations.
- For some PDE problems with finite-time blow-up, blow-up time related to exponent \( \alpha \) of exponential growth of IE solution, provided there is no-oscillation even with \( p \to q \) acceleration.
Conclusions

We have shown how Borel summation methods provides an alternate existence theory for PDE Initial value problems like N-S. With this integral equation (IE) approach, the PDE global existence is implied if known solution to IE has subexponential growth at $\infty$.

The solution to integral equation in a finite interval can be computed numerically with rigorously controlled errors. Integral equation in a suitable accelerated variable $q$ will decay exponentially for unforced N-S equation, unless there is a real time singularity of PDE solution.

The computation over a finite $[0, q_0]$ interval gives a refined bound on exponent $\alpha$ at $\infty$, and hence a longer existence time $[0, \alpha^{-1/n})$ to 3-D Navier-Stokes.

Approach should be useful in both regularity and singularity studies of more general PDE initial value problems.