Conformal Mapping and Boundary integral approach to Laplacian Growth Problems in 2-D

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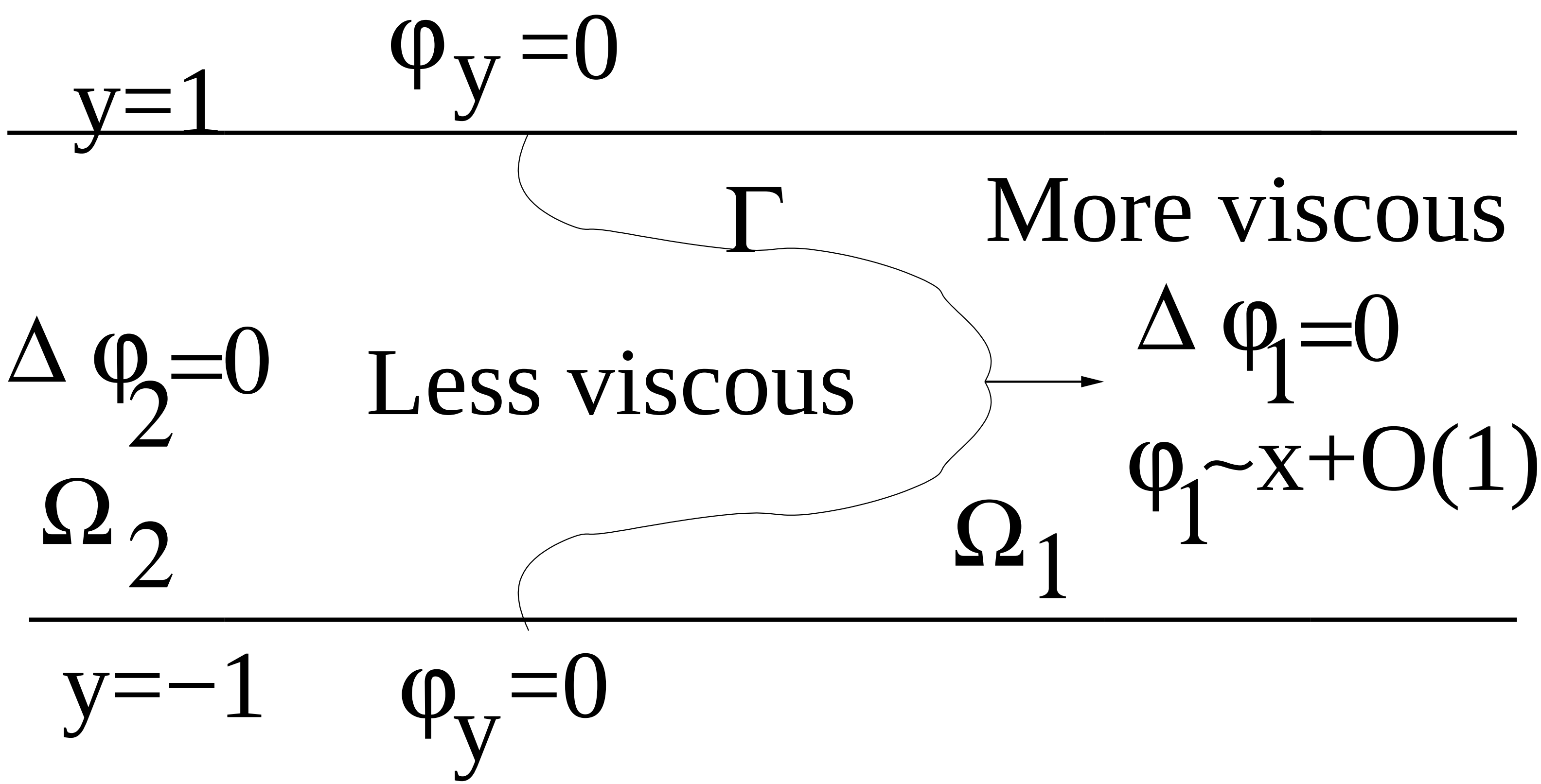
Idealized Hele-Shaw flow model

Gap averaged Stokes flow: $\mathbf{u}_1 = -\frac{b^2}{12\mu_1}\nabla p_1$ **in** Ω_1 **and** $\mathbf{u}_2 = -\frac{b^2}{12\mu_2}\nabla p_2$ **in** Ω_2**. With** $\phi_1 = -\frac{b^2}{12\mu_1}p_1$, $\phi_2 = -\frac{b^2}{12\mu_2}$, **incompressibility gives harmonic** ϕ_1, ϕ_2**. Nondimensionalizing:**



On Γ, $\phi_1 - \frac{\mu_2}{\mu_1}\phi_2 = \epsilon\kappa$, $\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n} = v_n$, **normal interface speed**

Zero viscosity ratio simplification

In this case, we only need consider one domain $\Omega = \Omega_1$, where

$\Delta\phi = 0$

Far-field and wall conditions in non-dimensionalized form:

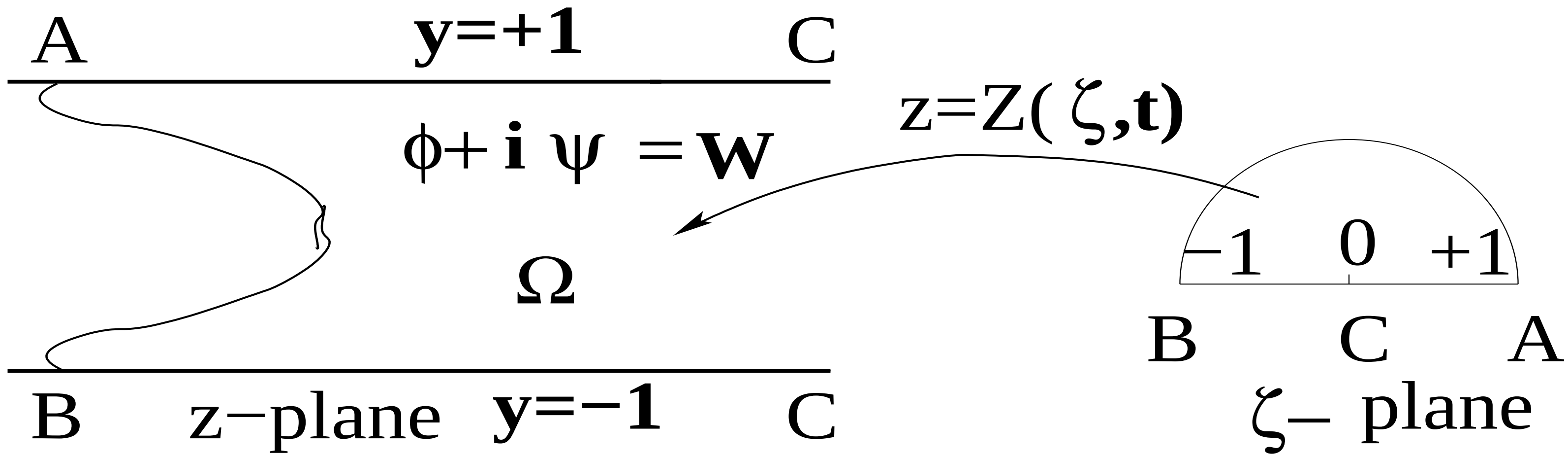
$$\phi \sim x + O(1) \text{ as } x \to +\infty \ , \ \text{ and } \frac{\partial \phi}{\partial y}(x, \pm 1) = 0$$

Interfacial conditions:

$$v_n = \frac{\partial \phi}{\partial n}, \ \text{ and } \ \phi = \epsilon\kappa$$

where κ is the curvature and ϵ surface tension coefficient. These interfacial conditions ignores 3-D thin-film effects. It turns out (Taylor and Saffman,'59), for steady flow, the problem with nonzero viscosity ratio is equivalent to a zero-viscosity problem with change of parameters.

Conformal map for 1-fluid-channel



Conformal map from unit−semi−circle to physical domain Ω in one fluid problem

$$Z(\zeta, t) = -2/\pi \log(\zeta) + f(\zeta, t)$$

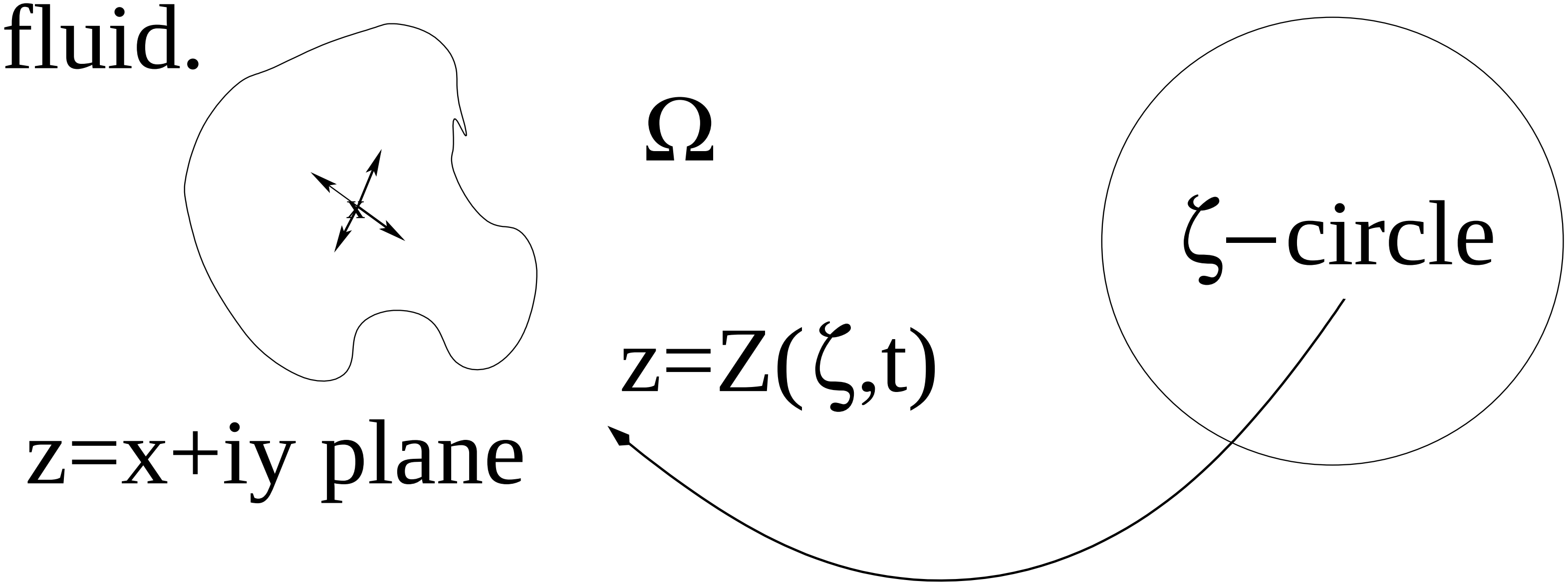
$$W(\zeta, t) = -2/\pi \log(\zeta) + \omega(\zeta, t)$$

$\mathrm{Im}\, f = 0 = \mathrm{Im}\, \omega$ **on (-1,1). Interface condition on** $|\zeta| = 1$**:**

$$\mathrm{Re}\left[\frac{Z_t}{\zeta Z_\zeta}\right] = \mathrm{Re}\, \frac{\zeta W_\zeta}{|Z_\zeta|^2}\ , \quad \mathrm{Re}\, W = -\frac{\epsilon}{|Z_\zeta|} \mathrm{Re}\left[1 + \frac{\zeta Z_{\zeta\zeta}}{Z_\zeta}\right]$$

Conformal map for 1 fluid-radial case

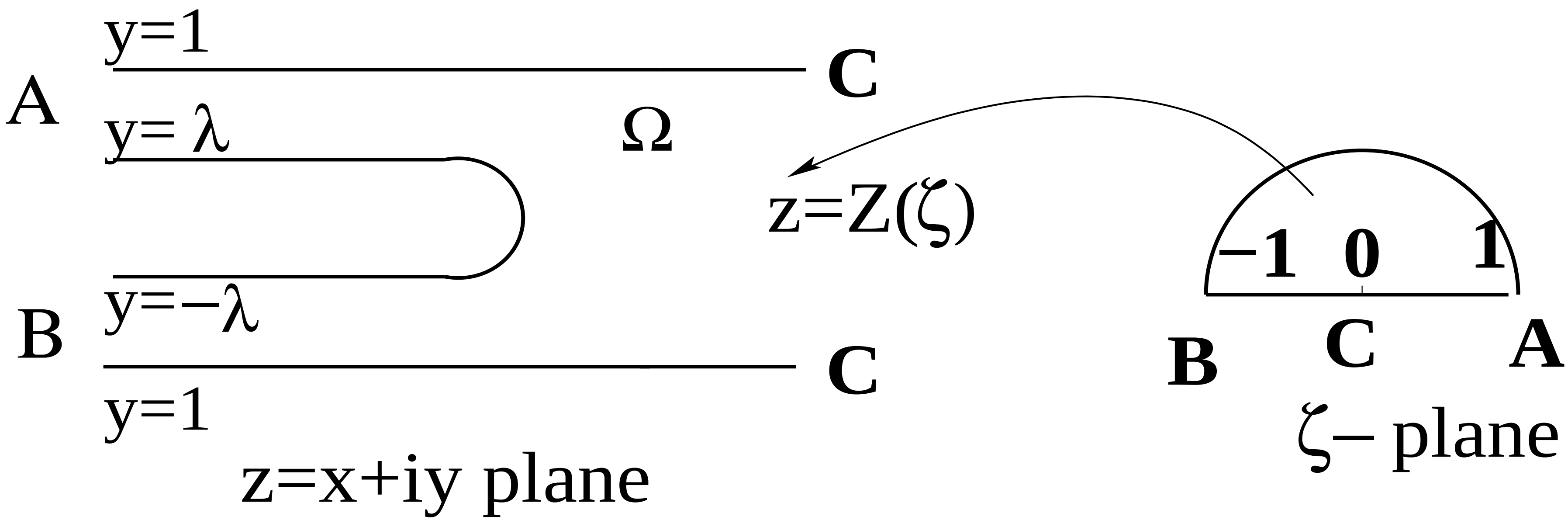
Viscous fluid displaced by injection of inviscid fluid.



$Z(\zeta,t) = \frac{a(t)}{\zeta} + f$, $W(\zeta,t) = -\log\zeta + \omega$, **where** f, ω **analytic in** $|\zeta| < 1$. **As before on** $|\zeta| = 1$:

$$\mathrm{Re}\left[\frac{Z_t}{\zeta Z_\zeta}\right] = \mathrm{Re}\,\frac{\zeta W_\zeta}{|Z_\zeta|^2} \;, \quad \mathrm{Re}\,W = -\frac{\epsilon}{|Z_\zeta|}\,\mathrm{Re}\left[1 + \frac{\zeta Z_{\zeta\zeta}}{Z_\zeta}\right]$$

Steady finger formulation in co-moving frame



With $Z(\zeta) = Z_0(\zeta;\lambda) + f(\zeta)$, where $Z_0(\zeta;\lambda)$ is the ZST solution, obtain wall condition $\mathrm{Im}\, f = 0$ on (-1,1) and on interface $|\zeta| = 1$:

$$\mathrm{Re}\, f = -\frac{\epsilon}{|f' + h|}\,\mathrm{Re}\left[1 + \zeta \frac{f'' + h'}{f' + h}\right], \quad \text{where}\; h(\zeta) = \frac{1 - (2\lambda - 1)\zeta^2}{\zeta(\zeta^2 - 1)}$$

Formal expansion $f \sim \epsilon f_1 + \epsilon^2 f_2 \; + \; ..$ consistent for $\lambda \in (0,1)$!

Formal symmetric steady-finger calculation

To determine λ, we first determine where $f \sim \epsilon f_1 + \epsilon^2 f_2 + ..$ is invalid. Analytic continuation to $|\zeta| > 1$, assuming boundary to be analytic (proved by T. and Xie, '03),

$$f = \epsilon \mathcal{I} - \frac{2\epsilon}{(f' + h)^{1/2} (f'(1/\zeta) + h(1/\zeta))^{1/2}}$$

$$\times \left[1 + \zeta \frac{f'' + h'}{2[f' + h]} + \frac{f''(1/\zeta) + h'(1/\zeta)}{2\zeta \left\{ f'(1/\zeta) + h(1/\zeta) \right\}} \right],$$

where $\mathcal{I}$ is a nonlocal but analytic term. It is seen $f \sim \epsilon f_1 + \epsilon^2 f_1 \; + \; ..$ not valid where $h(\zeta) = 0$, *i.e.* **$1 - (2\lambda - 1)\zeta^2 = 0$. Ignoring $\epsilon \mathcal{I}$, "inner-outer" Kruskal-Segur ('85) type asymptotic calculation determines $O\left(e^{-\epsilon^{-1/2}}\right)$ corrections that determine λ (Combescot** *et al***, '86, '87), Tanveer '87. Justification by Xie & T. '03, T. & Xie '03.**

Initial Value problem for $\epsilon = 0$

For $\epsilon = 0$, $\omega = 0$; then the equation on $|\zeta| = 1$ becomes

$$\mathrm{Re}\left[\frac{Z_t}{\zeta Z_\zeta}\right] = -\frac{1}{|Z_\zeta|^2}$$

$\epsilon = 0$ dynamics well-studied (Polabarinova-Kochina, '46, Galin, '46, Richardson, Gustaffson, ...) However, $\epsilon = 0$ evolution ill-posed *no continuity* **with respect to I.C. in a physically reasonable norm (say H^1) Howison ('86), Fokas & T. ('98).**

Analytic continuation to $|\zeta| > 1$ results in

$$Z_t = q_1 Z_\zeta + q_2 \quad , \quad q_1 = \frac{-\zeta}{2\pi i}\oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \frac{(\zeta + \zeta')}{(\zeta' - \zeta)} \frac{1}{|Z_{\zeta'}|^2} d\zeta'$$

$$q_2 = \frac{-2\zeta}{Z_\zeta(1/\zeta, t)}$$

$\epsilon = 0$ evolution problem

Since q_1 and q_2 analytic in $|\zeta| > 1$ for as long as a solution exists, $Z(\zeta, t)$ satisfying $Z_t = q_1 Z_\zeta + q_2$ has the same type of singularities as $Z(\zeta, 0)$; the singularities only move according to $\dot{\zeta}_s = -q_1(\zeta_s, t)$. Further, it is found $\frac{d}{dt} \oint_{C(t)} Z_\zeta d\zeta = 0$ for a moving curve $C(t)$ advected by the flow $\dot{\zeta} = -q_1(\zeta, t)$, *i.e.* there is no spontaneous generation of singularities.

Since approach of $|\zeta| = 1$ from the outside gives $\mathrm{Re}\, \frac{q_1}{\zeta} = \frac{1}{|Z_\zeta|^2} > 0$, from maximum principle, $\mathrm{Re}\, \frac{q_1(\zeta_s(t), t)}{\zeta_s(t)} > 0$, implying $\frac{\dot{\zeta}_s}{\zeta_s} < 0$. Therefore singularities on the outside continually approach $|\zeta| = 1$ which corresponds to the interface. This is one way to understand the origin of the ill-posedness (T., '93). Initial data infinitesimally close to each other for $|\zeta| \leq 1$ can have substantially different singular structure in $|\zeta| > 1$; yet these singularities affect the interface shape later in time.

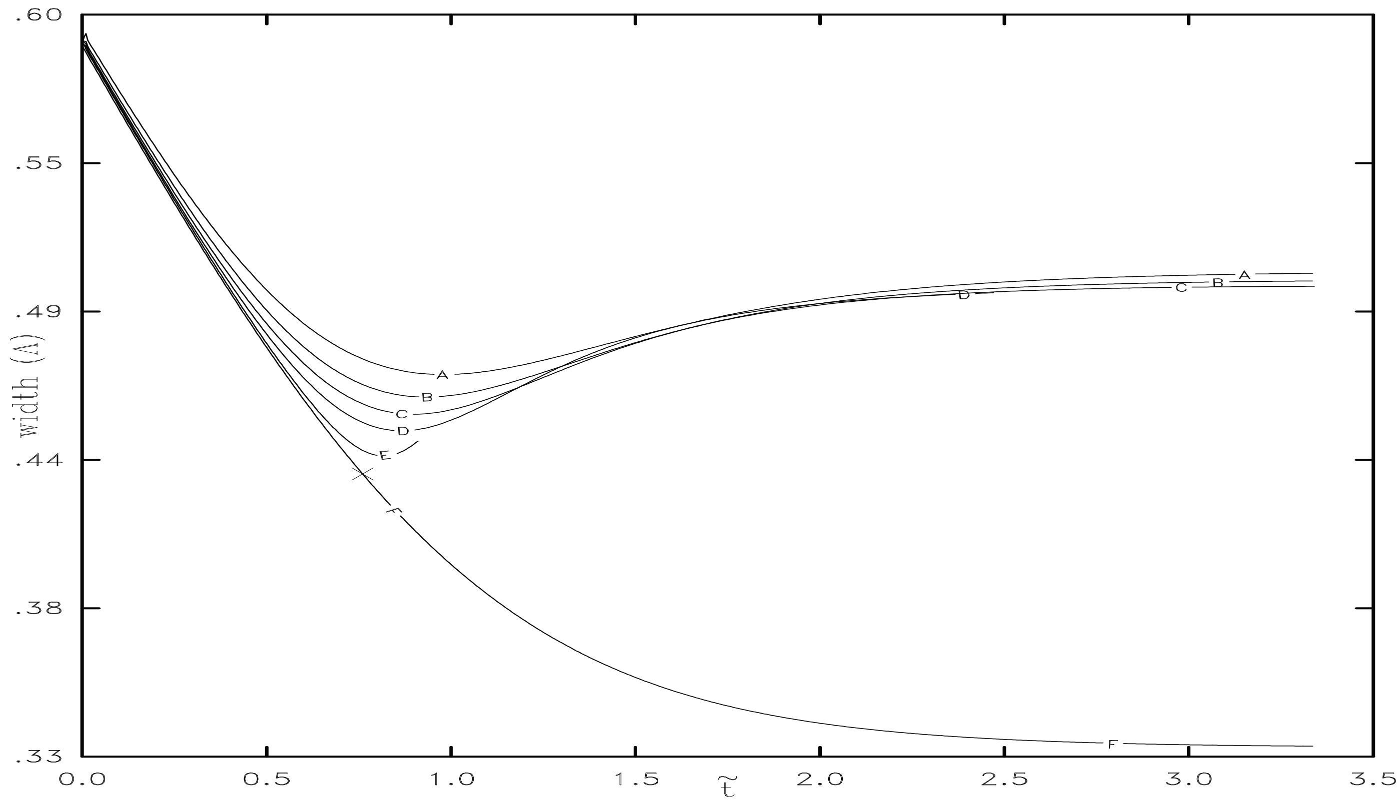
Effect of small ϵ on evolution

Since $\epsilon = 0$ is an ill-posed interface evolution problem, yet apparently well-posed when the domain is extended to $|\zeta| > 1$, it was suggested (T. '93) that a proper-framework to understand evolution for $\epsilon << 1$ perturbatively was to consider the evolution in $|\zeta| > 1$. This is complicated because our knowledge of nonlinear PDEs in the complex plane is very limited. However, some formal results have been obtained, some of which have been justified (Fokas & T., '98, Costin & T., '02).

For isolated singularities of particular types, nonzero ϵ creates a singularity cluster centered around a point ζ_s, where $\dot{\zeta}_s = -q_1$.

For a zero of $Z_\zeta(\zeta, 0)$, new singularity cluster is created that move away from a zero. This implies that generally $\lim_{\epsilon \to 0} Z^{(\epsilon)}(\zeta, t) \neq Z^{(0)}(\zeta, t)$, even when $Z^{(0)}(\zeta, t)$ is smooth on $|\zeta| = 1$, *i.e.* **corresponds to a smooth interface !**

Singular effect of $\epsilon \neq 0$ on finger evolution



Evolution of "finger width" $\Lambda = 1/\mathrm{tip\,speed}$ for a sequence of decreasing ϵ. Prediction based on formal asymptotics agree with numerical computation (Siegel *et al*, 96)

Toy problem for singular ϵ effect on time evolution

Consider the following PDE for $\mathrm{Im}\,\xi \geq 0$:

$$G_t + iG_\xi = 1 \, + \, 2i\epsilon \left[G^{-1/2}\right]_{\xi\xi\xi} \quad \text{with} \quad G(\xi, 0) = 1 - 2i\xi$$

Formal expansion $G \sim G^{(0)} + \epsilon G^{(1)} \, + \, ..$ gives:

$$G^0(\xi, t) \, = \, 2i \, (\xi_0(t) - \xi) \, , \quad \text{where} \;\; \xi_0(t) \, = \, -\frac{i}{2}(1 - t)$$

$$G^1_t + iG^1_\xi = 30(2i\xi_0(t) - 2i\xi)^{-7/2} \, , \;\; \text{where} \;\; G^1(\xi, 0) = 0$$

$$G^1(\xi, t) = -12 \, (2i\xi_0(t) - 2i\xi)^{-5/2} + 12 \, (2i\xi_d(t) - 2i\xi)^{-5/2} \, ,$$

$$\text{where} \quad \xi_d(t) = \xi_0(0) + it \, = \, -\frac{i}{2} + i \, t$$

Note $\xi_d(t)$ moves faster than $\xi_0(t)$ towards real axis

Inner scale and singular effects on real axis

When $\xi - \xi_d(t) \; = \; O(\mathcal{B}^{1/3}), t = O_s(1),$

$$G(\xi, t) \; \sim \; t \, M^{-2} \left\{ \mathcal{B}^{-1/3} [-i(\xi - \xi_d(t))] t^{1/6} \right\},$$

where $M(\eta)$ **satisfies**

$$-\frac{1}{2} M + \frac{1}{6} \eta M' = \left[-\frac{1}{2} + M''' \right] M^3 \text{ with matching condition}$$

The inner ODE admits $(\eta - \eta_s)^{2/3}$ **singularities; corresponding to** $(\xi - \xi_s)^{-4/3}$ **singularity for** G**, clustered near** $\xi = \xi_d$

These singularities affect evolution on real ξ **axis before** $\xi_0(t)$ **reaches real axis !**

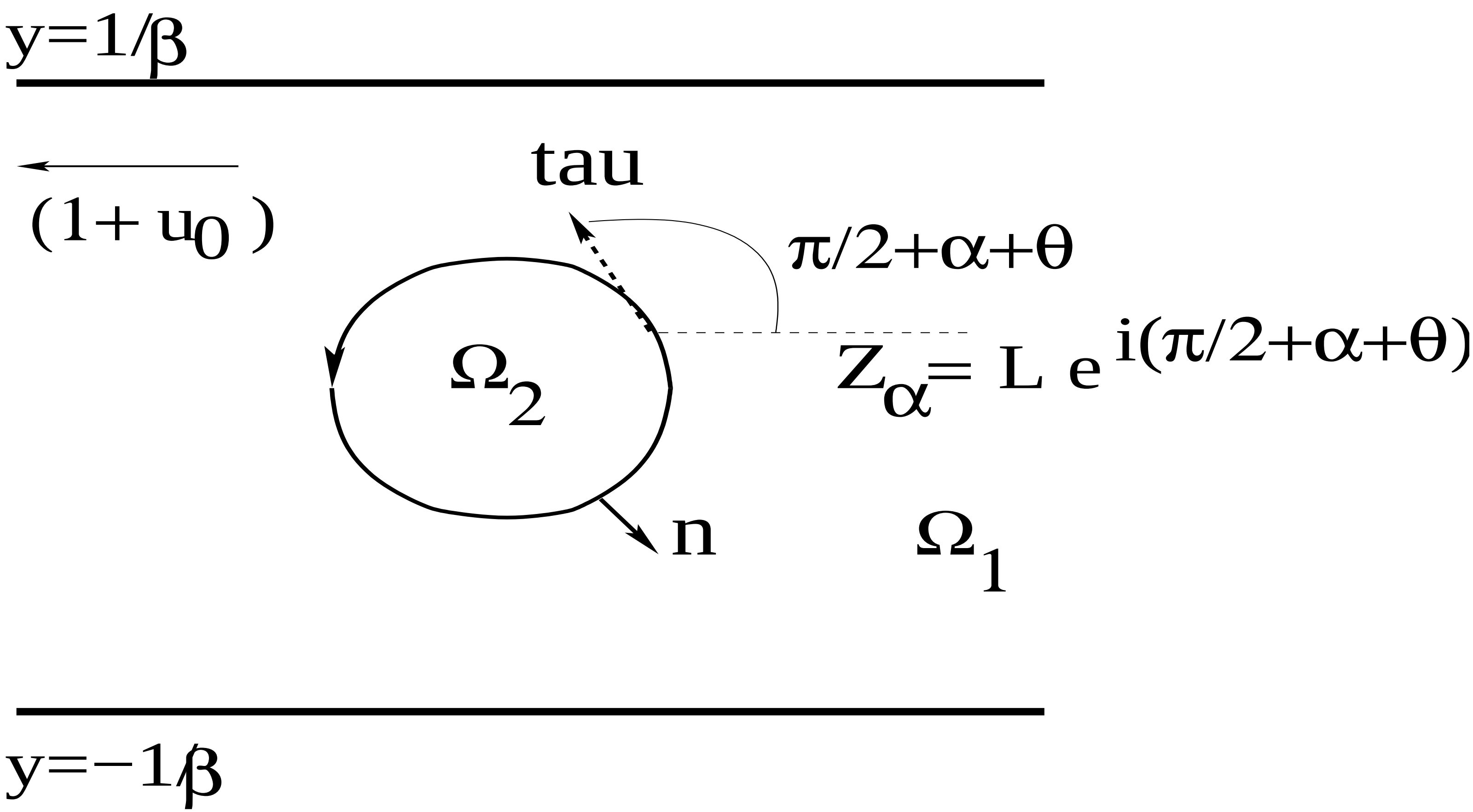
Similar singular effects occur for Hele-Shaw cell for small ϵ**. Other regularizations cause similar effect**

Bubble evolution in two fluid flow

The conformal mapping approach is not convenient when the viscosity ratio between the diplacing fluid and displaced fluid is nonzero. For steady state calculation, this is not an issue since Taylor & Saffman ’59 showed how steady flow with nonzero viscosity ratio is equivalent to a steady flow with zero viscosity ratio, simply by change of other parameters. However, this trick does not work for time-evolution problem

Therefore, we introduce a boundary integral formulation convenient in study of nonzero viscosity ratio. This was introduced by Hou *et al* **in 1993 and has been widely used in numerical computations. Ambrose ’04 used it for local existence analysis based on this so-called equal arclength formulation, which was exploited (Ye & T., ’11) for global existence proof of translating evolving bubbles.**

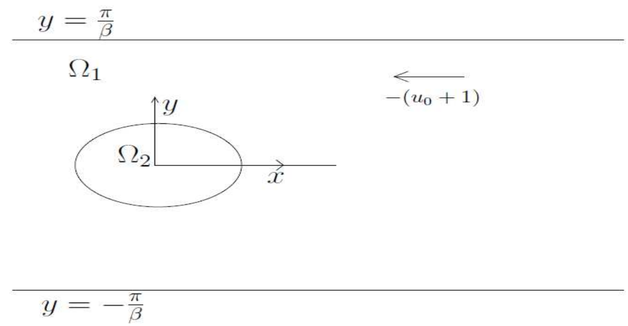
Geometry of the flow



Hele−Shaw Bubble evolution in the frame of the steady bubble.

Formulation of 2-fluid Hele-Shaw problem

Define harmonic ϕ_1, ϕ_2 in Ω_1, $\Omega_2 \subset \mathbb{R}^2$,



$$\phi_1 \sim -(u_0 + 1)x + O(1), \text{ as } x \to \infty, \ \frac{\partial \phi_1}{\partial y}(x, \pm\frac{\pi}{\beta}) = 0, \quad (1)$$

On $\partial\Omega_1 \cap \partial\Omega_2$:

$$(2 + u_0)x + \phi_1 - \frac{\mu_2}{\mu_1}\phi_2 = \epsilon\kappa \text{ and } \frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} = v_n \quad (2)$$

ϵ, n, v_n, $\frac{\mu_2}{\mu_1}$, $2 + u_0$ denote surface tension, inwards normal, interface speed, viscosity ratio and steady bubble speed

Boundary Integral Formulation

We seek representation of the velocity of fluid 1 and 2 in the form

$$u_{1,2} - iv_{1,2} = -(u_0 + 1) + \frac{1}{2\pi i}\int_0^{2\pi} \gamma(\alpha')\mathcal{M}(z,\alpha')d\alpha',$$

where $\mathcal{M}(z,\alpha') = \frac{1}{z - Z(\alpha')}$ **and for** $\beta \neq 0$**,**

$$\mathcal{M}(z,\alpha') = \frac{\beta}{4}\coth\left[\frac{\beta}{4}\left(z - Z(\alpha')\right)\right] - \frac{\beta}{4}\tanh\left[\frac{\beta}{4}\left(z - Z(\alpha')\right)\right]$$

As the free-boundary is approached from fluid 1 and fluid 2 respectively,

$$u_{1,2} - iv_{1,2} = -(u_0 + 1) + \frac{1}{2\pi i} PV \int_0^{2\pi} \gamma(\alpha')\kappa(\alpha,\alpha')d\alpha' \pm \frac{\gamma(\alpha)}{2Z_\alpha(\alpha)},$$

where $\kappa(\alpha,\alpha') = \mathcal{M}\left(Z(\alpha),\alpha'\right)$

Boundary Integral Formulation of Hou *et al*

Normal interface speed $U = (u_1, v_1) \cdot \mathrm{n} = (u_2, v_2) \cdot \mathrm{n}$ is

$$U = (u_0+1)\cos(\alpha + \theta(\alpha) + \mathrm{Re}\left(\frac{Z_\alpha}{2\pi s_\alpha} PV \int_0^{2\pi} \kappa(\alpha, \alpha')\gamma(\alpha')d\alpha'\right),$$

Tangent speed as interface is approached for two fluids:

$$\partial_\alpha \phi_{1,2} = \mathrm{Re}\left[Z_\alpha\left(u_{1,2} - iv_{1,2}\right)\right] = (u_0+1)s_\alpha \sin(\alpha + \theta(\alpha))$$

$$+\mathrm{Re}\left(\frac{Z_\alpha}{2\pi i} PV \int_0^{2\pi} \kappa(\alpha, \alpha')\gamma(\alpha')d\alpha'\right) \pm \frac{1}{2}\gamma(\alpha)$$

We use above relation in the α-derivative of interface relation $(2+u_0)x + \phi_1 - \frac{\mu_2}{\mu_1}\phi_2 = \epsilon\kappa$ to obtain a Fredholm integral equation for γ for given $Z(\alpha, t)$.

Since $v_n = \frac{\partial \phi}{\partial n}$, a boundary point $Z(\alpha, t) = X(\alpha, t) + iY(\alpha, t)$ must have normal speed U, with arbitrary tangent speed T, *i.e.* $(X_t(\alpha, t), Y_t(\alpha, t)) = U\mathrm{n} + T\tau$.

Hou *et al* equal arclength choice

Hou *et al* **'93 noted that if**

$$T(\alpha,t) = \int_0^{\alpha} (1+\theta_\alpha(\alpha',t))d\alpha' - \frac{\alpha}{2\pi}\int_0^{2\pi}(1+\theta_\alpha(\alpha',t))d\alpha',$$

then $|Z_\alpha| \equiv s_\alpha = \frac{L}{2\pi}$ **independent of** α**. In this equal arclength formulation, on** α**-differentiation of** (X_t, Y_t)**, obtain**

$$\theta_t(\alpha,t) = \frac{2\pi}{L}U_\alpha(\alpha,t) + \frac{2\pi}{L}T(\alpha,t)\left(1+\theta_\alpha(\alpha,t)\right),$$

$$L_t = -\int_0^{2\pi}\left(1+\theta_\alpha(\alpha,t)\right)U(\alpha,t)d\alpha$$

For given θ**,** L**,** $Z(\alpha,t) = \frac{iL}{2\pi}\int_0^{\alpha}\exp\left[i\alpha' + i\theta(\alpha',t)\right]d\alpha' + Z(0,t)$**, where** $Z(0,t) = X(0,t) + iY(0,t)$ **satisfies** $(X_t(0,t), Y_t(0,t)) = U(0,t)\mathrm{n}$**. This is the starting point for global existence in next lecture.**