Divergent series, Borel Summation, 3-D Navier-Stokes

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Navier-Stokes existence–background

· **Global Existence of smooth 3-D Navier-Stokes solution is an important open problem.**

· **Deviation from linear stress-strain relation or incompressibility is potentially important if N-S solutions are singular**

· **Usual numerical calculations do not address this issue because errors are not controlled, rigorously.**

· **Globally smooth solutions known only when Reynolds number small**

· **Generally, smooth solutions for smooth data on $[0, T]$ known to exist, for T scaling inversely with initial data/forcing.**

· **Global weak solutions known since Leray, but not known whether they are unique. For unforced problem in $\mathbb{T}^3$, such a solution becomes smooth again for $t > T_c$, T_c depends on IC**

Borel Summation–background and main idea

· Borel summation generates , under suitable conditions, a one-one correspondence between series and and functions that preserve algebraic operations (Ecalle, Costin,..).

· Borel sum can involve large or small variable(s)/ parameter(s).

· Formal expansion for $t << 1$: $v(x,t) = v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$ generally divergent for the initial value problem $v_t = \mathcal{N}[v]$, $v(x,0) = v_0$, $\mathcal{N}$ being some differential operator.

· Borel Sum of this series gives actual solution, which transcends restriction $t << 1$

· For Navier-Stokes, the Borel sum is given by

$$v(x,t) = v_0(x) + \int_0^{\infty} U(x,p) e^{-p/t} dp$$

Equation for U obtained by inverse-Laplace transforming N-S.

Incompressible 3-D Navier-Stokes in Fourier-Space

Consider 3-D N-S in infinite geometry or periodic box. Similar results expected for finite domain with no-slip BC using eigenfunctions of Stokes operator as basis. In Fourier-Space

$$\hat{v}_t + \nu |k|^2 \hat{v} = -i k_j P_k \left[\hat{v}_j \hat{*} \hat{v} \right] + \hat{f}(k)$$

$$P_k = \left(I - \frac{k(k\cdot)}{|k|^2} \right) \quad , \quad \hat{v}(k,0) = \hat{v}_0(k)$$

where P_k is the Hodge projection in Fourier space, $\hat{f}(k)$ is the Fourier-Transform of forcing $f(x)$, assumed divergence free and t-independent. Subscript j denotes the j-th component of a vector. $k \in \mathbb{R}^3$ or $\mathbb{Z}^3$. Einstein convention for repeated index followed. $\hat{*}$ denotes Fourier convolution.

Decompose $\hat{v} = \hat{v}_0 + \hat{u}(k,t)$, inverse-Laplace Transform in $1/t$ and invert the differential operator on the left side

Integral equation associated with Navier-Stokes

We obtain:

$$\hat{U}(k,p) = \int_0^p \mathcal{K}_j(p,p';k)\hat{H}_j(k,p')dp' + \hat{U}^{(0)}(k,p) \equiv \mathcal{N}\left[\hat{U}\right](k,p) \tag{1}$$

$$\mathcal{K}_j(p,p';k) = \frac{ik_j\pi}{z}\left\{z'Y_1(z')J_1(z) - z'Y_1(z)J_1(z')\right\}$$

$$z = 2|k|\sqrt{\nu p}\,,\; z' = 2|k|\sqrt{\nu p'}\,,\; \hat{H}_j = P_k\left\{\hat{v}_{0,j}\hat{*}\hat{U} + \hat{U}_j\hat{*}\hat{v}_0 + \hat{U}_j \overset{*}{*} \hat{U}\right\}$$

$$\hat{U}^{(0)}(k,p) = 2\frac{J_1(z)}{z}\hat{v}_1(k) \quad , \quad P_k = \left(I - \frac{k(k\cdot)}{|k|^2}\right)$$

$$\hat{v}_1(k) = \left(-\nu|k|^2\hat{v}_0 - ik_j\mathcal{P}_k\left[\hat{v}_{0,j}\hat{*}\hat{v}_0\right]\right) + \hat{f}(k),$$

$\hat{*}$, denotes Fourier Convolution, $*$ denotes Laplace convolution, while $\overset{*}{*}$ denotes Fourier followed by Laplace convolution. J_1 and Y_1 are the usual Bessel functions.

Results for Integral equation and Navier-Stokes-1

Introduce norm $\|.\|_{\mu,\beta}$ and $\|.\|$ for $\mu > 3$, $\beta \geq 0$ so that

$$\|\hat{w}\|_{\mu,\beta} = \sup_{k \in \mathbb{R}^3} e^{\beta|k|}(1+|k|)^{\mu}|\hat{w}(k)|$$

$$\|\hat{U}\| = \sup_{p>0} e^{-\alpha p}(1+p^2)\|\hat{U}(.,p)\|_{\mu,\beta}$$

Lemma 1: If $\|\hat{v}_0\|_{\mu+2,\beta}$ and $\|\hat{f}\|_{\mu,\beta}$ are finite, then an upper bound for α can be found interms of $\hat{v}_0$ and $\hat{f}$ so that the integral equation (1) has a unique solution for $p \in \mathbb{R}^+$ for which $\|\hat{U}\| < \infty$.

Theorem 1: Under same conditions as in Lemma 1, the 3-D Navier-Stokes has a unique solution for $\mathrm{Re}\,\frac{1}{t} > \alpha$. Furthermore, $\hat{v}(\cdot,t)$ is analytic for $\mathrm{Re}\,\frac{1}{t} > \alpha$ and $\|\hat{v}(\cdot,t)\|_{\mu+2,\beta} < \infty$ for $t \in [0,\alpha^{-1})$.

Theorem 2 deals with Borel Summability and the nature of the asymptotic expansion $\hat{v} \sim \hat{v}_0 + t\hat{v}_1..$ and will not be discussed.

Remarks on Theorem 1

Remark 1: Local existence results in Theorem 1 already known through classical methods. However, in the present formulation, global existence problem can be cast into a question of asymptotics of a known solution to integral equation. A sub-exponential growth as $p \longrightarrow \infty$ gives global existence.

Remark 2: Errors in Numerical solutions rigorously controlled, unlike usual N-S calculations. Discretization in p and Galerkin approximation in k results in:

$$\hat{U}_\delta(k, m\delta) = \delta \sum_{m'=0}^{m} \mathcal{K}_{m,m'} \mathcal{P}_N \mathcal{H}_\delta(k, m'\delta) + \hat{U}^{(0)}(k, m\delta)$$

$$\equiv \mathcal{N}_\delta\left[\hat{U}_\delta\right] \quad \text{for} \quad k_j = -N, \ldots N, \quad j = 1, 2, 3$$

$\mathcal{P}_N$ is the Galerkin Projection into N-Fourier modes. $\mathcal{N}_\delta$ has properties similar to $\mathcal{N}$. The continuous solution $\hat{U}$ satisfies $\hat{U} = \mathcal{N}_\delta\left[\hat{U}\right] + E$, where E is the truncation error. Thus, $\hat{U} - \hat{U}_\delta$ can be estimated using same tools as in Theorem 1.

Numerical Solutions to integral equation

We choose the Kida initial conditions and forcing

$$\mathbf{v}_0(\mathbf{x}) = (v_1(x_1, x_2, x_3, 0), v_2(x_1, x_2, x_3, 0), v_3(x_1, x_2, x_3, 0))$$

$$v_1(x_1, x_2, x_3, 0) = v_2(x_3, x_1, x_2, 0) = v_3(x_2, x_3, x_1, 0)$$

$$v_1(x_1, x_2, x_3, 0) = \sin x_3 \left(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3\right)$$

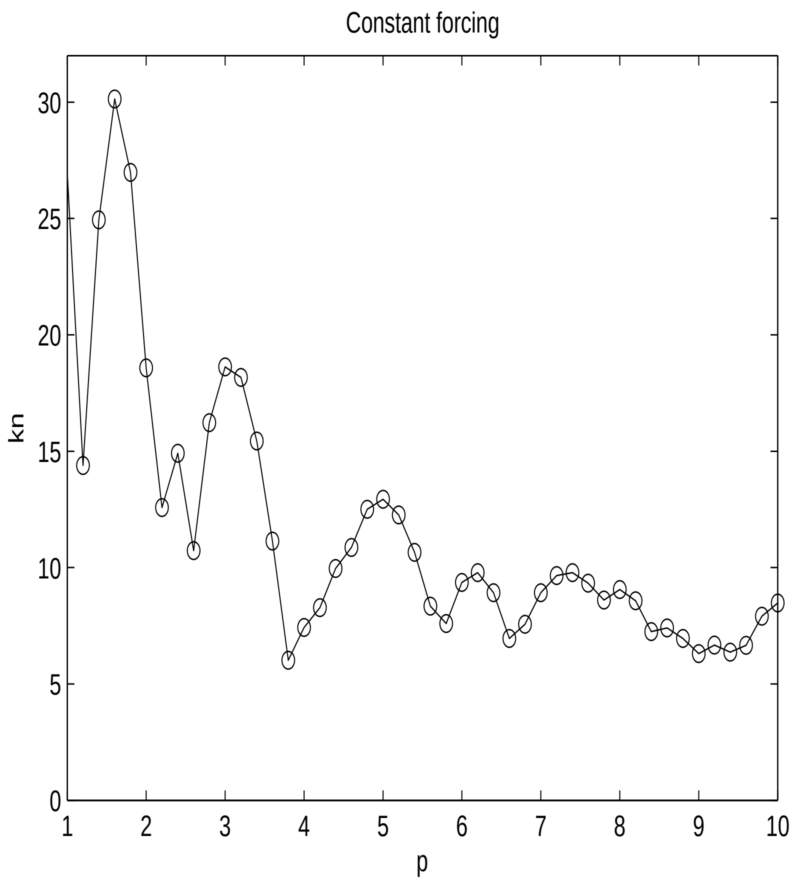
$$f_1(x_1, x_2, x_3) = \frac{1}{5} v_1(x_1, x_2, x_3, 0)$$

High Degree of Symmetry makes computationally less expensive

Corresponding Euler problem believed to blow up in finite time; so good candidate to study viscous effects

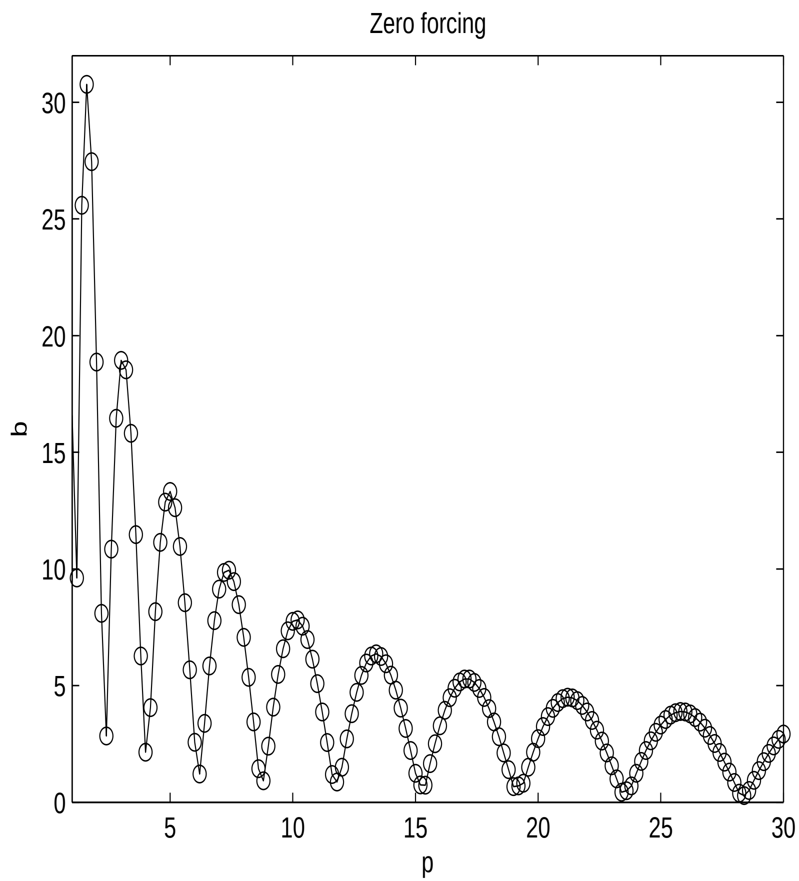
In the plots, "constant forcing" corresponds to $f = (f_1, f_2, f_3)$ as above, while zero forcing refers to $f = 0$. Recall sub-exponential growth in p corresponds to global N-S solution.

Numerical solution to integral equation-plot-1



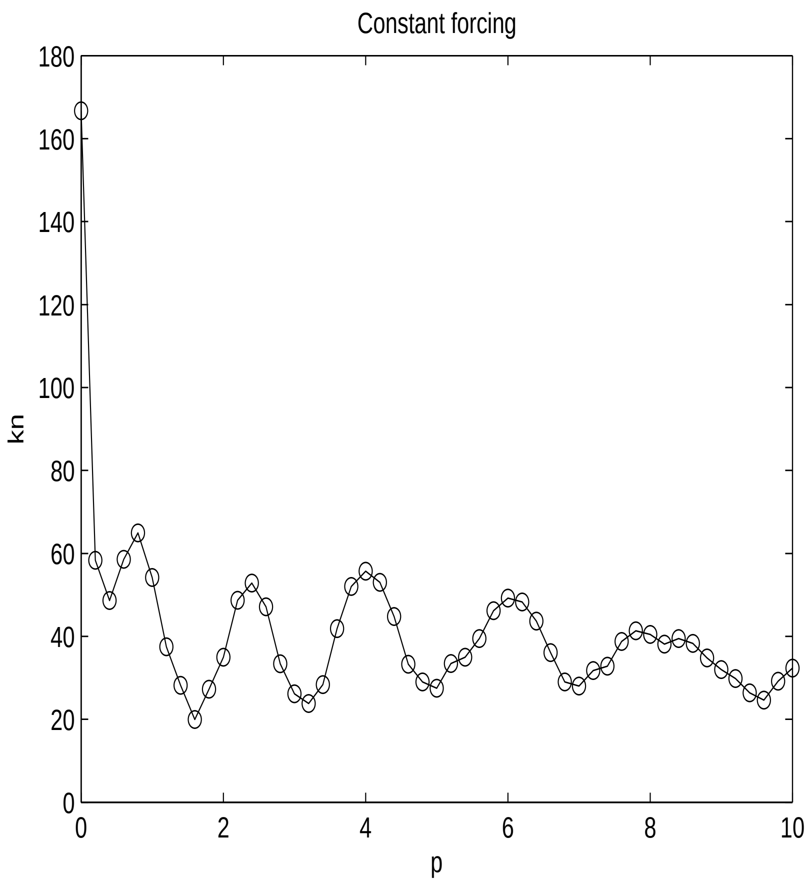
$\|\hat{U}(.,p)\|_{4,0}$ **vs.** p **for** $\nu = 1$**, constant forcing.**

Numerical solution to integral equation-plot-2



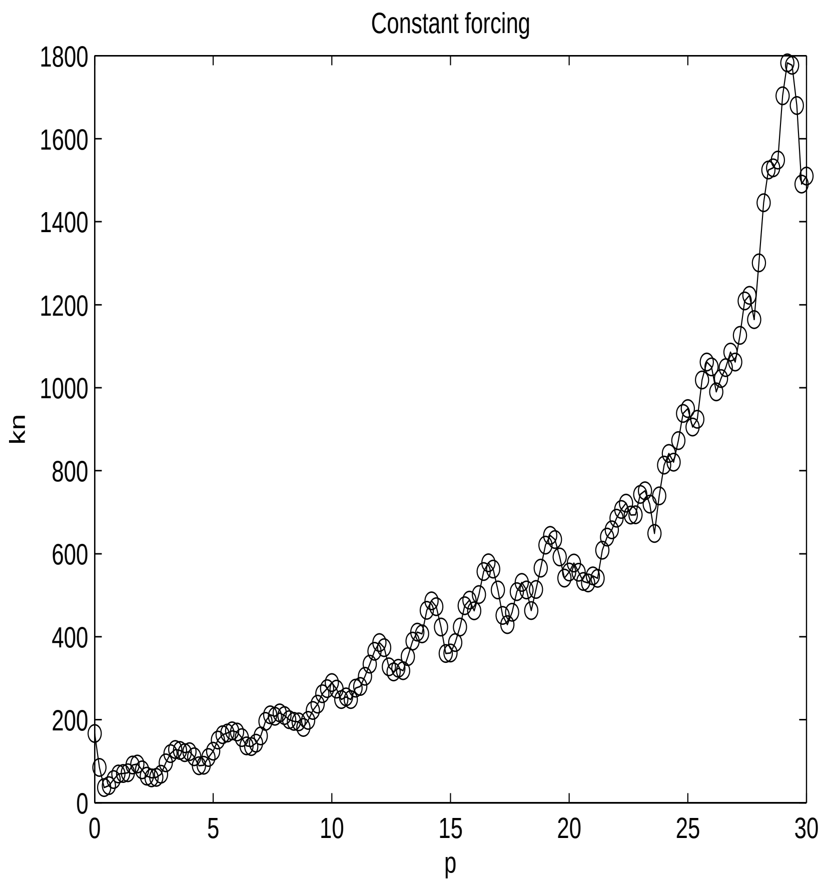
$\|\hat{U}(.,p)\|_{4,0}$ **vs.** p **for** $\nu = 1$**, no forcing**

Numerical solution to integral equation-plot-3



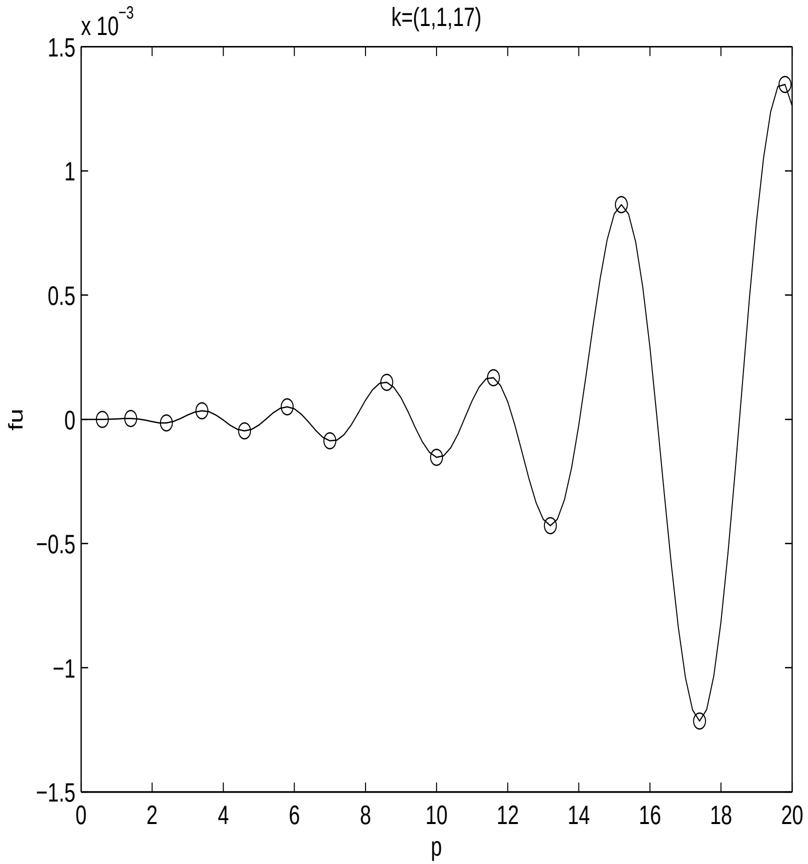
$\|\hat{U}(.,p)\|_{4,0}$ **vs.** p **for** $\nu = 0.16$**, constant forcing**

Numerical solution to integral equation-plot-4



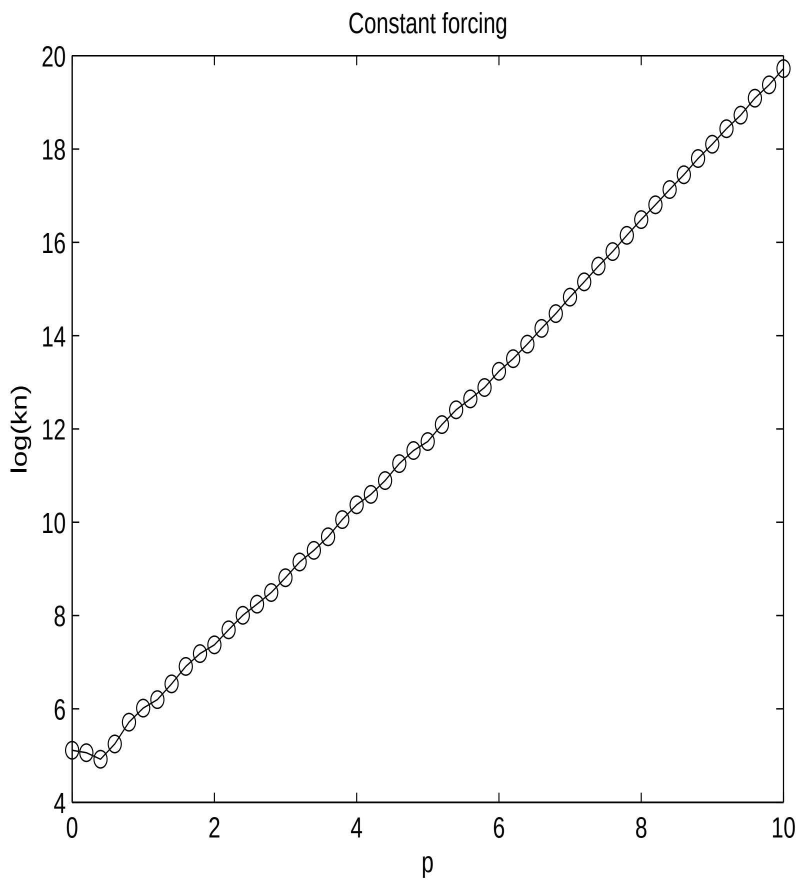
$\|\hat{U}(.,p)\|_{4,0}$ **vs.** p **for** $\nu = 0.1$**, constant forcing**

Numerical solution to integral equation-plot-5



$\hat{U}(k,p)$ **vs.** p **for** $k = (1,1,17)$**,** $\nu = 0.1$**, no forcing.**

Numerical solution to integral equation-plot-6



$\log \|\hat{U}(.,p)\|_{4,0}$ **vs.** $\log p$ **for** $\nu = 0.001$**, constant forcing**

Issues raised by numerical computations

Numerical solutions to integral equation available on finite interval $[0, p_0]$, yet N-S solution requires $[0, \infty)$ interval since $\hat{v}(k,t) = \hat{v}_0 + \int_0^\infty e^{-p/t}\hat{U}(k,p)dp$

Actually, the integral over $\int_0^{p_0}$ gives an approximate N-S solution, with errors that can be bounded for a time interval $[0, T]$, if computed solution to integral equation eventually decreases with p on a sufficiently large interval $[0, p_0]$.

Further, a non-increasing $\hat{U}$ over a sufficiently large interval $[0, p_0]$ gives smaller bounds on growth rate α as $p \to \infty$. Therefore, in such cases smooth NS solution exists over a long interval $[0, \alpha^{-1})$.

Recall for unforced problem in $\mathbb{T}^3$, even weak solution to NS becomes smooth for $t > T_c$, with T_c estimated from initial data. Hence global existence follows under some conditions.

Extending Navier-Stokes interval of existence

For $\alpha_0 \geq 0$, define

$$\epsilon = \nu^{-1/2} p_0^{-1/2} \, , \; a = \|\hat{v}_0\|_{\mu,\beta} \, , \; c = \int_{p_0}^{\infty} \|\hat{U}^{(0)}(.,p)\|_{\mu,\beta} e^{-\alpha_0 p} dp$$

$$\epsilon_1 = \nu^{-1/2} p_0^{-1/2} \left(2 \int_0^{p_0} e^{-\alpha_0 s} \|\hat{U}(.,s)\|_{\mu,\beta} ds + \|\hat{v}_0\|_{\mu,\beta} \right)$$

$$b = \frac{e^{-\alpha_0 p_0}}{\sqrt{\nu p_0} \alpha} \int_0^{p_0} \|\hat{U} \hat{*} \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{\mu,\beta} ds$$

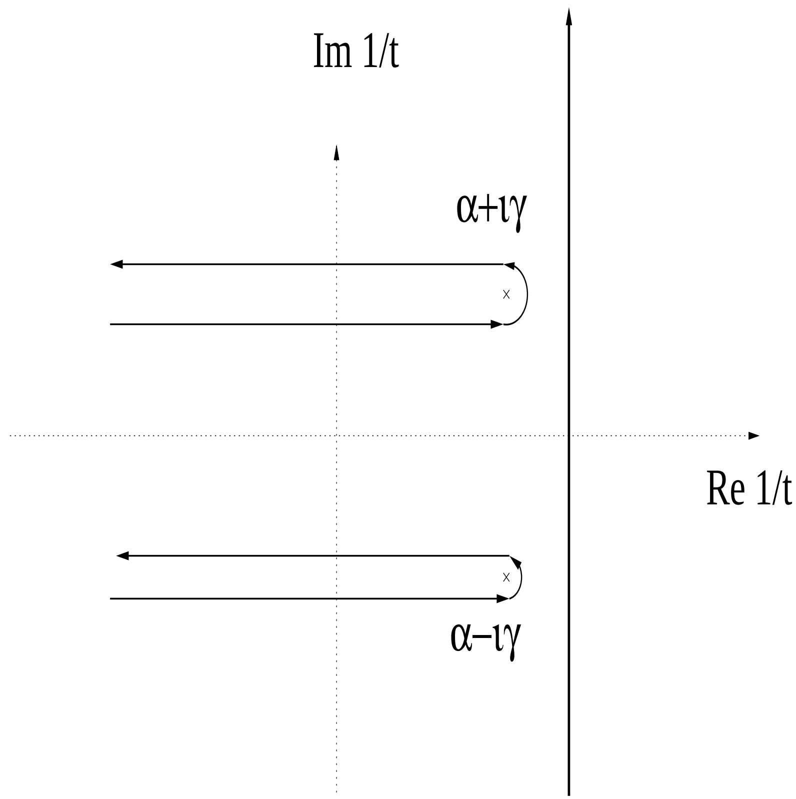
Theorem 3: A smooth solution to 3-D Navier-Stokes equation exists in the $\|.\|_{\mu,\beta}$ space on the interval $[0, \alpha^{-1})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

Remark: If p_0 is chosen large enough, ϵ, ϵ_1 is small when computed solution in $[0, p_0]$ decays with q. Then α can be chosen rather small.

Relation of Optimal α to Navier-Stokes singularities

$$\hat{U}(k,p) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{p/t}\left[\hat{v}(k,t) - \hat{v}_0(k)\right] d\left[\frac{1}{t}\right]$$



Rightmost singularity(ies) of NS solution $\hat{v}(k,t)$ in the $1/t$ plane determines optimal α. γ gives dominant oscillation frequency.

Laplace-transform and accelerated representation

To get rid of the effect of complex singularity, it is prudent to seek a more general Laplace-transform involves

$$\hat{v}(k,t) = \hat{v}_0(k) + \int_0^\infty e^{-q/t^n} \hat{U}(k,q) dq$$

We have arguments to show for at least the unforced problem, if there are complex singularities t_s in the right-half plane, but not on the real axis, then a a nonzero lower bound for $|\arg t_s|$ exists. Then, for sufficiently large n, no singularities in the $\tau = t^{-n}$ plane in the right-half plane. Hence, $\hat{U}(k,q)$ will not grow with q

$\hat{U}(k,q)$ satisfies an integral equation similar to the one satisfied by $\hat{U}(k,p)$ and Theorems similar to Theorem 1 follow. In the context of ODEs, change of variable $p \to q$ is called acceleration (Ecalle)

Extending Navier-Stokes interval of existence

For $\alpha_0 \geq 0$, define

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} \; , \; c = \int_{q_0}^{\infty} \|\hat{U}^{(0)}(.,q)\|_{\mu,\beta} e^{-\alpha_0 q} dq$$

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} \left(2 \int_0^{q_0} e^{-\alpha_0 s} \|\hat{U}(.,s)\|_{\mu,\beta} ds + \|\hat{v}_0\|_{\mu,\beta} \right)$$

$$b = \frac{e^{-\alpha_0 q_0}}{\sqrt{\nu} q_0^{1-1/(2n)} \alpha} \int_0^{q_0} \|\hat{U} \hat{*} \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{\mu,\beta} ds$$

Theorem 4: A smooth solution to 3-D Navier-Stokes equation exists in the $\|.\|_{\mu,\beta}$ *space on the interval* $[0, \alpha^{-1/n})$*, when* $\alpha \geq \alpha_0$ *is chosen to satisfy*

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

Remark: If q_0 *is chosen large enough,* ϵ*,* ϵ_1 *is small when computed solution in* $[0, q_0]$ *decays with* q*. Then* α *can be chosen rather small.*

Conclusions

We have shown how Borel summation methods provides an alternate existence theory for N-S equation

With this integral equation (IE) approach, the global existence of NS is implied if known solution to IE has subexponential growth.

The solution to integral equation in a finite interval can be computed numerically with errors controlled rigorously

Integral equation in an accelerated variable q expected to show no exponential growth unless there is singularity on the real t-axis.

The computation over a finite $[0, q_0]$ interval, gives a better upper bound on growth rate exponent α at ∞ and hence ensures a longer existence time $\left[0, \alpha^{-1/n}\right)$ to 3-D Navier-Stokes.

Unresolved issues include Rigorous control of round-off error and obtaining small enough bounds on truncation error for manageable step size.

Key points in the proof-I

$$\text{Define norm}: \|\hat{f}(\mathrm{k},p)\| = \sup_{p\geq 0} e^{-\alpha p}(1+p^2)\|\hat{f}(.,p)\|_{\mu,\beta}$$

Because of properties

$$\frac{e^{\alpha p}}{(1+p^2)} * \frac{e^{\alpha p}}{(1+p^2)} = e^{\alpha p}\int_0^p \frac{ds}{(1+s^2)[1+(p-s)^2]} \leq \frac{M_0 e^{\alpha p}}{1+p^2}$$

$$\left[e^{-\beta|\mathrm{k}|}(1+|\mathrm{k}|)^{-\mu}\right] \hat{*} \left[e^{-\beta|\mathrm{k}|}(1+|\mathrm{k}|)^{-\mu}\right] \leq \frac{C_0(\mu)e^{-\beta|\mathrm{k}|}}{(1+|\mathrm{k}|)^{-\mu}},$$

the following algebraic properties follow:

$$\|[\hat{f}(\mathrm{k},p)]\hat{*}[\hat{g}(\mathrm{k})]\|_{\mu,\beta} \leq C_0\|\hat{f}(\cdot,p)\|_{\mu,\beta}\|\hat{g}\|_{\mu,\beta}$$

$$\|\hat{u} \overset{*}{*} \hat{v}\| \leq M_0 C_0\|\hat{u}\|\|\hat{v}\| \ , \ \|\int_0^p |\hat{u}(\mathrm{k},s)|ds\| \leq C\alpha^{-1}\|\hat{u}\|$$

Key points in the proof-II

From these relations, it is possible to conclude from the integral equation that if

$$u(p) \equiv \|\hat{U}(.,p)\|_{\mu,\beta} \ , \ a = \|\hat{v}_0\|_{\mu,\beta} \ , \ u^0(p) = \|\hat{U}^{(0)}(.,p)\|_{\mu,\beta} \ ,$$

then

$$u(p) \leq \frac{C}{\sqrt{\nu p}} \int_0^p [u * u + au](s) ds + u^{(0)}(p)$$