MICHAEL'S SELECTION THEOREM IN D-MINIMAL EXPANSIONS OF THE REAL FIELD

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Abstract. Let $E \subseteq \mathbb{R}^n$. If T is a lower semi-continuous set-valued map from E to $\mathbb{R}^m$ and $(\mathbb{R}, +, \cdot, T)$ is d-minimal, then there is a continuous function $f\colon E \to \mathbb{R}^m$ definable in $(\mathbb{R}, +, \cdot, T)$ such that $f(x) \in T(x)$ for every $x \in E$.

For sets X and Y, we denote a map T from X to the power set of Y by $T\colon X \rightrightarrows Y$ and call such T a **set-valued map**. In 1956, E. Michael discussed problems on the existence of continuous selections of set-valued maps; see [9, 10]. Let $E \subseteq \mathbb{R}^n$ and $T\colon E \rightrightarrows \mathbb{R}^m$. A **continuous selection** of T is a continuous map $f\colon E \to \mathbb{R}^m$ such that $f(x) \in T(x)$ for every $x \in E$. Michael asserted: *If $T(x)$ is nonempty, closed and convex for every $x \in E$, and T is* **lower semi-continuous** *(that is, for $x_0 \in E$, $y_0 \in T(x_0)$ and a neighborhood V of y_0, there is a neighborhood U of x_0 such that for every $x \in U$, $T(x) \cap V \neq \emptyset$), then T has a continuous selection.* This theorem is known as Michael's Selection Theorem and has applications in various fields of mathematics (see, *e.g.*, Y. Benyamini and J. Lindenstrauss [4], S. Park [17] and M. Zippin [18]). The given construction involves an infinitary process that can produce a far more complicated selection than one would like. For example, even when T (as a set) is a polygon, the selection can have infinite oscillation; see 3.4. Thus, the question arises naturally:

If T is well behaved in some prescribed sense, is it possible to find a continuous selection that is similarly well behaved?

Here we employ first-order logic to study this question and restate it as follows:

Suppose T is lower semi-continuous, and each $T(x)$ is nonempty, closed and convex. Is there a continuous selection of T definable in $(\mathbb{R}; +, \cdot, T)$?

where $(\mathbb{R}; +, \cdot, T)$ is the expansion of the real field by T and "definable" means "definable possibly with parameters". (Readers not familiar with the notion may consult L. van den Dries and C. Miller [5] for an introduction.) Informally, can we define a continuous selection of T using *only* $+$, $\cdot$, T and finitely many real numbers?

Now let us study this new version of the main question in more specific cases. Let $E \subseteq \mathbb{R}^n$ and $T\colon E \rightrightarrows \mathbb{R}^m$. When $(\mathbb{R}; +, \cdot, T)$ defines $\mathbb{Z}$, every Borel set is definable in $(\mathbb{R}; +, \cdot, T)$ (see, *e.g.*, A. Kechris [8, 37.6]). By Michael's Selection Theorem, if E is Borel and $(\mathbb{R}; +, \cdot, T)$ defines $\mathbb{Z}$, then the answer is yes. At the other extreme, if $(\mathbb{R}; +, \cdot, T)$ is **o-minimal** (that is, every unary definable set is a finite union of points and open intervals), the answer is also yes (see M. Aschebrenner and A. Thamrongthanyalak [2]). The former type of expansions

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of the real field is so rich that they can define every Borel set no matter how complicated these sets are while the latter defines just only sets that we usually considered as *tame*. It is natural to ask whether the same result holds in the intermediate step.

From now on, let $\mathfrak{R}$ be an expansion of the real line $(\mathbb{R};<)$. We say that $\mathfrak{R}$ is **d-minimal** if for every definable family $\{A_x\}_{x\in\mathbb{R}^k}$ of subsets of $\mathbb{R}$, there is $N \in \mathbb{N}$ such that each A_x either has interior or is a union of N many discrete sets, equivalently, for every $\mathfrak{M}$ elementary equivalent to $\mathfrak{R}$, every unary definable set in $\mathfrak{M}$ is a disjoint union of open intervals and finitely many discrete sets (see C. Miller [11] for more information). In the context of this paper, we may restate the definition of d-minimality as follows: for every definable set-valued map $T\colon \mathbb{R}^k \rightrightarrows \mathbb{R}$, there is $N \in \mathbb{N}$ such that for each $x \in \mathbb{R}^k$, $T(x)$ either has interior or is a union of N many discrete sets. Obviously, every o-minimal expansion of the real line is also d-minimal. For examples of d-minimal expansions of the real line that are not o-minimal, we refer to H. Friedman and C. Miller [6,7], C. Miller and J. Tyne [15], and [11,13].

Here is the main result:

Theorem A. *Let $\mathfrak{R}$ be a d-minimal expansion of the real field, $E \subseteq \mathbb{R}^n$ be definable and $T\colon E \rightrightarrows \mathbb{R}^m$ be definable. If T is lower semi-continuous and each $T(x)$ is nonempty closed and convex, then T has a definable continuous selection.*

In other words, if T is lower semi-continuous, each $T(x)$ is nonempty closed and convex, and $(\mathbb{R};+,\cdot,T)$ is d-minimal, then T has a selection that is definable in $(\mathbb{R};+,\cdot,T)$.

Conventions and notations. Throughout, d, k, m, n and N will range over the set $\mathbb{N} = \{0,1,2,3,\dots\}$ of natural numbers.

For a set $S \subseteq \mathbb{R}^n$ we denote by $\operatorname{cl} S$ the closure, by $\operatorname{fr} S := \operatorname{cl} S \setminus S$ the frontier, by $\operatorname{int} S$ the interior of S, by $\operatorname{bd} S$ the boundary and by $\operatorname{isol} S$ the set of isolated points of S.

For a set $S \subseteq \mathbb{R}^{m+n}$ and $x \in \mathbb{R}^m$ we denote the fiber of S above x by $S_x := \{y \in \mathbb{R}^n : (x,y) \in S\}$.

We denote the Euclidean norm on $\mathbb{R}^n$ by $\| \ \|$. Given $x \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$, let $d(x,S) := \inf_{y\in S}\|x-y\|$ be the distance from x to S.

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1. D-minimal Michael's Selection Theorem

Throughout this section, assume $\mathfrak{R}$ is an expansion of the real field.

We say that a set-valued map $T\colon E \rightrightarrows \mathbb{R}^m$ is **continuous** if T is lower semi-continuous and closed (as a set) in $E \times \mathbb{R}^m$.

Let $E \subseteq \mathbb{R}^n$, $T\colon E \rightrightarrows \mathbb{R}^m$ and $f\colon E \to \mathbb{R}^m$. Let $T - f\colon E \rightrightarrows \mathbb{R}^m$ denote the set-valued map given by $x \mapsto \{y - f(x) : y \in T(x)\}$. Assume further that each $T(x)$ is nonempty closed and convex. We define the **least norm selection of** T, $\operatorname{lns}_T\colon E \to \mathbb{R}^m$, by $\operatorname{lns}_T(x) =$ the unique point $y \in T(x)$ such that $\|y\| = d(0,T(x))$. Note that the uniqueness is immediate from the convexity of each $T(x)$.

We begin by two straightforward results that proofs are left to the reader.

1.1. *Let* $E \subseteq \mathbb{R}^n$, $T\colon E \rightrightarrows \mathbb{R}^m$, *and* $f\colon E \to \mathbb{R}^m$. *Suppose* f *is continuous.*

(1) *If* T *is lower semi-continuous, then* $T - f$ *is lower semi-continuous*
(2) *If* T *is continuous, then* $T - f$ *is continuous.*

1.2. *Suppose* $T\colon E \rightrightarrows \mathbb{R}^m$ *is continuous and each* $T(x)$ *is nonempty closed and convex. Then* lns_T *is continuous, and definable if* T *is definable.*

Next is a minor variant (the proof is essentially the same) of the Definable Tietze Extension Theorem; see M. Aschenbrenner and A. Fischer [1, Lemma 6.6].

1.3. *Let* $A \subseteq B$ *be definable where* A *is closed in* B *and* $f\colon A \to \mathbb{R}^m$ *be definable and continuous. Then there is a definable continuous function* $g\colon B \to \mathbb{R}^m$ *extending* f.

For $d \leq n$, let $\Pi(n, d)$ denote the set of all coordinate projections $\mathbb{R}^n \to \mathbb{R}^d$:

$$(x_1, \dots, x_n) \mapsto (x_{i_1}, \dots, x_{i_d})$$

where $1 \leq i_1 < \cdots < i_d \leq n$. Let $S \subseteq \mathbb{R}^n$ be nonempty. The **dimension** of S (denoted by $\dim S$) is the largest $d \in \mathbb{N}$ such that πS has interior for some $\pi \in \Pi(n, d)$. Following [11], given $\pi \in \Pi(n, m)$, we say that S is a **π-special submanifold** if S is definable and, for every $y \in \pi S$ there is a box B about y such that π homeomorphically maps each connected component of $S \cap \pi^{-1}B$ onto B; and S is a **special submanifold** if S is a π-special submanifold for some $\pi \in \Pi(n, \dim S)$. Note that every special submanifold of dimension 0 is discrete.

Let $\mathcal{A}$ be a finite collection of subsets of $\mathbb{R}^n$ and $\mathcal{P}$ be a partition of $\mathbb{R}^n$. We say $\mathcal{P}$ is **compatible** with $\mathcal{A}$ if every $A \in \mathcal{A}$ is a union of sets in $\mathcal{P}$.

We need a modification of [11, Theorem 3.4.1]:

Theorem B (Decomposition Theorem). *Suppose* $\mathfrak{R}$ *is d-minimal. Let* $\mathcal{A}$ *be a finite collection of definable subsets of* $\mathbb{R}^n$. *Then there is a finite partition* $\mathcal{P}$ *of* $\mathbb{R}^n$ *into special submanifolds compatible with* $\mathcal{A}$ *such that, for each* $P \in \mathcal{P}$, *the frontier of* P *is a finite union of elements of* $\mathcal{P}$.

We postpone the proof until the next section.

The following is a consequence of the Definable Choice (see [12]).

1.4. *Suppose* $\mathfrak{R}$ *is d-minimal. Let* $E \subseteq \mathbb{R}^n$ *and* $T\colon E \rightrightarrows \mathbb{R}^n$. *Then there is a definable function* $f\colon E \to \mathbb{R}^n$ *such that* $f(x) \in T(x)$ *for every* $x \in E$.

For any set $S \subseteq \mathbb{R}^n$, the phrase "S has interior" means "S has nonempty interior", while "S has no interior" means "S has empty interior".

We are now ready for the

Proof of Theorem A. Suppose $\mathfrak{R}$ is d-minimal. Let $E \subseteq \mathbb{R}^n$ be definable and $T\colon E \rightrightarrows \mathbb{R}^m$ be definable. Suppose T is lower semi-continuous and each $T(x)$ is nonempty closed and convex. First, we will show that there is a finite partition $\mathcal{P}$ of E into special submanifolds such that for every $P \in \mathcal{P}$, $T\restriction P$ is continuous and $\operatorname{fr} P$ is a finite union of elements in $\mathcal{P}$. We proceed by induction on $\dim E$. If $\dim E = 0$, this follows immediately from d-minimality and that every special submanifold of dimension 0 is discrete. Assume $\dim E > 0$ and the result holds for sets of dimension less than $\dim E$. By the Decomposition Theorem, it suffices to assume that E is a special submanifold. Let $S = \pi(\operatorname{fr} T\restriction E)$ where $\pi \in \Pi(n+m, n)$ denote projection on the first n coordinates. Note that if S is nowhere dense in E, then $\dim S < \dim E$ because E is a special submanifold. Therefore, it is enough to show that S has no interior in E. Suppose not. By the Definable Choice, let $f\colon S \to \mathbb{R}^m$ be a definable map such that $f \subseteq \operatorname{fr} T$. Since each $T(x)$ is closed, $d(f(x), T(x)) > 0$. By decomposition and compactness, there is $\delta > 0$ and a compact set $B \subseteq S$ such that B has interior, $f\restriction B$ is continuous, and $d(f(x), T(x)) > \Delta$. Hence, $f\restriction B \nsubseteq \operatorname{fr} T$, which is absurd.

Now, let $\mathcal{P}$ be a finite partition of E into special submanifolds such that, for every $P \in \mathcal{P}$, $T\restriction P$ is continuous and $\operatorname{fr} P$ is a finite union of elements in $\mathcal{P}$. We proceed by induction on the cardinality, d, of $\mathcal{P}$. If $d = 1$, then $\mathcal{P} = \{E\}$, and the result is immediate from 1.2. Suppose $d > 1$ and the result holds for any partition whose cardinality is less than d. Let $P_0 \in \mathcal{P}$ such that P_0 is not contained in $\operatorname{fr} P$ for any $P \in \mathcal{P}$. Then there is a definable continuous function $f_1\colon E \setminus P_0 \to \mathbb{R}^n$ such that $f_1(x) \in T(x)$ for every $x \in E \setminus P_0$. Note that $E \setminus P_0$ is closed in E. By 1.3, there is a definable continuous extension $f_2\colon E \to \mathbb{R}^n$ of f_1. Replacing T by $T - f_2$, we may assume that $f_2 = 0$. Since $T\restriction P_0$ is continuous, $\operatorname{lns}_{T\restriction P_0}\colon P_0 \to \mathbb{R}^m$ is continuous. Define $f\colon E \to \mathbb{R}^n$ by

$$f(x) = \begin{cases} \operatorname{lns}_{T\restriction P_0}(x), & \text{if } x \in P_0; \\ 0, & \text{if } x \in E \setminus P_0. \end{cases}$$

We can easily see that $f(x) \in T(x)$ for every $x \in E$ and f is continuous on $E \setminus \operatorname{fr} P_0$. Hence, we now need only consider $x_0 \in \operatorname{fr} P_0$ and show that f is continuous at x_0. Let $\epsilon > 0$ and $y_0 \in T(x_0)$. By lower semi-continuity, $\limsup_{x\to x_0} \|f(x)\| \leq \|f(x_0)\| = 0$; that is, $\lim_{x\to x_0} f(x) = 0$. Therefore, f is continuous at x_0. □

2. Proof of Theorem B

As mentioned earlier, Theorem B is a modification of [11, Theorem 3.4.1]; there are two issues that need to be addressed before we start working toward the proof. First, the statement of [11, Theorem 3.4.1] does not include the frontier condition, which is used in the proof of Theorem A. More serious is that we cannot actually *use* [11, Theorem 3.4.1] because there is a nontrivial mistake in the proof that has hitherto gone unrepaired; we shall remedy this.

We begin with some preliminary results that hold in greater generality. (Recall $\mathfrak{R}$ is an expansion of $(\mathbb{R}; <)$, not necessarily the real field.)

2.1 ([11, Section 7]). *If every definable subset of $\mathbb{R}$ has interior or is nowhere dense, then every definable set has interior or is nowhere dense.*

2.2 (Almost continuity [11, Theorem 3.3]). *Suppose every definable subset of $\mathbb{R}$ has interior or is nowhere dense. Let $U \subseteq \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^m$ be definable. Then there is a definable open dense V subset of U such that $f\restriction V$ is continuous.*

Let $S \subseteq \mathbb{R}^n$. For $\pi \in \Pi(n, \dim S)$, we say that S is π-**good** if

— S is definable;

— πS is open;

— for every open box $B \subseteq \mathbb{R}^n$, $\pi(S \cap B)$ either has interior or is empty;
— $\operatorname{cl} S \cap \pi^{-1}x = \operatorname{cl}(S \cap \pi^{-1}x)$ and $\dim(S \cap \pi^{-1}x) = 0$ for each $x \in \pi S$.

2.3. *Suppose every definable subset of $\mathbb{R}$ has interior or is nowhere dense. Let $S \subseteq \mathbb{R}^n$ be definable and $\pi \in \Pi(n, \dim S)$. If πS has interior, then there is definable, open and dense $U \subseteq \mathbb{R}^{\dim S}$ such that $S \cap \pi^{-1}U$ is π-good.*

(This is essentially a corollary of the proof of [11, Partition Lemma].)

Proof. The result is trivial if $\dim S = n$, so assume that $\dim S < n$. Let Y be the set of all $a \in S$ such that $\pi(S \cap B)$ has interior for every box B containing a. It is routine to show that Y is definable and $\pi(S \setminus Y)$ is nowhere dense. In addition, put

$$Z = \{x \in \pi(S \setminus Y) : \dim Y_x > 0 \text{ or } \operatorname{cl} Y_x \neq (\operatorname{cl} Y)_x\}$$

and $U = \operatorname{int}(\mathbb{R}^{\dim S} \setminus Z)$. Observe that $\mathbb{R}^{\dim S} \setminus U$ is nowhere dense and $S \cap \pi^{-1}U$ is π-good. □

Let $S \subseteq \mathbb{R}^n$ and $d \in \{0, \dots, n\}$. For each $\pi \in \Pi(n, d)$, let $\operatorname{reg}_\pi S$ denote the set of $a \in S$ such that there is a box B about x such that $\pi{\restriction}B \cap S$ homeomorphically maps $B \cap S$ onto an open subset of $\mathbb{R}^d$. Note that: (1) if S is a π-special submanifold, then $S = \operatorname{reg}_\pi S$; (2) if S is connected and $S = \operatorname{reg}_\pi S$, then S is path connected; and (3) if $S = \operatorname{reg}_\pi S$ and X is a connected component of S with $\dim X = \dim S$, then $X = \operatorname{reg}_\pi X$.

2.4. *Suppose every definable subset of $\mathbb{R}$ has interior or is nowhere dense. Let $d \in \{0, \dots, n\}$, $S \subseteq \mathbb{R}^n$ be definable such that S_x is discrete for every $x \in \mathbb{R}^d$, and $\pi \in \Pi(n, d)$ denote projection on the first d coordinates. Then $\pi(S \setminus \operatorname{reg}_\pi S)$ is nowhere dense.*

Proof. Let $C = S \setminus \operatorname{reg}_\pi S$. Suppose to the contrary that πC is somewhere dense; then it has interior. Since each S_x is discrete, for each $x \in \pi C$ there exist $y \in C_x$ and a box $V \subseteq \mathbb{R}^{n-d}$ (with rational vertices) such that $S_x \cap V = \{y\}$. By the Baire Category Theorem, there is a box $V \subseteq \mathbb{R}^{n-d}$ such that $\{x \in \mathbb{R}^d : S_x \cap V = \{y\} \text{ for some } y \in C_x\}$ is somewhere dense; therefore, contains a box $U \subseteq \mathbb{R}^d$. For each $x \in U$, let $f(x)$ be the unique point in $S_x \cap V$. Then $f \subseteq C$. By 2.2, there is a box $B \subseteq U$ such that $f{\restriction}B$ is continuous. Then $f{\restriction}B \subseteq \operatorname{reg}_\pi S$, which is absurd. □

Let $\mathcal{U}(0) = \{\mathbb{R}^0\}$, and $\mathcal{U}(n+1)$ be the collection of all open definable $U \subseteq \mathbb{R}^{n+1}$ such that the projection πU of U on the first n coordinates belongs to $\mathcal{U}(n)$, and if X is a connected component of U, then X is a cell of $(\overline{\mathbb{R}}, \mathbb{Z})$ and πX is a connected component of πU. Note that $\mathcal{U}(1)$ is the collection of all definable open subsets of $\mathbb{R}$. For $0 \le d \le n$ and $\pi \in \Pi(n, d)$, let $\mathcal{M}(n, d, \pi)$ be the collection of all definable $M \subseteq \mathbb{R}^n$ for which there are finitely many coordinate permutations $\sigma_1, \dots, \sigma_m$ of $\mathbb{R}^d$ and $U_1, \dots, U_m \in \mathcal{U}(d)$ such that:

— $\sigma_1 U_1, \dots, \sigma_m U_m$ are pairwise disjoint;
— $\pi M = \sigma_1 U_1 \cup \dots \cup \sigma_m U_m$;
— for all $x \in \mathbb{R}^d$, $M \cap \pi^{-1}x$ is discrete;
— if X is a connected component of M, then πX is a connected component of πM and $\pi{\restriction}X \colon X \to \pi X$ is a homeomorphism.

Let $\mathcal{M}(n,d) = \bigcup_{\pi\in\Pi(n,d)} \mathcal{M}(n,d,\pi)$, and $\mathcal{M}(n) = \bigcup_{0\le d\le n} \mathcal{M}(n,d)$. Every connected component of $M \in \mathcal{M}(n)$ is simply connected. Every $M \in \mathcal{M}(n)$ is a special submanifold but not conversely.

For the rest of this section, we assume that $\mathfrak{R}$ is d-minimal. Consider the following conditions:

(I_n) If $A \subseteq \mathbb{R}^n$ is definable and bounded, $\dim A < n$ and $\pi \in \Pi(n, \dim A)$, then there exist definable, open $U \subseteq \mathbb{R}^{\dim A}$ and a finite pairwise disjoint $\mathcal{Q} \subseteq \mathcal{M}(n, \dim A, \pi)$ such that (1) U is dense in $\mathbb{R}^{\dim A}$, (2) $A \cap \pi^{-1}U = \bigcup \mathcal{Q}$ and (3) for every $Q \in \mathcal{Q}$, the projection under π of each connected component of Q is a connected component of U and $\operatorname{fr} Q \cap \pi^{-1}U$ is a finite union of elements in $\mathcal{Q}$.

(II_n) If $\mathcal{A}$ is a finite collection of definable and bounded subsets of $\mathbb{R}^n$, then there is a finite partition $\mathcal{P}$ of $\mathbb{R}^n$ by elements of $\mathcal{M}(n)$ such that $\mathcal{P}$ is compatible with $\mathcal{A}$, and for each $P \in \mathcal{P}$, $\operatorname{fr} P$ is a finite union of elements in $\mathcal{P}$.

Observe that Theorem B is immediate from (II_n) and the existence of semialgebraic homeomorphism from $\mathbb{R}$ to the interval $(-1, 1)$. We shall establish (I_n) and (II_n) in turn by induction. The case $n = 1$ is immediate from d-minimality (and [11, 2.4]). Let $n \ge 1$ and assume (I_m) and (II_m) hold for every $m \le n$. We must show both (I_{n+1}) and (II_{n+1}); the proof is lengthy and involves subsidiary inductions. Before proceeding, we think it is important to investigate the flaw in the proof of [11, Theorem 3.4.1]. The idea is if $\mathfrak{R}$ is d-minimal, the set of all *topological singularities* in any definable sets is supposed to be relatively small. The Baire Category Theorem is the key in the argument of the proof of [11, Theorem 3.4.1] because it can be used to test nowhere density of sets. However, the Baire Category Theorem was accidentally applied to a possibly uncountable family of sets; this is where the proof breaks down.

To fix this problem, we need to detect other singularities that are not detectable by the argument in [11]. We notice that there are two more types of singularities: the first type (to be described later) can be detected from the first n coordinates; on the other hand, the second type can only be detected in the last coordinate. Since we already have the result for every $m \le n$, the first type can be easily detected (we only need to apply the assumption in the right order). To illustrate how to detect the second kind, we motivate by arguing in $\mathbb{R}^2$. When $\dim A = 0$, (I_2) and (II_2) follow immediately from d-minimality. On the other hand, the case $\dim A = 2$ is immediate from the lower dimension case. Therefore, we need only consider when $\dim A = 1$. Now let us consider $A = ((-1,1)\times 2^{-\mathbb{N}}) \cup \{(x, -x^2) : 0 < x < 1\}$. Each connected component of A is a C^1-manifold; however, A is not a covering space of its projection on the first coordinate. Let $A' = A \cap (\mathbb{R} \times (-\infty, 0]) = \{(x, -x^2) : 0 < x < 1\}$. Then $(\operatorname{cl} A')_0 = \{0\} \ne \emptyset = \operatorname{cl} A'_0$. This is an example of singularities that we will handle.

For $Y \subseteq \mathbb{R}^{n+1}$, let

$$Y^+ := \{(z,t) : s < t \text{ for all } (z,s) \in Y\} \text{ and } Y^- := \{(z,t) : t < s \text{ for all } (z,s) \in Y\}.$$

2.5. *Let $U \subseteq \mathbb{R}^d$ and $\mathcal{Q} \subseteq \mathcal{M}(n,d,\pi)$ satisfy the conclusion of (I_{n+1}). Then if $V \subseteq U$ is a finite union of open elements in $\mathcal{M}(d)$, V is dense in U and $\mathcal{Q} \subseteq \mathcal{M}(n,d,\pi)$, then V and $\{Q \cap \pi^{-1}V : Q \in \mathcal{Q}\}$ satisfy the conclusion of (I_{n+1}).*

The above result will be used repeatedly in the proof of (I_{n+1}).

Let $S \subseteq \mathbb{R}^n$. The **rank of** S (denoted by $\operatorname{rank} S$) is the infimum of $k \in \mathbb{N}$ such that S is the union of k-many discrete sets. Note that if S is closed and $\operatorname{rank} S$ is finite, then $\operatorname{rank} S$ is equal to the Cantor-Bendixson rank of S. It is known that if $\operatorname{rank} S$ is finite and S is nonempty, then $\operatorname{rank} S = 1 + \operatorname{rank}(S \setminus \operatorname{isol} S)$ see, *e.g.*, [7, §1].

2.6. *If* $\{A_x\}_{x\in\mathbb{R}^k}$ *is a definable family of subsets of* $\mathbb{R}^n$ *such that either* $A_x = \emptyset$ *or* $\dim A_x = 0$ *for every* $x \in \mathbb{R}^k$, *then there exists* $N \in \mathbb{N}$ *such that* $\operatorname{rank} A_x \leq N$ *for every* $x \in \mathbb{R}^k$.

This follows immediately from definability of the rank.

We now begin the

Proof of (I_{n+1}). If $d = 0$, this immediately follows from d-minimality and the definitions of $\mathcal{M}(n+1, 0, \pi)$ and rank. Suppose $d > 0$. Let $A \subseteq \mathbb{R}^{n+1}$ be definable and bounded. By 2.3 and 2.6, we reduce to the case A is π-good and for each $x \in \pi A$, $\operatorname{rank}(\operatorname{cl} A_x) \leq N$. We proceed by induction on N.

Assume $\operatorname{rank}(\operatorname{cl} A_x) = 1$ for every $x \in \pi A$. Then each $\operatorname{cl} A_x$ is finite; so $\operatorname{cl} A_x = A_x$ for every $x \in \pi A$. By 2.4 and (II_d), we reduce to the case $A = \operatorname{reg}_\pi A$ and πA is a finite disjoint union of open elements in $\mathcal{M}(d)$. We will show that $A \in \mathcal{M}(n+1, d, \pi)$.

Let X be a connected component of A. We will prove that πX is a connected component of πA. Since $X = \operatorname{reg}_\pi X$, πX is open. Suppose πX is not closed in πA and let $x \in \pi A \cap \operatorname{fr} \pi X$. Since $\operatorname{cl} X$ is bounded, there is $y \in (\operatorname{cl} X)_x \subseteq (\operatorname{cl} A)_x = \operatorname{cl} A_x = A_x$. Then $X \cup \{(x, y)\}$ is a connected subset of M. This is absurd because X is a connected component of M. Thus, πX is closed in πA. Since πX is both open and closed in πA, πX is a connected component of πA.

Therefore, it remains to show that $\pi{\restriction}X\colon X \to \pi X$ is a homeomorphism. Note that if $p\colon C \to D$ is a covering map where C is path-connected and D is simply connected, by the unique lifting theorem for covering maps (see, *e.g.*, J. Munkres [16, 8.4.2]), p is a homeomorphism. Hence, we need only show that $\pi{\restriction}X$ is a covering map. Let $x \in \pi X$. Then $X \cap \pi^{-1}x$ is finite; let $X \cap \pi^{-1}x = \{(x, y_1), \dots, (x, y_k)\}$. Since $X = \operatorname{reg}_\pi X$, for each $i \in \{1, \dots, k\}$, there is a box B_i about (x, y_i) such that $\pi{\restriction}X \cap B_i \to \pi B_i$ is a homeomorphism. For each $\epsilon > 0$, let $B^\epsilon := \prod_{j=1}^{d}(x_j - \epsilon, x_j + \epsilon)$ and $B_i^\epsilon := B_i \cap \pi^{-1}B^\epsilon$. It suffices to prove that there is $\epsilon > 0$ such that $X \cap \pi^{-1}B^\epsilon = \bigcup_{i=1}^{k} B_i^\epsilon$. Suppose there is no such ϵ. Then, for each $\epsilon > 0$, there is $(x_\epsilon, y_\epsilon) \in (X \cap \pi^{-1}B^\epsilon) \setminus \bigcup_{i=1}^{k} B_i^\epsilon$. Therefore, $(x_\epsilon, y_\epsilon) \notin \bigcup_{i=1}^{k} B_i$. Since $\operatorname{cl} X$ is closed and bounded, there is $y \in \mathbb{R}^{n+1-d}$ such that $(x, y) \in \operatorname{cl}\{(x_\epsilon, y_\epsilon) : \epsilon > 0\}$. Since A is π-good and X is a connected component of A, $(x, y) \in X$. Therefore, $y = y_i$ for some $i \in \{1, \dots, k\}$. Hence, there is $\epsilon > 0$ such that $(x_\epsilon, y_\epsilon) \in B_i$, which is absurd.

This completes the case $N = 1$.

Suppose $N \geq 1$ and the result holds for N. Assume $\operatorname{rank}(\operatorname{cl} A_x) \leq N + 1$ for every $x \in \pi A$. By 2.4 and the inductive hypothesis, we reduce to the case $A = \operatorname{reg}_\pi A$ and for all $x \in \pi A$, A_x is discrete and $\operatorname{rank}(\operatorname{cl} A_x) = N + 1$. Now, it suffices to find $U \subseteq \pi A$ definable, open and dense such that $A \cap \pi^{-1}U \in \mathcal{M}(n+1, d, \pi)$.

First, consider $\operatorname{fr} A$. Note that $\operatorname{rank}(\operatorname{fr} A_x) = N$. By the induction hypothesis, let $U_1 \subseteq \pi A$ be definable, open and dense in πA, and $\mathcal{Q}_1 \subseteq \mathcal{M}(n+1, d, \pi)$ be finite and pairwise disjoint such that

— $\operatorname{fr} A \cap \pi^{-1} U_1 = \bigcup \mathcal{Q}_1$;
— πX is a connected component of U for each connected component X of $Q \in \mathcal{Q}_1$;
— for every $Q \in \mathcal{Q}_1$, $\operatorname{fr} Q \cap \pi^{-1} U$ is a finite union of elements in $\mathcal{Q}_1$.

Let $\pi_1 \in \Pi(n+1, n)$ and $\pi_2 \in \Pi(n, d)$ be projections on the first n and d coordinates, respectively. By 2.5, (I_n) and (II_d), we reduce to the case that there is finite and pairwise disjoint $\mathcal{P} \subseteq \mathcal{M}(n, d, \pi_2)$ such that

— $\pi_1(\operatorname{cl} A) \cap \pi_2^{-1} U_1 = \bigcup \mathcal{P}$;
— $\pi_2 Y$ is a connected component of U_1 for each connected component Y of $P \in \mathcal{P}$;
— for every $P \in \mathcal{P}$, $\operatorname{fr} P \cap \pi_2^{-1} U_1$ is a finite union of elements in $\mathcal{P}$;
— for every $Q \in \mathcal{Q}_1$ there exists $P \in \mathcal{P}$ such that $\pi_1 Q = P$.

Note that if X is a connected component of $A \cap \pi^{-1} U_1$, then $\pi_1 X$ is contained in a connected component of some $P \in \mathcal{P}$. For $Q \in \mathcal{Q}_1$, we define sets $\operatorname{fr}^+ Q$ and $\operatorname{fr}^- Q$ as follows:

$$(z,t) \in \operatorname{fr}^+ Q \text{ iff } (z,t) \in Q \;\&\; \text{ if } X \text{ is the connected component of } Q \text{ containing } (z,t), \text{ then } (z,t) \in \operatorname{fr}(A \cap X^+);$$

$$(z,t) \in \operatorname{fr}^- Q \text{ iff } (z,t) \in Q \;\&\; \text{ if } X \text{ is the connected component of } Q \text{ containing } (z,t), \text{ then } (z,t) \in \operatorname{fr}(A \cap X^-).$$

It is an exercise to see that both $\operatorname{fr}^+ Q$ and $\operatorname{fr}^- Q$ are definable.

For each $z \in \mathbb{R}^n$ and $Q \in \mathcal{Q}_1$, let

$$\operatorname{fr}^+ Q_z := \{t \in \mathbb{R} : (z,t) \in Q \,\&\, t \in \operatorname{fr}(A_z \cap (t, +\infty))\},$$
$$\operatorname{fr}^- Q_z := \{t \in \mathbb{R} : (z,t) \in Q \,\&\, t \in \operatorname{fr}(A_z \cap (-\infty, t))\}.$$

Let $Q \in \mathcal{Q}_1$. We will show that

these sets $\{x \in \pi A : \exists y \in \mathbb{R}^{n-d}, (\operatorname{fr}^+ Q)_{(x,y)} \neq \operatorname{fr}^+ Q_{(x,y)}\}$ *and* $\{x \in \pi A : \exists y \in \mathbb{R}^{n-d}, (\operatorname{fr}^+ Q)_{(x,y)} \neq \operatorname{fr}^+ Q_{(x,y)}\}$ *are nowhere dense.*

Let $D := \{(z,t) \in \operatorname{fr}^+ Q : t \notin \operatorname{fr}^+ Q_z\}$. Then $\pi D = \{x \in \pi A : \exists y \in \mathbb{R}^{n-d}, (\operatorname{fr}^+ Q)_{(x,y)} \neq \operatorname{fr}^+ Q_{(x,y)}\}$. Since Q is a special submanifold, it is enough to show that D is nowhere dense in Q. Suppose to the contrary that D is somewhere dense in Q. Then D has interior in Q. By the Baire Category Theorem, there exists a box $B \subseteq \mathbb{R}^{n+1}$ such that $D \cap B \not\subseteq \operatorname{fr}^+ Q$, which is absurd. Therefore, $\{x \in \pi A : \exists y \in \mathbb{R}^{n-d}, (\operatorname{fr}^+ Q)_{(x,y)} \neq \operatorname{fr}^+ Q_{(x,y)}\}$ is nowhere dense. Similarly, we have $\{x \in \pi A : \exists y \in \mathbb{R}^{n-d}, (\operatorname{fr}^- Q)_{(x,y)} \neq \operatorname{fr}^- Q_{(x,y)}\}$ is nowhere dense.

By (II_d), let U be a finite disjoint union of open sets in $\mathcal{M}(d)$ such that U is dense in U_1 and, for $Q \in \mathcal{Q}_1$ and $(x,y) \in \pi_1(\operatorname{cl} A) \cap \pi_2^{-1} U$, $(\operatorname{fr}^+ Q)_{(x,y)} = \operatorname{fr}^+ Q_{(x,y)}$ and $(\operatorname{fr}^- Q)_{(x,y)} = \operatorname{fr}^- Q_{(x,y)}$. Next, we will prove that $M := A \cap \pi^{-1} U \in \mathcal{M}(n+1, d, \pi)$; that is, if X is a connected component of M, then πX is a connected component of U and $\pi{\restriction}X \colon X \to \pi X$ is a homeomorphism.

Let X be a connected component of M. First, we will show that πX is a connected component of U. Since X is connected, there is a connected component V of U containing πX. To prove that $\pi X = V$, it suffices to show that πX is both open and closed in V. Since $\pi{\restriction}X$ is a local homeomorphism, πX is open. Note that if πX is not closed in V, then, by the compactness of $\operatorname{cl} X$, $\operatorname{fr} X \cap \pi^{-1}V \neq \emptyset$. Therefore, it is enough to show that $\operatorname{fr} X \cap \pi^{-1}V = \emptyset$. Let $(x, y, t) \in \operatorname{fr} X \cap \pi^{-1}V$ where $x \in V$, $y \in \mathbb{R}^{n-d}$ and $t \in \mathbb{R}$. Hence, there is $Q \in \mathcal{Q}_1$ containing (x, y, t). We will prove that $\pi_1 X \subseteq \pi_1 Q$. Note that either $\pi_1 X \cap \pi_1 Q = \emptyset$ or $\pi_1 X \subseteq \pi_1 Q$. Suppose to the contrary that $\pi_1 X \cap \pi_1 Q = \emptyset$. Then there is $P \in \mathcal{P}$ such that $\pi_1 X \subseteq P$. Then $P \cap \pi_1 Q = \emptyset$. Since $\emptyset \neq \pi X \subseteq V \cap \pi_2 P$ and $\pi_2 P$ is connected, $V \subseteq \pi_2 P$. Since $(x, y) \in \operatorname{cl}(\pi_1 X)$ and $x \in V \subseteq \pi_2 P$, $(x, y) \in P$. This is impossible since $P \cap \pi_1 Q = \emptyset$. Therefore, $\pi_1 X \subseteq \pi_1 Q$. Let Z be a connected component of Q containing (x, y, t). Then either $X \subseteq Z^+$ or $X \subseteq Z^-$. Without loss of generatlity, assume $X \subseteq Z^+$. Then $(x, y, t) \in \operatorname{fr}^+ Q$; so $t \in \operatorname{fr}^+ Q_{(x,y)}$. Let $S := \{s \in \mathbb{R} : (x, y, s) \in \operatorname{fr} A \setminus Q \,\&\, s > t\}$.

Case1. Suppose $t \in \operatorname{cl} S$. Let $(s_i)_{i\in\mathbb{N}}$ be a decreasing sequence in S converging to t. For each $i \in \mathbb{N}$, let Y_i be a connected component of $\operatorname{fr} A \cap \pi^{-1}V$ such that $(x, y, s_i) \in Y_i$. Therefore, for $i < j$, $\pi_1 Y_i = \pi_1 Y_j = \pi_1 Q$ and $Y_i \subseteq Y_j^+$. Let $Y' = \operatorname{fr}(\bigcup_{i\in\mathbb{N}} Y_i) \subseteq \operatorname{fr} A$. Then Y' is connected. Since $(x, y, t) \in Y' \cap Z$, $Y' \subseteq Z$.

Let $(z, s) \in X$. Since $X \subseteq Z^+$, there is $u < s$ such that $(z, u) \in Z$. For each $i \in \mathbb{N}$, let $u_i \in \mathbb{R}$ such that $(z, u_i) \in Y_i$. Since each Y_i is connected and $\pi_1 X \subseteq \pi_1 Y_i$, $u < s < u_i$ for every $i \in \mathbb{N}$. Therefore, $(z, \lim_{i\to\infty} u_i) \in Y' \setminus Z$. This is a contradiction.

Case2. Suppose $t \notin \operatorname{cl} S$. Then there exists a box $B \subseteq \mathbb{R}^{n+1}$ about (x, y, t) such that $\pi B \subseteq \pi Q$ and $B \cap Z^+ \cap \operatorname{fr} A = \emptyset$. Let $(t_i)_{i\in\mathbb{N}}$ be a decreasing sequence such that $t_i \in (B \cap A)_{(x,y)}$ and $\lim_{i\to\infty} t_i = t$. For each $i \in \mathbb{N}$, let Z_i be the connected component of M containing (x, y, t_i). Since $Z_0 = \operatorname{reg}_\pi Z_0$, there is a box $B' \subseteq \pi B$ about x such that $(Z_0)_{x'} \neq \emptyset$ for every $x' \in \operatorname{cl}(B')$.

Let $i \in \mathbb{N} \setminus \{0\}$. We will show that $\operatorname{cl} B' \subseteq \pi Z_i$. Let $x' \in \operatorname{fr}(\pi Z_i) \cap \operatorname{cl} B'$. Since $\operatorname{cl} Z_i$ is bounded, there exists $(x', y', t') \in \operatorname{cl} Z_i \cap B$. Since $(x, y) \in \pi_1 Z_i \cap \pi_1 Q$, $\pi_1 Z_i \subseteq \pi_1 Q$. Pick $(x', y', s') \in Z$. Since Z and Z_i are connected, $s' < t'$. Then $(x', y', t') \in M$, which is absurd. Therefore, $\operatorname{cl} B' \subseteq \pi Z_i$.

Let $x_0 \in B' \cap \pi X$, $(x_0, y_0, t_X) \in X$, $(x_0, y_0, t_Z) \in Z$ and $(x_0, y_0, s_i) \in Z_i$ for every $i \in \mathbb{N}$. By connectedness of X, Z and Z_i, we have $t_Z < t_X < s_{i+1} < s_i$ for every $i \in \mathbb{N}$. Therefore, $(x_0, y_0, \lim_{i\to\infty} s_i) \in \operatorname{fr} A$. This is absurd because $B \cap Z^+ \cap \operatorname{fr} A = \emptyset$. Hence, πX is closed in V; and so $\pi X = V$.

To complete the proof of (I_{n+1}), we will prove the injectivity of $\pi\restriction X$. We first show that $\pi\restriction X$ satisfies the unique lifting property. Let $a_0 \in X$. We show that:

(a) Every path $\alpha\colon [0, 1] \to V$ with $\alpha(0) = \pi a_0$ has a unique path $\tilde{\alpha}\colon [0, 1] \to X$ such that $\tilde{\alpha}(0) = a_0$ and $\pi \circ \tilde{\alpha} = \alpha$.

(b) Let $\alpha_1, \alpha_2\colon [0, 1] \to V$ be paths with $\alpha_1(0) = \alpha_2(0) = \pi a_0$ and $H\colon [0, 1] \times [0, 1] \to V$ be a homotopy between these two paths such that $H(0, t) = \pi a_0$ for every $t \in [0, 1]$. Then there is a unique homotopy $\tilde{H}\colon [0, 1] \times [0, 1] \to X$ between $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$.

(a). Since $\pi\restriction X$ is a local homeomorphism, the uniqueness is immediate. Let $\alpha\colon [0, 1] \to V$ be continuous with $\alpha(0) = \pi a_0$. Consider the set

$$I := \{t \in [0, 1] : \exists \text{ path } \tilde{\alpha}_t\colon [0, t] \to X, \tilde{\alpha}_t(0) = a_0 \,\&\, \forall s \in [0, t], \pi(\tilde{\alpha}_t(s)) = \alpha(s)\}.$$

It is enough to prove that $I = [0, 1]$. Since $X = \operatorname{reg}_\pi X$, Z is open in $[0, 1]$. Obviously, I is a subinterval of $[0, 1]$ containing 0. Therefore, it suffices to show that $t_0 := \sup I \in I$. By the uniqueness, for each $t < t_0$, there is a unique path $\tilde{\alpha}_t$ such that $\tilde{\alpha}_t(0) = a_0$ and

$\pi(\tilde{\alpha}_t(s)) = \alpha(s)$ for every $s \in [0,t]$. Thus, $\tilde{\alpha}_s = \tilde{\alpha}_t \restriction [0,s]$ for $s \le t < t_0$. Define $\beta\colon [0,t_0) \to X$ by $\beta(t) = \tilde{\alpha}_t(t)$. Then β is continuous. Let $\Gamma := \mathrm{cl}\{\beta(x) : x \in [0,t_0)\}$. Since Γ is connected, $\Gamma \subseteq \mathrm{cl}\, X$. Recall that $\mathrm{cl}\, X \cap \mathrm{fr}\, A \cap \pi^{-1}V = \emptyset$; therefore, $\emptyset \neq \Gamma_{\alpha(t_0)} \subseteq X_{\alpha(t_0)}$. Since $X = \mathrm{reg}_\pi X$, $\Gamma_{\alpha(t_0)}$ is a singleton; so $t_0 \in I$.

(b). The uniqueness follows immediately from (a). Next, let $\alpha_1, \alpha_2\colon [0,1] \to \pi X$ be paths with $\alpha_1(0) = \alpha_2(0) = \pi a_0$ and $H\colon [0,1] \times [0,1] \to V$ be a homotopy between α_1 and α_2 such that $H(0,t) = \pi a_0$ for every $t \in [0,1]$. For each $t \in [0,1]$, let $\beta_t\colon [0,1] \to V$ be a path such that $\beta_t(s) = H(s,t)$ for every $s \in [0,1]$. Define $\tilde{H}\colon [0,1] \times [0,1] \to V$ by $\tilde{H}(s,t) = \tilde{\beta}_t(s)$. Then $\tilde{H}$ is continuous.

Let $x \in \pi X$ and $y_1, y_2 \in X_x$. We will show that $y_1 = y_2$. Since X is path connected, pick a path $\alpha\colon [0,1] \to X$ such that $\alpha(0) = (x,y_1)$ and $\alpha(1) = (x,y_2)$. Let $\beta_1, \beta_2\colon [0,1] \to \pi X$ be two paths such that $\beta_1 = \pi \circ \alpha$ and β_2 is the constant path $t \mapsto x$. Since V is simply connected, let H be a homotopy between β_1 and β_2. This H can be lifted to a homotopy $\tilde{H}$ between $\tilde{\beta_1}$ and $\tilde{\beta_2}$, which are α and the constant path $t \mapsto (x,y_1)$, respectively. Therefore, we also have that $(x,y_2) = \alpha(1) = \tilde{\beta_2}(1) = (x,y_1)$; so $y_1 = y_2$. $\square$

For $A \subseteq \mathbb{R}^n$, the **full dimension** of a set A, denoted by $\mathrm{fdim}\, A$, is the ordered pair (d,k) where $d = \dim A$ and k is the cardinality of the set $\{\pi \in \Pi(n,d) : \pi A \text{ has interior}\}$. The full dimension is ordered by the lexicographical order.

Proof of (II_{n+1}). We may assume that $\mathcal{A}$ is pairwise disjoint and set $S = \mathrm{cl}(\bigcup \mathcal{A})$. Observe that for a box $B \subseteq \mathbb{R}^{n+1}$, there is a finite partition of $\mathbb{R}^n \setminus B$ by elements of $\mathcal{M}(n+1)$. Therefore, we need only prove that there is a finite partition $\mathcal{P}$ of S by elements of $\mathcal{M}(n)$ such that $\mathcal{P}$ is compatible with $\mathcal{A}$, and for each $P \in \mathcal{P}$, $\mathrm{fr}\, P$ is a finite union of elements in $\mathcal{P}$. We proceed by induction on $(d,k) = \mathrm{fdim}\, S$. The case $\dim S = 0$ is trivial. The case $0 < \dim S < n+1$ follows from (I_{n+1}) and the inductive hypothesis. (Note that the full dimension is used in this case.)

Suppose $\dim S = n+1$. By the inductive hypothesis and (II_n), it is enough to assume that $\mathcal{A} = \{W\}$ where $W \subseteq \mathbb{R}^{n+1}$ is open. Hence, $S = \mathrm{cl}\, W$. Note that $\dim(\mathrm{bd}\, S) \le n$. Let $\pi \in \Pi(n+1,n)$ denote projection on the first n coordinates. By (I_{n+1}), let $U \subseteq \pi S$ be a finite disjoint union of open sets in $\mathcal{M}(n)$ such that U is dense in πS and $\mathcal{Q} \subseteq \mathcal{M}(n+1,n,\pi)$ be a finite partition of $\mathrm{bd}\, S \cap \pi^{-1}U$ by elements of $\mathcal{M}(n+1,n,\pi)$ such that each connected component of $Q \in \mathcal{Q}$ is a connected component of U and for each $Q \in \mathcal{Q}$, $\mathrm{fr}\, Q \cap \mathrm{bd}\, S \cap \pi^{-1}U$ is a union of elements in $\mathcal{Q}$.

It suffices to show that $M := W \cap \pi^{-1}U \in \mathcal{U}(n+1)$. Let X be a connected component of M. For $Q \in \mathcal{Q}$ and a connected component Y of Q, if $\pi Y \cap \pi X \neq \emptyset$, then either $X \subseteq Y^+$ or $X \subseteq Y^-$. Consider

$$\mathcal{X}_1 = \{Y : X \subseteq Y^+ \text{ and there exists } Q \in \mathcal{Q} \text{ such that } Y \text{ is a connected component of } Q\};$$
$$\mathcal{X}_2 = \{Y : X \subseteq Y^- \text{ and there exists } Q \in \mathcal{Q} \text{ such that } Y \text{ is a connected component of } Q\}.$$

Observe that there exist $Y_1 \in \mathcal{X}_1$ and $Y_2 \in \mathcal{X}_2$ such that $Z_1 \subseteq \mathrm{cl}\, Y_1^-$ and $Z_2 \subseteq \mathrm{cl}\, Y_2^+$ for all $Z_1 \in \mathcal{X}_1$ and $Z_2 \in \mathcal{X}_2$. Therefore, X is a cell of $(\overline{\mathbb{R}}, \mathbb{Z})$ and πX is a connected component of U. Therefore, $M \in \mathcal{U}(n+1)$. This ends the proof of (II_{n+1}). $\square$

3. Concluding remarks

3.1. In Theorem B, we can achieve differentiability up to any fixed order $p \in \mathbb{N}$. Let $S \subseteq \mathbb{R}^n$ and $\pi \in \Pi(n, d)$ where $d \leq n$. We say that S is a π-special C^p-submanifold if S is definable and, for every $y \in \pi S$ there is a box B about y such that $\pi \restriction S \cap \pi^{-1} B$ is a C^p-diffeomorphism from $S \cap \pi^{-1} B$ onto B and other definitions are defined in the obvious way. Then Theorem B holds when we replace the word "special submanifolds" by "special C^p-submanifolds".

3.2. In the proof of Theorem B, the only use of the definability of multiplication was to obtain a definable homeomorphism $\tau\colon \mathbb{R} \to (-1, 1)$, and this was used only to reduce to dealing with bounded sets. Hence, if $\mathfrak{R}$ is a d-minimal expansion of $(\mathbb{R}; <)$, then the conclusion of Theorem B holds if $\mathfrak{R}$ defines a bijection from a bounded interval to $\mathbb{R}$. (Theorem 3.4.1 of [11] was stated for d-minimal expansions of $(\mathbb{R}; <)$; it would appear that new ideas would be needed in order to repair its proof in this generality.)

3.3. For contrast, Theorem A fails without multiplication even in the o-minimal case (see [3]).

3.4. Consider $T\colon \mathbb{R} \rightrightarrows \mathbb{R}$ such that each $T(x)$ is the closed interval $[x, x+1]$. The identity map is obviously a continuous selection for T. However, Michael's construction can produce a selection that differs from the identity map by a function that has infinite oscillation.

We end this paper with some model-theoretical remarks.

3.5. Theorem A is independent of parameters. That is, we can replace all the occurrences of the word "definable" by "$\emptyset$-definable" in the definitions and the statements of both theorems. In particular, *for every set $E \subseteq \mathbb{R}^n$, if T is a lower semi-continuous set-valued map from E to $\mathbb{R}^m$ and $(\mathbb{R}; +, \cdot, T)$ is d-minimal, then there is a continuous function $f\colon E \to \mathbb{R}^m$ $\emptyset$-definable in $(\mathbb{R}; +, \cdot, T)$ such that $f(x) \in T(x)$ for every $x \in E$.* Rather than tracking the parameters, see [14, 2.5] for an easier approach.

3.6. We may ask whether Theorem A holds for d-minimal expansions of arbitrary ordered fields. From the above proofs, we see that this would hold so long as the structure satisfies Theorem B. But our proof of Theorem B uses intensively connectedness, simple connectedness and the path lifting property on subsets of $\mathbb{R}^n$, which are first-order condition. Therefore, we still do not know whether an appropriate analog of Theorem B holds in the more abstract setting.

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