This page corrects details of a proof from §8.2 of Burns, Climenhaga, Fisher, Thompson, "Unique equilibrium states for geodesic flows in nonpositive curvature", GAFA 28 (2018), 1209–1259. The only changes are in the blue text following (8.3). In particular, the statement of Theorem 8.1 is unchanged. The rest of §8.2 is included for context.

8.2. **Replacing singular orbit segments with regular ones.** Fix $\eta_0 > 0$ small enough that $\mathrm{Reg}(\eta_0)$ has nonempty interior. By Lemma 2.12, there exists $R > 0$ such that for every $v \in T^1M$ we have both $W^s_R(v) \cap \mathrm{Reg}(\eta_0) \neq \emptyset$ and $W^u_R(v) \cap \mathrm{Reg}(\eta_0) \neq \emptyset$. In particular, we can define maps $\Pi^s, \Pi^u \colon T^1M \to \mathrm{Reg}(\eta_0)$ such that $\Pi^\sigma(v) \in W^\sigma_R(v)$ for every $v \in T^1M$ and $\sigma = s, u$. Given $t > 0$, we use these to define a map $\Pi_t \colon \mathrm{Sing} \to \mathrm{Reg}$ by

$$\Pi_t = f_{-t} \circ \Pi^u \circ f_t \circ \Pi^s. \tag{8.2}$$

That is, given $v \in \mathrm{Sing}$ we choose $v' = \Pi^s(v) \in W^s_R(v)$ with $\lambda(v') \geq \eta_0$, and $w = f_{-t}(\Pi^u(f_t v'))$ such that $f_t w \in W^u_R(f_t v')$ and $\lambda(f_t w) \geq \eta_0$, as shown in Figure 3.

Theorem 8.1. *For every $\delta > 0$ and $\eta \in (0, \eta_0)$, there exists $L > 0$ such that for every $v \in \mathrm{Sing}$ and $t \geq 2L$, the image $w = \Pi_t(v)$ has the following properties:*

1. *$w, f_t(w) \in \mathrm{Reg}(\eta)$;*
2. *$d_{\mathrm{K}}(f_s(w), \mathrm{Sing}) < \delta$ for all $s \in [L, t-L]$;*
3. *for every $s \in [L, t-L]$, $f_s(w)$ and v lie in the same connected component of $B(\mathrm{Sing}, \delta) := \{w \in T^1M : d_{\mathrm{K}}(w, \mathrm{Sing}) < \delta)\}$.*

We emphasize that Theorem 8.1 does *not* allow us to conclude that $f_s(w)$ is close to $f_s(v)$; all we know is that $f_s(w)$ is close to *some* singular vector for $s \in [L, t-L]$. For example, if $f_s(v)$ is in the middle of a flat strip on a surface, then $f_s(w)$ will be close to the edge of the flat strip for $t \in [L, t-L]$.

Proof of Theorem 8.1. Let δ, η, η_0 be as in the statement of the theorem. For property (1), it is immediate from the definition of Π_t that $\lambda(f_t w) \geq \eta$. By uniform continuity of λ, we can take ϵ_0 sufficiently small such that if $v_2 \in W^u_{\epsilon_0}(v_1)$ and $\lambda(v_1) \geq \eta_0$, then $\lambda(v_2) \geq \eta$. By Corollary 3.14, ,there exists $T_0 > 0$ such that if $t \geq T_0$ and $f_t(w) \in W^u_R(f_t v')$, then $w \in W^u_{\epsilon_0}(v')$. Thus, if $\lambda(v') \geq \eta_0$, then $\lambda(w) \geq \eta$. Thus, item (1) of the theorem holds for any $t \geq T_0$.

We turn our attention to item (2). By Proposition 3.4, there are $\eta', T_1 > 0$ such that

$$\text{if } \lambda^u(f_s v) \leq \eta' \text{ for all } |s| \leq T_1, \text{ then } d_{\mathrm{K}}(v, \mathrm{Sing}) < \delta. \tag{8.3}$$

Given $v \in \mathrm{Sing}$, we have $\Pi^s(v) = v' \in W^s_R(v)$, and $\lambda(f_s v) = 0$ for all s.

By continuity of λ^u, we can take ϵ_1 sufficiently small such that if $v_2 \in W^s_{\epsilon_1}(v_1)$, then $|\lambda^u(v_1) - \lambda^u(v_2)| < \eta'/2$. Applying Proposition 3.13 to the compact set $\{v : \lambda^u(v) \geq \eta'/2\} \subset \mathrm{Reg}$ gives $T_2 > 0$ such that if $\lambda^u(v_1) \geq \eta'/2$ and $\tau \geq T_2$, then $f_{-\tau} W^s_{\epsilon_1}(v_1) \supset W^s_R(f_{-\tau} v_1)$ and $f_\tau W^u_{\epsilon_1}(v_1) \supset W^u_R(f_\tau v_1)$.

Suppose for a contradiction that $\lambda^u(f_s v') \geq \eta'/2$ for some $s \geq T_2$. Applying the previous paragraph with $v_1 = f_s v'$ gives $f_s v \in f_s W^u_R(f_s v') \subset W^s_{\epsilon_1}(f_s v')$. By our choice of ϵ_1, this gives $\lambda^u(f_s v) > 0$, contradicting the fact that $v \in \mathrm{Sing}$, and we conclude that $\lambda^u(f_s v') < \eta'/2$ for $s \geq T_2$.

Similarly, if there is $s \in [T_2, t - T_2]$ such that $\lambda^u(f_s w) \geq \eta'$, then the same argument with $v_1 = f_s w$ and $\tau = t - s$ gives $f_s v' \in f_{-(t-s)} W^u_R(f_t w) \subset W^u_{\epsilon_1}(f_s w)$, and our choice of ϵ_1 gives $\lambda^u(f_s v') \geq \lambda^u(f_s w) - \eta'/2 \geq \eta'/2$, a contradiction since $\lambda^u(f_s v') < \eta'/2$ for all $s \geq T_2$. Thus $\lambda^u(f_s w) < \eta'$ for all $s \in [T_2, t - T_2]$.

Applying (8.3) gives $d_{\mathrm{K}}(f_s w, \mathrm{Sing}) < \delta$ for all $s \in [T_2 + T_1, t - T_2 - T_1]$. Thus, taking $L = \max(T_0, T_1 + T_2)$, assertions (1) and (2) follow for $s \geq 2L$.

For item (3) of the theorem, we observe that v and w can be connected by a path $u(r)$ that follows first $W^s_R(v)$, then $f_{-t}(W^u_R(f_t v'))$ (see Figure 3), and that the arguments giving $d_{\mathrm{K}}(f_s w, \mathrm{Sing}) < \delta$ also give $d_{\mathrm{K}}(f_s u(r), \mathrm{Sing}) < \delta$ for every $s \in [L, t-L]$ and every r. We conclude that $f_s v$ and $f_s w$ lie in the same connected component of $B(\mathrm{Sing}, \delta)$ for every such s. □