Solution #3

5.4.5. (a) Suppose \( e^{At} = S e^{\lambda t} S^{-1} \). Then
\[
e^{A(t+t')} = S e^{\lambda(t+t')} S^{-1} = S e^{At} e^{A't'} S^{-1} = S e^{At} S^{-1} S e^{A't'} S^{-1} = e^{At} e^{A't'}
\]

(b) \( A^2 = 0 \), \( A^k = 0 \), \( k \geq 2 \). \( e^A = I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)
\( B^2 = 0 \), \( B^k = 0 \), \( k \geq 2 \). \( e^B = I + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \)
\( A+B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} T & DT' \end{bmatrix} \)
\( e^{A+B} = T e^{DT'} \neq e^A e^B \)

5.6.30. (a) \( J^k = \begin{bmatrix} 2^k & 2^{k+1} \\ 0 & 2^k \end{bmatrix} \) And \( J^{10} = \begin{bmatrix} 2^{10} & 10(2^9) \\ 0 & 2^{10} \end{bmatrix} \)

(b) \( A^{10} = M J^{10} M^{-1} = 2^{10} \begin{bmatrix} 61 & 45 \\ -80 & -89 \end{bmatrix} \)

(c) \( e^J = \begin{bmatrix} e^1 & e^2 \\ 0 & e^2 \end{bmatrix} \) \( e^A = M e^J M^{-1} = e^2 \begin{bmatrix} 13 & 9 \\ -16 & -11 \end{bmatrix} \)

5.13 (a) \( \frac{dx(t)}{dt} = \frac{d}{dt} \left( e^{At} x(0) + e^{Bt} B x(0) \right) = A e^{At} x(0) e^{Bt} + e^{At} x(0) B e^{Bt} \)

Notice \( B e^{Bt} = e^{Bt} B \) from the definition of \( e^{Bt} \).
\[
\frac{dx(t)}{dt} = A (e^{At} x(0) e^{Bt}) + (e^{At} x(0) e^{Bt}) B
\]
\[= A x(t) + x(t) B
\]
So \( x(t) \) is a solution.
From (a), \( x(t) = e^{At} x(0) e^{-At} \) is the solution.

\[
\det (X(t) - \lambda I) = \det (e^{At} x(0) e^{-At} - \lambda I)
= \det (e^{At}) \det (x(0) - \lambda I) \det (e^{-At})
= \det (x(0) - \lambda I)
\]

So the eigenvalues are always the same.

5.20. (a) \( \det (k - \lambda I) = 0 \)

\[\Rightarrow \lambda = \overline{a} + ia, \ a \in \mathbb{R}.\]

So for \( \det ((k - I) - \lambda I) = 0 \)

\[\lambda = -1 + 2ia \neq 0, \text{ so it's invertible.}\]

(b) Let \( \{v_1, v_2, \ldots, v_n\} \) be the eigenvectors,

let \( U = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \)

Then \( k = UDU^H \) where \( D \) is a diagonal matrix.

And \( U \) is unitary since \( \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \) form an orthonormal basis.

(c) \( e^{At} e^{Bt} = e^{(A+B)t} = e^{At} e^{-Bt} = I \)

Eigenvalues of \( k \) is imaginary.

(d) \( e^{kt}, (e^{kt})^H = e^{kt} e^{-kt} = e^{kt} e^{-kt} = I \)

where, \( e^{-kt} = \left( \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} \right)^H = \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} = e^{kt} \).

\( k^H = -k \) because \( k \) is.
Q. (a) \[ A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \ \lambda_1 = 1; \ (1) \], \ [ B = \begin{bmatrix} 3 & -8 \\ 0 & -1 \end{bmatrix}, \ \lambda_1 = 3; \ (1) \]
\[ \lambda_2 = 2; \ (2) \]
\[ \lambda_2 = -1; \ (2) \]

Choose \[ T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]. \[ D_A = T^T A T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \]. \[ D_B = T^T B T = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \]

Q. (b) \[ A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ \lambda_1 = 0; \ (1) \]
\[ \lambda_2 = 2; \ (1) \]

Let \[ T = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \]