April 24, 2013

Math 3345

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Spring Semester 2013

Final Exam

Name: ....

Class time: ....

Instructions: Use the blank paper given to respond to exactly two problems in each of the four sections on the test form, for a total of eight problems. On each sheet that you turn in, make sure to include your name, the number of the problem you are doing, and a page number if you require multiple pages for a given problem. Each problem must be on a separate sheet; turn in each sheet to its appropriate pile. For each problem that involves writing a proof, you are to explicitly formulate a claim and then give a proof. Please read each problem carefully before attempting it. No writing will be graded if you indicate not to do so. Each of the problems is given equal weight in grading. Attempts at more than two problems in a section will result in a random selection of problems being graded.

Grading and Hints: The test will be scored on a 200 point scale, with each completed problem contributing up to 25 points toward this score. If you wish to request a hint or a definition, you may do so by agreeing to a reduction in your score. Each hint or definition received will result in a reduction of between 5 and 15 points for the total score of the section of the problem for which the hint is given. You will be informed of the reduction before the hint or definition is given. This reduction cannot reduce your score for a section to below 0 points.

Rules: You may not use any outside materials or aids during this test except for writing implements or erasers. In particular, you may not use any notes, books, or electronic devices, including calculators, cellular phones, smart watches, and AR glasses, during the test period, and you may not look at the work of others. You may not cause any major distractions during the test, and time travel during the test is not permitted. If possible, you must not sit next to other test-takers while taking this test.

I swear that I understand the instructions and rules stated above, and I will follow them. I understand that violation of any of the above rules is considered academic misconduct and is grounds for having my test score not counted toward my course grade.

Signature: .... Date: ....

Check this box if you plan to graduate at the end of this semester. ☐

A. Formal Logic and Basic Proof Techniques

1. **Prove or Disprove.** Let $P(x)$ and $Q(x)$ be unary predicate symbols. Then we have the following logical equivalence:

$$\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists y Q(y)$$

2. Let P and Q be propositional forms, neither of which are tautologies.
   (a) **Prove or Disprove.** The propositional form $P \to Q$ must not be tautology.
   (b) **Prove or Disprove.** The propositional form $P \wedge Q$ must not be tautology.

3. **Prove or Disprove.** Let a and b be positive integers and suppose that, for every positive integer c, that $a \equiv b \mod c$. Then $a = b$.

B. Mathematical Induction

4. **Prove by induction.** Let $n \in \mathbb{N}$ be such that $n > 1$. Then there is a prime number $p \in \mathbb{N}$ such that p divides n.

5. **Prove by induction.** Let $d, n \in \mathbb{N}$ such that $d \neq 0$. Then then are numbers $r, k \in \mathbb{N}$ such that $0 \leq r < d$ and $n = dk + r$.

6. **Prove by induction.** For any positive integer n,

$$\frac{2 + 4 + 6 + \ldots + 2n}{1 + 3 + 5 + \ldots + (2n - 1)} = \frac{n + 1}{n}.$$

[Hint: Determine formulas for the numerator and denominator separately.]

C. Abstract Sets and Functions

7. **Prove or Disprove.** Let A, B, and C be nonempty sets and let $f : A \to B$ and $g : B \to C$. Let $h : A \to C$ be the composition $h = g \circ f$. Suppose that h is invertible. Then both f and g are invertible and $h^{-1} = f^{-1} \circ g^{-1}$.

8. **Prove or Disprove.** For any sets A, B, and C, we have that $A \setminus (B \setminus C)) = (A \setminus B) \cup (A \cap C)$.

9. **Prove or Disprove.** Let A, B, C, and D be sets such that A and B are disjoint, and let $f : A \to C$ and $g : B \to D$ both be surjections onto their respective targets. Let $X = A \cup B$ and let $Y = C \cup D$. Then the rule

$$h(x) := \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}$$

defines a function $h : X \to Y$, and this function h is a surjection from X to Y.

10. **Prove.** Let A and B be sets and let $\langle B_\alpha \rangle_{\alpha \in A}$ be a nonempty indexed family of nonempty subsets of B. For each $\alpha \in A$ let $C_\alpha = \{(\alpha, \beta) : \beta \in B_\alpha\}$. Then

    (a) $\langle C_\alpha \rangle_{\alpha \in A}$ an indexed family of subsets of $A \times B$.

    (b) The family $\langle C_\alpha \rangle_{\alpha \in A}$ is pairwise disjoint.

    (c) There is a function $f : A \times B \to B$ such that for any choice function $g : A \to A \times B$ for the family $\langle C_\alpha \rangle_{\alpha \in A}$, the function $f \circ g : A \to B$ is a choice function for the family $\langle B_\alpha \rangle_{\alpha \in A}$.

D. Principles of Counting

11. **Prove or Disprove.** For any set A, its power set $\mathcal{P}(A)$ is equinumerous to the set $\{0, 1\}^A$.

12. **Prove or Disprove.** Let A and B be sets and suppose that $a \in A$ and $b \in B$. Then $A \setminus \{a\}$ is equinumerous to $B \setminus \{b\}$.

13. **Prove or Disprove.** Let A and B be finite sets. Then $\overline{\overline{A \cup B}} + \overline{\overline{A \cap B}} = \overline{\overline{A}} + \overline{\overline{B}}$.

14. **Prove or Disprove.** Let S be an infinite set. Then there is a set $A \subseteq S$ such that A is equinumerous to $\mathbb{N}$.