# Solutions for Homework 4, Math 3345

Zhuang He

September 20, 2013

9(f) In the sentence $P(x, y)$, both $x$ and $y$ are free variables.

In $(\forall y)P(x, y)$, only $x$ is free.

In $(\exists x)(\forall y)P(x, y)$, neither $x$ nor $y$ is free.

In $(\exists x)P(x, y)$, only $y$ is free.

In $(\forall y)(\exists x)P(x, y)$, neither $x$ nor $y$ is free.

10(e) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 1)$ means "there exists $y$ a real number, such that for every real number $x$, the product $xy$ equals to 1".

Claim:$(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 1)$ is false.

Proof: It is sufficient to show for every real number $y$, the sentence $(\forall x \in \mathbb{R})(xy = 1)$ is false. If we fix $y$ a real number, then there is a real number $x_0 = 0$ such that $x_0 y = 0 \cdot y = 0 \neq 1$. Therefore $x_0 = 0$ is a counterexample for the sentence $(\forall x \in \mathbb{R})(xy = 1)$, and hence $(\forall x \in \mathbb{R})(xy = 1)$ is false. Now since $y$ is an arbitrary element of $\mathbb{R}$, we proved the claim.

10(f) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)$ means "for all real number $x$, there exists $y$ a real number, such that the product $xy$ equals to 1".

Claim:$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)$ is false.

Proof: It is sufficient to exhibit a value of $x$ such that $(\exists y \in \mathbb{R})(xy = 1)$ is false. In fact, this value can be chosen as $x = 0$. For $x = 0$ fixed, for every $y$ a real number, we have $x \cdot y = 0 \cdot y = 0 \neq 1$. Therefore for $x = 0$ the sentence $(\exists y \in \mathbb{R})(xy = 1)$ is false. This proves our claim.

11(a) Let $S$ be the set of real numbers. Then $S$ is not bounded above.

Proof: $S$ is bounded above if and only if $(\exists b \in \mathbb{R})(\forall x \in S)(x < b)$. Now we prove it is false. Then we need to prove for every $b \in \mathbb{R}$, $(\forall x \in S)(x < b)$ is false. In fact, if $b$ is fixed, then $b + 1$ is a real number such that $b + 1 < b$ is false. Thus $(\forall x \in S)(x < b)$ is false. And we proved the claim that $S$ is not bounded above.

11(b) Let $S$ be the set of all number $x$ such that some person on earth has $x$ hairs on his or her head. Then $S$ is bounded above.

Proof: $S$ is a finite set since there are only finite people on earth. Therefore, $S$ always have a maximal element. Let us assume the maximal element is $n$. Then for all $x \in S$, $x \leq n$. Therefore $(\exists b \in \mathbb{R})(\forall x \in S)(x < b)$ is true since $b = n$ is an example. Therefore $S$ is bounded above.

13 Proof: We will denote $\mathbb{R}_+$ as the set of all positive real numbers. Then $f$ is continuous at $a$ iff $(\forall \epsilon \in \mathbb{R}_+)(\exists \delta \in \mathbb{R}_+)(\forall x \in \mathbb{R})(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)$.

Therefore $f$ is not continuous at $a$

iff $\neg(\forall \epsilon \in \mathbb{R}_+)(\exists \delta \in \mathbb{R}_+)(\forall x \in \mathbb{R})(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)$

iff $(\exists \epsilon \in \mathbb{R}_+)\neg(\exists \delta \in \mathbb{R}_+)(\forall x \in \mathbb{R})(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)$ (by De Morgan's Law)

iff $(\exists \epsilon \in \mathbb{R}_+)(\forall \delta \in \mathbb{R}_+)\neg(\forall x \in \mathbb{R})(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)$ (by De Morgan's Law)

iff $(\exists \epsilon \in \mathbb{R}_+)(\forall \delta \in \mathbb{R}_+)(\exists x \in \mathbb{R})\neg(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)$ (by De Morgan's Law)

iff $(\exists \epsilon \in \mathbb{R}_+)(\forall \delta \in \mathbb{R}_+)(\exists x \in \mathbb{R})(|x-a| < \delta \wedge \neg(|f(x) - f(a)| < \epsilon))$
(by negation of conditional sentences)

iff $(\exists \epsilon \in \mathbb{R}_+)(\forall \delta \in \mathbb{R}_+)(\exists x \in \mathbb{R})(|x-a| < \delta \wedge |f(x) - f(a)| \geq \epsilon)$.

14(a) $(\exists! x \in \mathbb{R})(2x+7=3)$ means "there exists a unique real number $x$ such that $2x+7$ is equal to 3".

This sentence is true. By solving the equation $2x+7=3$, we find a unique root $x = \frac{3-7}{2} = -2$. Therefore, for $x=-2$, $2 \times (-2) + 7 = 3$, which means there exists a value of $x$ such that $2x+7=3$ is true, and this value is unique for $2x+7=3$ to be true. Therefore the sentence $(\exists! x \in \mathbb{R})(2x+7=3)$ is true.

14(b) $(\exists! x \in \mathbb{R})(x^2 - 4x + 3 < 0)$ means "there exists a unique real number $x$ such that $x^2 - 4x + 3$ is less than 0".

This sentence is false. Let $x = 2$, we see $2^2 - 4 \times 2 + 3 = -1 < 0$. Let $x = 2.5$, we see $2.5^2 - 4 \times 2.5 + 3 = -0.75 < 0$. Therefore the values of $x$ such that $x^2 - 4x + 3$ is less than 0 is not unique. Thus $(\exists! x \in \mathbb{R})(x^2 - 4x + 3 < 0)$ is false.

14(c) $(\exists! x \in \mathbb{Z})(x^2 - 4x + 3 < 0)$ means "there exists a unique integer $x$ such that $x^2 - 4x + 3$ is less than 0".

This sentence is true. If we factor $x^2 - 4x + 3$ on $\mathbb{R}$ we get $x^2 - 4x + 3 = (x-1)(x-3)$. Therefore, $x^2 - 4x + 3 < 0$ if and only if $(x-1)(x-3) < 0$. Note that the product of two real number is negative if and only if one of them is positive and the another is positive. Thus if $(x-1)(x-3) < 0$, then $x - 1 < 0$ and $x - 3 > 0$, or $x - 1 < 0$ and $x - 3 > 0$. That is, $x < 1$ and $x > 3$ or $x > 1$ and $x < 3$. The first case is impossible since $x < 1$ and $x > 3$ are contradictory. Therefore if $(x-1)(x-3) < 0$ is true, then $1 < x < 3$. Now we find the only integer $n$ such that $1 < n < 3$ is $n = 2$, and when $n = 2$, $2^2 - 4 \times 2 + 3 = -1 < 0$. Therefore $n = 2$ is the unique integer that makes $(x-1)(x-3) < 0$ true. Thus $(\exists! x \in \mathbb{Z})(x^2 - 4x + 3 < 0)$ is true.

14(d) $(\exists! x \in \mathbb{R})(x^2 - 4x + 4 = 0)$ means "there exists a unique real number $x$ such that $x^2 - 4x + 4$ equals to 0".

This sentence is true. By completing the square, we see $x^2 - 4x + 4 = (x-2)^2$. Therefore $x^2 - 4x + 4 = 0$ iff $(x-2)^2 = 0$ iff $x = 2$. Therefore $x = 2$ is the unique real number such that $x^2 - 4x + 4 = 0$. Thus $(\exists! x \in \mathbb{R})(x^2 - 4x + 4 = 0)$ is true.

14(e) $(\exists! x \in \mathbb{R})(x^2 - 4x + 5 = 0)$ means "there exists a unique real number $x$ such that $x^2 - 4x + 5$ equals to 0".

This sentence is false. By completing the square, we see $x^2-4x+5=(x-2)^2+1$. Since $(x-2)^2\geq 0$ is true for all $x\in\mathbb{R}$, we find $(x-2)^2+1\geq 0+1=1>0$ for all $x\in\mathbb{R}$. Thus no real numbers $x$ satisfies $x^2-4x+5=0$. Thus $(\exists! x\in\mathbb{R})(x^2-4x+5=0)$ is false.

14(f) $(\forall x\in\mathbb{R})(\exists! y\in\mathbb{R})(x+y=0)$ mean "for all real number $x$, there exists a unique real number $y$, such that $x+y$ is equal to 0".

This sentence is true. We will prove for all real number $x$, $(\exists! y\in\mathbb{R})(x+y=0)$ is true. Because $x$ is fixed, $y=-x$ is a real number that satisfies $x+y=0$, and hence an example for $(\exists! y\in\mathbb{R})(x+y=0)$. Also, if $x+y=0$, then $y=-x$. Therefore $y=-x$ is the only real number satisfies $x+y=0$. Thus for the $x$ we choose, $(\exists! y\in\mathbb{R})(x+y=0)$ is true. Since $x$ is an arbitrary real number, $(\forall x\in\mathbb{R})(\exists! y\in\mathbb{R})(x+y=0)$ is true.

14(g) $(\forall x\in\mathbb{R})(\exists! y\in\mathbb{R})(xy=1)$ mean "for all real number $x$, there exists a unique real number $y$, such that $xy$ is equal to 1".

This sentence is false. To prove that it is false, it is sufficient to exhibit a value of $x$ such that $(\exists! y\in\mathbb{R})(xy=1)$ is false. We choose $x=0$. For $x=0$, every real number $y$ makes $xy=0\times y=0\neq 1$. Therefore, $x=0$ is a counterexample for $(\forall x\in\mathbb{R})(\exists! y\in\mathbb{R})(xy=1)$, which is then false.

14(h) $(\forall x\in\mathbb{R})[\text{if } x\neq 0, \text{then } (\exists! y\in\mathbb{R})(xy=1)]$ means "for all real number $x$, if $x$ is not zero, there exists a unique real number $y$, such that $xy$ is equal to 1".

This sentence is true. We will prove for all real number $x$, $(x\neq 0)\Rightarrow(\exists! y\in\mathbb{R})(xy=1)$ is true.

By conditional proof, we need only assume $x\neq 0$, and then prove that $(\exists! y\in\mathbb{R})(xy=1)$ is true. Now $x\neq 0$, therefore $y=\frac{1}{x}$ exists(Note: if $x=0$, then $\frac{1}{x}$ does not exist). And $y=\frac{1}{x}$ is the unique real number such that $xy=1$. Thus $(x\neq 0)\Rightarrow(\exists! y\in\mathbb{R})(xy=1)$ is true. Since $x$ is an arbitrary real number, $(\forall x\in\mathbb{R})[\text{if } x\neq 0, \text{then } (\exists! y\in\mathbb{R})(xy=1)]$ is true.

14(i) $(\forall x\in\mathbb{R})(\exists! y\in\mathbb{R})(xy=0)$ mean "for all real number $x$, there exists a unique real number $y$, such that $xy$ is equal to 0".

This sentence is false. To prove that it is false, it is sufficient to exhibit a value of $x$ such that $(\exists! y\in\mathbb{R})(xy=0)$ is false. We choose $x=0$. For $x=0$, every real number $y$ makes $xy=0\times y=0$. That is, for $x=0$, such $y$ that satisfies $xy=0$ is not unique. Therefore, $x=0$ is a counterexample for $(\forall x\in\mathbb{R})(\exists! y\in\mathbb{R})(xy=0)$, which is then false.

14(j) $(\forall x\in\mathbb{R})[\text{if } x\neq 0, \text{then } (\exists! y\in\mathbb{R})(xy=0)]$ means "for all real number $x$, if $x$ is not zero, there exists a unique real number $y$, such that $xy$ is equal to 0".

This sentence is true. We will prove for all real number $x$, $(x\neq 0)\Rightarrow(\exists! y\in\mathbb{R})(xy=0)$ is true.

By conditional proof, we need only assume $x\neq 0$, and then prove that $(\exists! y\in\mathbb{R})(xy=0)$ is true. Recall that for any two real number $a,b$, $ab=0$ iff $a=0$ or $b=0$. Now $x\neq 0$, and $y=0$ satisfies $xy=x\times 0=0$. For other $y\neq 0$, we find $xy\neq 0$. Thus $y=0$ is the unique $y$ such that $xy=0$ true. Then $(x\neq 0)\Rightarrow(\exists! y\in\mathbb{R})(xy=0)$ is true. Since $x$ is an arbitrary real number, $(\forall x\in\mathbb{R})[\text{if } x\neq 0, \text{then } (\exists! y\in\mathbb{R})(xy=0)]$ is true.