EXTRA CREDIT PROJECTS

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The following extra credit projects are to be turned in by the end of the day of the last day before finals start. Each successfully completed extra credit project is worth 5 points, added to your midterm grade. Some partial credit may be given for incomplete solutions, but it will not be much. You are encouraged to attend office hours to discuss your solutions with me before the due date so that you can make corrections. Each of the numbered problems is considered part of the project. These projects require you to have far-reaching knowledge of the topics covered in this class and as such each assignment may require a good deal of time to complete.

Note. As this assignment sheet is a work in progress, it may contain errors; please contact me if you think that you have found any.

1. Propositional Logic

In this project you will prove that any truth table is the truth table of some propositional form built up from propositional variables using only the connectives $\wedge$, $\vee$, and $\neg$. To do this, we will need to go into the theory of propositional logic at least far enough to give rigorous definitions of propositional forms and truth tables. For the problem, we fix countably many distinct propositional variables $A_0, A_1, A_2, \ldots$ and for each $n \in \mathbb{N}$ we define the set $\mathcal{A}_n$ of the first n atoms by $\mathcal{A}_n = \{A_0, A_1, \ldots A_{n-1}\}$.

We define the set $\mathcal{P}_n$ of propositional forms involving the first n propositional variables $\mathcal{A}_n$ recursively as follows. We say that an object $P \in \mathcal{P}_n$ if and only if one of the following holds:

- P is one of the propositional variables $A_0, A_1, \ldots A_{n-1}$.
- There is a propositional form $Q \in \mathcal{P}_n$ such that P is $\neg Q$.
- There are propositional forms $Q \in \mathcal{P}_n$ and $R \in \mathcal{P}_n$ such that P is $Q \vee R$.
- There are propositional forms $Q \in \mathcal{P}_n$ and $R \in \mathcal{P}_n$ such that P is $Q \wedge R$.

For each $n \in \mathbb{N}$, we say that a truth assignment is a function $v : \mathcal{P}_n \to \{T, F\}$ such that for any propositional form $P \in \mathcal{P}_n$,

- If there is a propositional form $Q \in \mathcal{P}_n$ such that P is $\neg P$, then $v(P) = T$ if and only if $v(Q) = F$.
- If there are propositional forms $Q \in \mathcal{P}_n$ and $R \in \mathcal{P}_n$ such that P is $Q \vee R$, then $v(P) = T$ if and only if either $v(Q) = T$ or $v(R) = T$.
- If there are propositional forms $Q \in \mathcal{P}_n$ and $R \in \mathcal{P}_n$ such that P is $Q \wedge R$, then $v(P) = T$ if and only if both $v(Q) = T$ and $v(R) = T$.

1.1. Show that a truth assignment is completely determined by what it does to propositional variables. That is, show that if v and u are truth assignments such that $v(A_k) = u(A_k)$ for every $k \in \{0, 1, \ldots, n-1\}$, then $v = u$.

Now let $\mathcal{V}_n$ be the set of all truth assignments on $\mathcal{P}_n$. Let $\mathcal{F}_n$ be the set of all functions from $\mathcal{V}_n$ to $\{T, F\}$. We call $\mathcal{F}_n$ the set of all truth tables. Each truth assignment corresponds to a specific row on a truth table, and a function in $\mathcal{F}_n$ corresponds to a column in which each row contains either a T or an F. For each $P \in \mathcal{P}_n$, we define the function $f_P : \mathcal{V}_n \to \{T, F\}$ by $f_P(v) = v(P)$, and let $\mathcal{E}_n$ be the set of all such f_P. We call the set $\mathcal{E}_n$ the set of all truth tables that come from propositional forms. Each function in $\mathcal{E}_n$ corresponds to a column of a truth table that is calculated from a specific propositional form, and the truth assignments to the propositional variables are listed in the far left columns as usual for truth tables.

1.2. Verify that each truth table that comes from a propositional form is an official truth table. That is, show that for each $n \in \mathbb{N}$ that $\mathcal{E}_n \subseteq \mathcal{F}_n$.

The set $\mathcal{E}_n$ here allows us to think about propositional forms and their relationship with truth assignments in a different way. Rather than thinking of the propositional form as the object and the truth value assignment as the function, we can think of the proposition as the function and the truth value assignment as the object. Again, when one draws a table of values for this function, it has the form of a truth table as we discussed in the first weeks of class. A question arises here: is it possible that there are functions in $\mathcal{F}_n$ that don't arise from propositional forms? In other words, are there truth tables that don't come from propositional forms? In what follows, you will prove that in fact $\mathcal{E}_n = \mathcal{F}_n$, which means that all possible truth tables not only come from propositional forms, but that we need at most the three connectives we started with to get all possible truth tables.

1.3. There is a propositional form $P \in \mathcal{P}_n$ such that for all truth assignments $v \in \mathcal{V}_n$, $v(P) = F$.

1.4. Let $v \in \mathcal{V}_n$ be a truth value assignment such that there is a $m \in \{0, 1, \ldots, n-1\}$ such that for all $k \in \{0, 1, \ldots, n-1\}, v(A_k) = T$ if and only if $k = m$. Then there is a propositional form $P \in \mathcal{P}_n$ such that for all truth assignments $u \in \mathcal{P}_n$, $u(P) = T$ if and only if $u = v$. [Hint: Do a proof by induction on n. For the induction step, take an appropriate propositional form $Q \in \mathcal{P}_{n-1}$ and define P to be either $Q \wedge A_n$ or $Q \wedge (\neg A_n)$.]

1.5. Let $V \subseteq \mathcal{V}_n$ be a set of truth value assignments. Then there is a propositional form $P \in \mathcal{P}_n$ such that for all truth assignments $v \in \mathcal{V}_n$, $v(P) = T$ if and only if $v \in V$. [Hint: Prove this by induction on the number of elements of V, forming your base case from 1.3 and 1.4.]

1.6. For each truth table $f \in \mathcal{F}_n$, there is a propositional form $P \in \mathcal{P}_n$ such that $f = f_P$. Thus $\mathcal{E}_n = \mathcal{F}_n$. [Hint: Let V be the set of all truth value assignments v such that $f(v) = T$, and then apply 1.5 to the set V. Use 1.2 to conclude that $\mathcal{E}_n = \mathcal{F}_n$.]

We now know that $\mathcal{E}_n = \mathcal{F}_n$ for any natural number n. A natural question to ask at this point is this: what about infinite truth tables? It turns out that not all infinite truth tables are given by finite propositional forms. To prove this we need a few more definitions. Let $\mathcal{P} = \cup_{n=0}^{\infty} \mathcal{P}_n$ and let $\mathcal{V}$ be the set of all truth assignments on $\mathcal{P}$. Let $\mathcal{F}$ be the set of all functions from $\mathcal{V}$ to $\{T, F\}$. For each $P \in \mathcal{P}$, define $f_P \in \mathcal{F}$ by $f_P(v) = v(P)$.

1.7. Let $f \in \mathcal{F}$ be the infinite truth table defined by

$$f(v) = \begin{cases} T, & \text{if } v(A_n) = T \text{ for all } n \in \mathbb{N}, \\ F, & \text{otherwise,} \end{cases}$$

for each $v \in \mathcal{V}$. (*i.e.* the first row of the truth table f has a T in it, and all other rows have F.) Show that there is no propositional form $P \in \mathcal{P}$ such that $f = f_P$. [Hint: Suppose toward a contradiction that P is a propositional form such that $f = f_P$. Show by the definition of $\mathcal{P}$ that $P \in \mathcal{P}_n$ for some natural number n. But P is also an element of $\mathcal{P}_{n+1}$. Use this to show that for any truth assignment, changing only the truth value of A_{n+1} cannot affect the truth value of P. Show that this fact contradicts $f = f_P$ because there is a truth assignment such that changing its value at any propositional variable changes the value of f.]

2. The Ackermann Hierarchy

In this project you will analyze some of the behavior of the so-called Ackermann hierarchy of functions. There is a useful Wikipedia page on this function which you may refer to, though all proofs in this assignment must be your own. The m^{th} function in the Ackermann hierarchy is defined recursively as follows:

$$A_m(n) = \begin{cases} n+1, & \text{if } m = 0 \\ A_{m-1}(1), & \text{if } m > 0 \text{ and } n = 0 \\ A_{m-1}(A_m(n-1)), & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

Notice that this definition requires the use of double induction to make sense of it. In fact

2.1. Show that the function $A_m : \mathbb{N} \to \mathbb{N}$ is defined for each $n \in \mathbb{N}$. [Hint: do a double complete induction on n and m.]

To understand how the Ackermann functions work, it's a good idea to work formulas for the first few functions explicitly.

2.2. Prove that for each $n \in \mathbb{N}$:

(1) $A_0(n) = n + 1$

(2) $A_1(n) = n + 2$

(3) $A_2(n) = 2n + 3$

(4) $A_3(n) = 2^{n+3} - 3$

Beyond $m = 3$, the functions A_m increase so fast that they require new notation to make work with. We won't go into that notation (you can check out the Wikipedia page on Knuth's up-arrow notation if you are curious), but we will go over some basic properties of the Ackermann hierarchy.

2.3. Prove that each function A_m is increasing, and that for each $k, m, n \in \mathbb{N}$, if $k \leq m$ then $A_k(n) \leq A_m(n)$.

2.4. Define the function $A : \mathbb{N} \to \mathbb{N}$ by $A(n) = A_n(n)$. Show that A cannot be one of the functions in the Ackermann hierarchy by showing that for each $m \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that $A(N) > A_m(N)$.

2.5. Show that for any $m \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that for all $n > N$, we have that $A(n) > A_m(n)$.

The function A is an extremely fast growing function, and an extremely hard function to compute. This is especially true if you use the recursion given above to compute it, because the only operations used in the recursion are the addition and subtraction of the number 1. This leads us to this final result that may be of interest to anyone who wants to work with hard-to-compute functions:

2.6. Prove that for each $n \in \mathbb{N}$ that the computation of $A(n)$ requires more than $A(n)$ allowed operations, where an allowed operation is either the addition or subtraction of the number 1. [Hint: since there is no explicit recursion given for A, it may be helpful to look at the entire hierarchy of functions to prove this result.]

Now that you have worked with this double recursion, you can apply some of the ideas used in this problem to recursively define addition, multiplication, and exponentiation of integers in terms of adding and subtracting the number 1.

2.7. For each part of this problem, give the requested definition under the requested conditions. You need not prove that the definition you give is correct.

(1) Give a recursive definition for $n + m$ using only iteration and the addition and subtraction of the number 1.
(2) Give a recursive definition for $n \cdot m$ using only using only iteration, addition, and subtraction.
(3) Give a recursive definition for n^m using only using only iteration, addition, subtraction, and multiplication.

3. The Axiom of Choice

In this project you will prove that the axiom of choice is equivalent to the statement that all surjective functions have right inverses. Since we already did part of this proof in class, it only remains to show that the existence of right inverses for surjective functions implies that the axiom of choice is true. Given an indexed family of sets, we are going to try to set up a function so that any right inverse for it is either the required choice function, or something very much like it. The complication here is that we can't use the indexing function directly, so we have to make up a new indexing function to do the job by forcing it to carry around a bunch of extra information *in the index.* To do this, we'll make the set of indices a lot larger without changing what sets are indexed. At the end of the proof, we'll apply the right inverse property of surjective functions to this augmented indexing function to carve away most of that extra stuff, leaving behind just enough of it to establish a choice function for the family of sets we started with.

3.1. Let B be a set and let $\langle B_\alpha \rangle_{\alpha \in A}$ be an indexed family of nonempty subsets of B with index set A. Let $\mathscr{B} = \{B_\alpha : \alpha \in A\}$ be the range of this indexed family of sets. Let

$M \subseteq A \times B$ be given by $M = \{(\alpha, \beta) : \alpha \in A \text{ and } \beta \in B_\alpha\}$. For each $(\alpha, \beta) \in M$, let $B_{(\alpha,\beta)} = B_\alpha$. Show that $\langle B_m \rangle_{m \in M}$ is an indexed family of sets with index set M and that its range is $\mathscr{B}$.

3.2. Since the indexing function is surjective onto its range, it must have a right inverse $g : \mathscr{B} \to M$. Let $f : A \to B$ be the function that sends each $\alpha \in A$ to the $\beta \in B$ such that β is the second coordinate of $g(B_\alpha)$. Prove that the operation just described actually defines f as a function, and show that f is in fact a choice function for $\langle B_\alpha \rangle_{\alpha \in A}$.

It turns out that there are many useful equivalent statements of the axiom of choice. Some versions, such as Tychonoff's theorem and Zorn's lemma, are in common use in many areas of mathematics. As the theoretical build-up needed to make sense of these versions of the axiom of choice is outside the scope of this class, we'll finish with a version that only requires notions that we have previously discussed. Note that this version does not make explicit mention of functions, but it does rely on the fact that the axiom of choice is always true whenever every set in an indexed family is finite.

3.3. Show that the following statement is equivalent to the axiom of choice: For any nonempty set M and any nonempty pairwise disjoint collection $\mathscr{M}$ of nonempty subsets of M, there is a set $X \subseteq M$ such that for each set $A \in \mathscr{M}$ the intersection $A \cap X$ has exactly one element. [Hint: To go from the axiom of choice to this result, use $\mathscr{M}$ as your index set and let X be the image of your choice function. For the converse, you'll have to somehow turn your indexed family of sets into a pairwise disjoint collection of sets. It may be useful to think about the index set M defined in 3.1.]

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