Math 8440—Homework 13, due November 20

(1) Prove that geodesic flow is NOT ergodic over $G$. (We proved it was ergodic over $\Gamma \setminus G$.)

(2) For our connection between geodesics and continued fractions we used the coordinates $(y, \tilde{y}, \epsilon)$. Consider the map $\phi$ which acts by $\phi(y, \tilde{y}, \epsilon) = (y, 1/(y + \tilde{y}), \epsilon)$. (This is a bijection, although you do not need to prove this.) Prove that

$$\phi T \phi^{-1}(y, z, \epsilon) = \left( \left\{ \frac{1}{y} \right\}, y(1 - yz), -\epsilon \right)$$

Let $\tilde{T} = \phi T \phi^{-1}$. Show that $\tilde{T}$ preserves the measure $m$ which is given by Lebesgue measure in the first two coordinates and the normalized counting measure in the third.

(3) Use the previous problem to prove that the Gauss map preserves the measure $\mu(A) = \int_A \frac{dx}{(1 + x) \log 2}$.

(4) Find a $(z, \zeta) \in F$ such that the geodesic containing $(z, \zeta)$ is periodic in $\Gamma \setminus T^1 \mathbb{H}$.

(5) Classify all geodesics traveling through the modular surface $\Gamma \setminus T^1 \mathbb{H}$ such that if $(z, \zeta) \in F$ is on the geodesic, then $(-z, -\zeta)$ is also on the geodesic. Give your classification in terms of properties of the standard coordinates $(y, \tilde{y}, \epsilon)$.

(6) Consider $(z, \zeta) \in F$ and the forward geodesic ray from this point traveling through the modular surface seen as $F = \Gamma \setminus T^1 \mathbb{H}$. Prove that if $(z, \zeta)$ has standard coordinates $(y, \tilde{y}, \epsilon)$ with $y$ having bounded continued fraction digits, then there exists a constant $M$ such that each point $(z_t, \zeta_t)$ along the forward geodesic ray will have $\Im(z_t) \leq M$. 
