Math 8440—Homework 3, due September 11

(1) Suppose \( M_1, M_2 \in \text{SL}_2(\mathbb{Z}) \), \( x \in \mathbb{R} \setminus \mathbb{Q} \). Prove that \((M_1M_2)x = M_1(M_2x)\).

(Note: requiring \( x \) be irrational removes the case where we might see infinity.)

(2) In class we showed that, provided \( T^{n-1}x \neq 0 \), then
\[
x - \frac{p_n}{q_n} = \frac{(-1)^n T^n x}{q_n(q_{n-1}T^n x + q_n)}.
\]

Show that, for the continued fraction expansion, \( a_n(x) = \lfloor 1/T^n x \rfloor \), and then prove that
\[
\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_{n}q_{n+1}}.
\]

(3) Suppose that \( x = \langle a_0; a_1, a_2, \ldots \rangle \) has an eventually periodic continued fraction expansion—that is, there exist \( N \geq 0, k \geq 1 \), such that \( a_n = a_{n+k} \) for \( n \geq N \). Prove that \( x \) must be a quadratic irrational. (Hint: first prove it for purely periodic expansions in [0,1) and then extend.)

(4) Let \( (X, \mu, T) \) be a measure-preserving dynamical system on a probability space. Prove that \( T \) is ergodic if and only if, for every measurable \( A \subset X \) of positive measure, we have that
\[
\mu \left( \bigcup_{n=0}^{\infty} T^{-n} A \right) = 1
\]

(5) Let \( (X, \mu, T) \) be a measure-preserving dynamical system on a probability space. Prove that \( T \) is ergodic if and only if, for every measurable sets \( A, B \subset X \) of positive measure, there exists a non-negative integer \( n \) such that
\[
\mu \left( T^{-n} A \cap B \right) > 0.
\]

(You may use the characterization of ergodicity given in the previous problem.)

(6) Prove that the continued fraction map \( T \) preserves the Gauss measure
\[
\mu(A) = \int_A \frac{1}{(\log 2)(1 + x)} \, dx.
\]

(Note: this won’t be as immediately obvious as some others we’ve investigated. Be prepared to spend a little bit of time on it.)