Math 8440—Homework 5, due September 25

(1) Prove that a continued fraction expansion of a number is finite if and only if the number is rational. (Thus the set of points that land in \(X_\infty\) are the rational points.)

(2) Consider the continued fraction expansion. Prove that for almost all \(x\), we have that

\[
\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \infty,
\]

where \(x = \langle a_1, a_2, a_3, \ldots \rangle\).

Note: you cannot directly apply the pointwise ergodic theorem to \(f(x) = a_1(x)\) since this is not an \(L^1(X, \mu)\) function! It may be helpful to remember, in this problem and others that

\[
\mu(C_k) = \frac{1}{\log 2} \log \left(1 + \frac{1}{k(k + 2)}\right).
\]

(3) Consider the continued fraction expansion. Let \(\delta < 1\). Prove that for almost all \(x\) we have that

\[
\lim_{n \to \infty} \frac{a_1^\delta + a_2^\delta + \cdots + a_n^\delta}{n}
\]

converges to something finite. Here again, \(x = \langle a_1, a_2, a_3, \ldots \rangle\).

(4) Consider the continued fraction expansion. Prove that for almost all \(x\), we have that

\[
\lim_{n \to \infty} \left(\frac{a_1 a_2 \cdots a_n}{n}\right)^{1/n} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k + 2)}\right)^{\log k / \log 2},
\]

where \(x = \langle a_1, a_2, a_3, \ldots \rangle\). Don’t forget the integrability requirement in the pointwise ergodic theorem!

(5) Consider the continued fraction expansion. Let \(k\) be a positive integer. Prove that for almost all \(x\), we have that

\[
\lim_{n \to \infty} \frac{\#\{1 \leq i \leq n : a_i(x) \in \{2, 4, 6, \ldots, 2k\}\}}{n} = \frac{1}{\log 2} \log \left(\frac{4\Gamma(k + \frac{3}{2})^2}{\pi \Gamma(k + 1) \Gamma(k + 2)}\right),
\]

where \(\Gamma\) is the usual gamma function. (You may make use of basic facts about the gamma function, such as those found on the corresponding Wikipedia page.)

If you were to let \(k\) go to infinity, this would show that the expected frequency of a continued fraction digit being even is \(\log(4/\pi)/\log 2\).

(6) Let \(S\) be any dense subset of \([0, 1)\) containing both 0 and 1. Prove that the collection

\[
\{[a, b] : a, b \in S\}
\]

is a semi-algebra that generates the Borel \(\sigma\)-algebra.

Some facts that may help you: Recall the Borel \(\sigma\)-algebra is generated by all open interval \((a, b)\) intersected with \([0, 1)\) (or equivalently closed or clopen). If you can show that the \(\sigma\)-algebra generated by the collection above contains all such open intervals, then you are done.

A set is dense in \([0, 1)\) if every subinterval of \([0, 1)\) contains a point from the set.