Math 8440—Homework 8, due October 16

(1) Suppose that \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are sequences. Prove that

\[
\sup_n (a_n + b_n) \leq (\sup_n a_n) + (\sup_n b_n).
\]

The following definition may prove useful: \( \sup_n a_n = L \) if \( a_n \leq L \) for all \( n \) and if for any \( \epsilon > 0 \) there exists an \( n \) such that \( a_n \geq L - \epsilon \).

(2) Following similar lines as the previous problem, let \( \{a_{n,m}\}_{n,m=1}^{\infty} \) consist of positive numbers. Prove that

\[
\sup_n \sum_{m=1}^{\infty} a_{n,m} \leq \sum_{m=1}^{\infty} \sup_n a_{n,m}
\]

provided that all sums are convergent.

(3) For an arbitrary fibred system with transformation \( T \) and a cylinder set \( C_s \), prove that, for any \( r \in \mathbb{N} \), we have

\[
\lim_{n \to \infty} \frac{\# \{0 \leq i \leq n-1 : T^i x \in C_s \}}{n} = \lim_{n \to \infty} \frac{\# \{0 \leq i \leq rn-1 : T^i x \in C_s \}}{rn}
\]

provided the latter limit exists.

(4) Using the bounded convergence theorem, show that for an arbitrary fibred system on a probability space with transformation \( T \) and a disjoint collection of cylinder sets \( C_s \), we have

\[
\lim_{n \to \infty} \frac{\# \{0 \leq i \leq n-1 : T^i x \in \bigcup C_s \}}{n} = \sum_{C_s} \lim_{n \to \infty} \frac{\# \{0 \leq i \leq n-1 : T^i x \in C_s \}}{n}
\]

(5) Suppose that \( x \) is base-\( b \) normal and that \( r \) is rational. Using Weyl’s criterion and not Pyatetskii-Shapiro, prove that \( x + r \) is base-\( b \) normal.

(6) Suppose that \( x \) is base-\( b \) normal and that \( m \) is a non-zero positive integer. Without using Weyl’s criterion or Pyatetskii-Shapiro prove that \( mx \) is also base-\( b \) normal.