# HOMEWORK 2

(1) Exercise 3 of chapter 9 of Rudin.
(2) Exercise 5 of chapter 9 of Rudin.
(3) Define $F = \prod_{i=1}^{\infty} \mathbb{R}$ to be the set of infinite sequences $(x_1, x_2, x_3, \ldots)$ of real numbers. Set $E = \bigoplus_{i=1}^{\infty} \mathbb{R}$ to consist of all elements in $F$ in which all but finitely many of the slots are zeroes. Prove that $E$ and $F$ are vector spaces.
   a) Prove that $E$ has a countable basis.
   b) Prove that if $S$ is a countable subset of $F$, then $S$ cannot span all of $F$.
(4) Let $E$ be a vector space such that it has a countable basis $B = \{e_i\}_{i \in I}$. Assume we can find any another basis $S = \{f_j\}_{j \in J}$. Prove that the cardinality of $I$ and $J$ is the same.
   Hint: Prove by cases, first for $I$ finite. And then for $I$ infinite. Assume cardinality of $J$ is bigger than $I$. Then assign for each index of $J$ a finite subset of elements of $I$ (why?) and take the union of all these finite sets.
(5) Prove that every vector space has a basis.
   Hint: We need to use Zorn's Lemma to prove this. Zorn's Lemma: Assume $C$ is a collection of subsets of some fixed unnamed set, and assume that $C$ has the property that whenever there is a chain $S_1 \subset S_2 \subset \ldots$ of sets in $C$, the union of this chain also belongs to $C$. Then Zorn's Lemma says that $C$ contains a maximal element. This means that $C$ contains some set $M$ which is not properly contained in any other set in the collection $C$. In fact, Zorn's lemma implies that every set $S$ in $C$ is contained in some maximal set $M$, because we can apply Zorn's lemma to the subcollection of sets in $C$ containing $S$. Now consider $C$ to be the collection of all linearly independent subsets of a vector space $V$.