MATH 5202 - HOMEWORK 3

Homework:
(Notation: We write 0 to represent the number 0 as well as the zero vector in $\mathbb{R}^m$.)

(1) (a) Let $F : \mathbb{R}^m \to \mathbb{R}$ such that $F(0) = 0$ and $F(t\overline{x}) = tF(\overline{x})$ for any $\overline{x} \in \mathbb{R}^m$ and $t \neq 0$. Prove that F has all directional derivatives at the origin and that:
$$\frac{\partial F}{\partial \overline{v}}(0) = F(\overline{v}).$$
(b) Let $F : \mathbb{R}^m \to \mathbb{R}$ such that $F(t\overline{x}) = |t|F(\overline{x})$ for any $\overline{x} \in \mathbb{R}^m$ and $t \in \mathbb{R}$. Assume F is differentiable at the origin. Prove that $F(\overline{x}) = 0$ for all $\overline{x} \in \mathbb{R}^m$.

(2) Let $U \subset \mathbb{R}^m$ open and $F : U \to \mathbb{R}$ differentiable at a point $\overline{a} \in U$.
(a) Prove that there exist $\epsilon > 0$ and $M > 0$ such that for all $|\overline{h}| < \epsilon$ then $\overline{a} + \overline{h} \in U$ and $|F(\overline{a} + \overline{h}) - F(\overline{a})| < M|\overline{h}|$.
(b) Prove that the following modification is not true: There exist $\epsilon > 0$ and $M > 0$ such that if $|\overline{y} - \overline{a}| < \epsilon, |\overline{x} - \overline{a}| < \epsilon$ then $|F(\overline{y}) - F(\overline{x})| < M|\overline{y} - \overline{x}|$.

(3) Let $F : \mathbb{R}^m \to \mathbb{R}$ differentiable at any point such that $F(\overline{x}/2) = F(\overline{x})/2$ for all $\overline{x} \in \mathbb{R}^m$. Prove that F is linear.

(4) Let $F : U \subset \mathbb{R}^m \to \mathbb{R}$ for U open subset of $\mathbb{R}^m$. Define $F^k : U \to \mathbb{R}$ as $F^k(\overline{x}) := (F(\overline{x}))^k$ for any $k \geq 1, k \in \mathbb{Z}$. Prove that $DF^k(\overline{x})(\overline{v}) = kF^{k-1}(\overline{x})DF(\overline{x})(\overline{v})$.

(5) Problem 12 of Chapter 9 of Rudin.

Challenge problem:

Let $F : \mathbb{R}^m \to \mathbb{R}$ a continuous function such that F has all directional derivatives at any point of $\mathbb{R}^m$. Assume $\frac{\partial F}{\partial \overline{u}}(\overline{u}) > 0$ for all $\overline{u} \in \mathbb{R}^m$ for which $|\overline{u}| = 1$. Prove that exists a point $\overline{a} \in \mathbb{R}^m$ such that $\frac{\partial F}{\partial \overline{v}}(\overline{a}) = 0$ for all $\overline{v} \in \mathbb{R}^m$.