MATH 5202 - HOMEWORK 4

(1) Problem 21 of chapter 9 of Rudin.

(2) Problem 23 of chapter 9 of Rudin.

(3) Prove that:
(a) There exists a continuously differentiable real function f defined in a neighborhood U of 1 such that $f(1) = 1$ and

$$x^{f(x)} + (f(x))^x = 2.$$

where x belongs to U.
(b) Find the line tangent to the graph of f at the point $(1, f(1))$.

(4) Prove that:
(a) There exists a continuously differentiable real function f defined in a neighborhood U of $(3, -2)$ such that $f(3. - 2) = 1$ and for any $(x, y) \in U$ we have:

$$(f(x, y))^6 + x(f(x, y))^2 + 5y(f(x, y)) + y^2 + 2 = 0.$$

(b) If $\overline{u} \in \mathbb{R}^2$ with $|\overline{u}| = 1$ and $(D_{\overline{u}} f)(3, -2) = 0$. Find $\overline{u}$.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ a C^1 function such that $|f'(t)| \leq k < 1$ for all $t \in \mathbb{R}$. Define the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(x, y) = (x + f(y), y + f(x))$. Prove that F is a diffeomorphism of $\mathbb{R}^2$ on itself.

Bonus problem:

Let $F : U \to \mathbb{R}^m$ a C^1 function in the open set $U \subset \mathbb{R}^m$. Prove that if all the singularities of F (points for which the Jacobian is 0) are isolated points and $m > 1$ then F is an open map. Use this to prove the Fundamental Theorem of Algebra.