MATH 5202 - HOMEWORK 8

Let E be a measurable set. Recall the following definition: We say a family $\mathcal{F}$ of measurable functions on E is said to be **uniformly integrable over** E provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$, if $A \subseteq E$ is measurable and $\lambda(A) < \delta$, then $\int_A |f| d\lambda < \epsilon$.

(1) Prove the following: Assume E has finite measure. Let the sequence of functions $\{f_n\}$ be uniformly integrable over E. If $f_n \to f$ pointwise a.e. on E, then f is integrable over E.

(2) Show that the proposition above is false if $E = \mathbb{R}$.

(3) Let E be a set of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on E to $h \equiv 0$. Then prove: $\lim_{n\to\infty} \int_E h_n d\lambda = 0$ if and only if $\{h_n\}$ is uniformly integrable over E.

(4) Show that the statement above is false without the assumptions of h_n are non negative. What about when E has infinite measure?

(5) Problem 52 of Royden chapter 4.