MATH 6222 - HOMEWORK 1

(due on February 11th)

Exercises 1.1.2, 1.2.3, 1.2.4, 1.2.11(a), 1.3.3, 1.3.4, 1.4.1 (see below), 1.4.4 and 1.5.3 on J. Lebl's notes at: http://www.jirka.org/scv/scv.pdf

Let $\mathbb{B}_n \subset \mathbb{C}^n$ be the unit ball and $G_n = \{(w', w_n) \in \mathbb{C}^n, \mathrm{Im} w_n > |w'|^2\}$. G_n is called the Siegel upper half-space. Define $\phi(z) = (w_1, \ldots, w_n)$ by $w_j = z_j/(1+z_n)$ for $1 \leq j \leq n-1$ and $w_n = i(1-z_n)/(1+z_n)$. Show that:

(i) Show that $\phi : \mathbb{B}_n \to G_n$ is biholomorphic. (ϕ is called the **Cayley transform**).

(ii) The boundary $bG_n = \{(z', t + i|z'|^2), z' \in \mathbb{C}^{n-1}, t \in \mathbb{R}\}$ is naturally identified with $\mathbb{C}^{n-1} \times \mathbb{R} = \{(z', t)\}$. Show that the multiplication:

$$(z', t) \cdot (w', u) = (z' + w', t + u + 2\mathrm{Im} < z', w' >)$$

(where $< z', w' >= \sum_{i=1}^{n-1} z_i \overline{w_i}$) turns bG_n into a group which is non-abelian if $n > 1$. (This group is called the Heisenberg group of order $n-1$).

Problem 1.4.1 (typos corrrected):

a) Let Δ be an analytic disc and $\Delta \cap \partial\mathbb{B}_n \neq \emptyset$. Prove Δ contains points not in $\overline{\mathbb{B}_n}$.

b) Let Δ be an analytic disc. Prove that $\Delta \cap \partial\mathbb{B}_n$ is nowhere dense in Δ.

c) Find an analytic disc, such that $(1,0) \in \Delta, \Delta \cap \mathbb{B}_2 = \emptyset$ and locally near $(1,0) \in \partial\mathbb{B}_2$, the set $\Delta \cap \partial\mathbb{B}_2$ is the curve defined by $\mathrm{Im} z_1 = 0, \mathrm{Im} z_2 = 0, (\mathrm{Re} z_1)^2 + (\mathrm{Re} z_2)^2 = 1$.

bonus problem:

Find a concrete power series whose convergence domain is the two-dimensional unit ball $\{(z_1, z_2), |z_1|^2 + |z_2|^2 < 1\}$.