For a function f defined on E, we have the associated functions $|f|$, f^+, and f^- defined on E by

$$|f|(x) = \max\{f(x), -f(x)\},\ f^+(x) = \max\{f(x), 0\},\ f^-(x) = \max\{-f(x), 0\}.$$

If f is measurable on E, then, by the preceding proposition, so are the functions $|f|$, f^+, and f^-. This will be important when we study integration since the expression of f as the difference of two nonnegative functions,

$$f = f^+ - f^- \text{ on } E,$$

plays an important part in defining the Lebesgue integral.

PROBLEMS

1. Suppose f and g are continuous functions on $[a, b]$. Show that if $f = g$ a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E?
2. Let D and E be measurable sets and f a function with domain $D \cup E$. We proved that f is measurable on $D \cup E$ if and only if its restrictions to D and E are measurable. Is the same true if "measurable" is replaced by "continuous"?
3. Suppose a function f has a measurable domain and is continuous except at a finite number of points. Is f necessarily measurable?
4. Suppose f is a real-valued function on $\mathbf{R}$ such that $f^{-1}(c)$ is measurable for each number c. Is f necessarily measurable?
5. Suppose the function f is defined on a measurable set E and has the property that $\{x \in E \mid f(x) > c\}$ is measurable for each rational number c. Is f necessarily measurable?
6. Let f be a function with measurable domain D. Show that f is measurable if and only if the function g defined on $\mathbf{R}$ by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.
7. Let the function f be defined on a measurable set E. Show that f is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable. (Hint: The collection of sets A that have the property that $f^{-1}(A)$ is measurable is a σ-algebra.)
8. (Borel measurability) A function f is said to be **Borel measurable** provided its domain E is a Borel set and for each c, the set $\{x \in E \mid f(x) > c\}$ is a Borel set. Verify that Proposition 1 and Theorem 6 remain valid if we replace "(Lebesgue) measurable set" by "'Borel set." Show that: (i) every Borel measurable function is Lebesgue measurable; (ii) if f is Borel measurable and B is a Borel set, then $f^{-1}(B)$ is a Borel set; (iii) if f and g are Borel measurable, so is $f \circ g$; and (iv) if f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.
9. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable?
10. Suppose f and g are real-valued functions defined on all of $\mathbf{R}$, f is measurable, and g is continuous. Is the composition $f \circ g$ necessarily measurable?
11. Let f be a measurable function and g be a one-to-one function from $\mathbf{R}$ onto $\mathbf{R}$ which has a Lipschitz inverse. Show that the composition $f \circ g$ is measurable. (Hint: Examine Problem 38 in Chapter 2.)