The following generalization of the Lebesgue Dominated Convergence Theorem, the proof of which we leave as an exercise (see Problem 32), is often useful (see Problem 33).

Theorem 19 (General Lebesgue Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ on E in the sense that*

$$|f_n| \leq g_n \text{ on } E \text{ for all } n.$$

$$\text{If } \lim_{n\to\infty} \int_E g_n = \int_E g < \infty, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Remark *In Fatou's Lemma and the Lebesgue Dominated Convergence Theorem, the assumption of pointwise convergence a.e. on E rather than on all of E is not a decoration pinned on to honor generality. It is necessary for future applications of these results. We provide one illustration of this necessity. Suppose f is an increasing function on all of* **R**. *A forthcoming theorem of Lebesgue (Lebesgue's Theorem of Chapter 6) tells us that*

$$\lim_{n\to\infty} \frac{f(x+1/n)-f(x)}{1/n} = f'(x) \text{ for almost all } x. \tag{22}$$

From this and Fatou's Lemma we will show that for any closed, bounded interval $[a, b]$,

$$\int_a^b f'(x)\,dx \leq f(b)-f(a).$$

In general, given a nondegenerate closed, bounded interval $[a, b]$ and a subset A of $[a, b]$ that has measure zero, there is an increasing function f on $[a, b]$ for which the limit in (22) fails to exist at each point in A (see Problem 10 of Chapter 6).

PROBLEMS

28. Let f be integrable over E and C a measurable subset of E. Show that $\int_C f = \int_E f \cdot \chi_C$.

29. For a measurable function f on $[1, \infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number n. Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges? Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

30. Let g be a nonnegative integrable function over E and suppose $\{f_n\}$ is a sequence of measurable functions on E such that for each n, $|f_n| \leq g$ a.e. on E. Show that

$$\int_E \liminf f_n \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E \limsup f_n.$$

31. Let f be a measurable function on E which can be expressed as $f = g + h$ on E, where g is finite and integrable over E and h is nonnegative on E. Define $\int_E f = \int_E g + \int_E h$. Show that this is properly defined in the sense that it is independent of the particular choice of finite integrable function g and nonnegative function h whose sum is f.

32. Prove the General Lebesgue Dominated Convergence Theorem by following the proof of the Lebesgue Dominated Convergence Theorem, but replacing the sequences $\{g - f_n\}$ and $\{g + f_n\}$, respectively, by $\{g_n - f_n\}$ and $\{g_n + f_n\}$.

33. Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f - f_n| \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: Use the General Lebesgue Dominated Convergence Theorem.)

34. Let f be a nonnegative measurable function on $\mathbf{R}$. Show that

$$\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbf{R}} f.$$

35. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and is a measurable function of x for each fixed value of y. Suppose for each fixed value of x, $\lim_{y\to 0} f(x, y) = f(x)$ and that for all y, we have $|f(x, y)| \le g(x)$, where g is integrable over $[0, 1]$. Show that

$$\lim_{y\to 0} \int_0^1 f(x, y)dx = \int_0^1 f(x)dx.$$

Also show that if the function $f(x, y)$ is continuous in y for each x, then

$$h(y) = \int_0^1 f(x, y)dx$$

is a continuous function of y.

36. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and is a measurable function of x for each fixed value of y. For each $(x, y) \in Q$ let the partial derivative $\partial f / \partial y$ exist. Suppose there is a function g that is integrable over $[0, 1]$ and such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \le g(x) \text{ for all } (x, y) \in Q.$$

Prove that

$$\frac{d}{dy}\left[\int_0^1 f(x, y)\,dx\right] = \int_0^1 \frac{\partial f}{\partial y}(x, y)\,dx \text{ for all } y \in [0, 1].$$

4.5 COUNTABLE ADDITIVITY AND CONTINUITY OF INTEGRATION

The linearity and monotonicity properties of the Lebesgue integral, which we established in the preceding section, are extensions of familiar properties of the Riemann integral. In this brief section we establish two properties of the Lebesgue integral which have no counterpart for the Riemann integral. The following countable additivity property for Lebesgue integration is a companion of the countable additivity property for Lebesgue measure.

Theorem 20 (the Countable Additivity of Integration) *Let f be integrable over E and $\{E_n\}_{n=1}^{\infty}$ a disjoint countable collection of measurable subsets of E whose union is E. Then*

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f. \tag{23}$$