Solutions to #6.1, #6.2, #6.3, #6.5, and #6.6.

#6.1

Under the risk free measure, we know that
\[ dS(t) = (r - \delta)S(t)dt + \sigma S(t)d\tilde{Z}(t) \]

This stock pays dividends at 4%, so that means that \( r = 0.11 \). The variance gives us that \( \sigma = 0.5 \). This stochastic differential equation represents a geometric brownian motion with

\[
S(t) = 10e^{(0.07 - 0.125)t + 0.5\tilde{Z}(t)} \\
S(2) = 10e^{-0.11 + 0.5\sqrt{2}\tilde{N}(0,1)} \\
S(2)^{-1} = \frac{1}{10}e^{0.11 - 0.5\sqrt{2}\tilde{N}(0,1)}
\]

Using our formula \( \mathbb{E}(e^{a+b\tilde{N}(0,1)}) = e^{a+b^2/2} \), we have that
\[
\mathbb{E}(S^{-1}(2)) = \frac{1}{10}e^{0.11+25} = 0.14333
\]

This tells us that our expected payoff is \([14.333]\). To compute the cost of this option we discount the expected payoff by the risk free rate.

\[
14.333e^{-0.11*2} = [11.5]
\]

#6.2

The stochastic differential equation gives us that \( r - \delta = 0.03 \) and \( \sigma = 0.2 \). Since \( \delta = 0.02 \), we can deduce that \( r = 0.05 \). Now we can solve for part b) first (since it is immediately accessible using the Black-Scholes formula).

\[
d_1 = \frac{\ln \left( \frac{17}{21} \right) + (0.03 + 0.02) * 0.5}{0.2 \sqrt{0.5}} = 1.67 \\
d_2 = d_1 - 0.2 \sqrt{0.5} = 1.53
\]

Thus the price of the call is
\[
C = \mathcal{N}(d_1)S(0)e^{-0.02*0.5} - \mathcal{N}(d_2) * 17 * e^{-0.05*0.5} \\
= 0.95254 * 21 * e^{-0.01} - 0.93699 * 17 * e^{-0.025} \\
= 4.27
\]

The answer to part a) is \( 4.27 * e^{0.05*0.5} = [4.38] \). To determine the solution to part c) we can use the Black-Scholes formula for puts, or we can use put-call parity. I choose put-call parity.

\[
c - p = 21e^{-0.01} - 17e^{-0.025} \\
4.27 - p = 20.79 - 16.58 \\
p = 0.06
\]
Finally, part d) has a payoff that is the same as a portfolio containing one call and one put (that we have already computed). The resulting price is \(4.27 + 0.06 = 4.33\)

\#6.3

From the dynamics we know that \(X(t) = 100e^{(0.06t - 0.045)t + 0.3Z(t)}\), so
\[
X(1)^2 = 10000e^{0.03 + 0.6Z(1)}
\]

Using our original Black-Scholes formula for a call with strike 10,000, we have the following:
\[
d_2 = \frac{\ln \left( \frac{10,000}{10,000} \right) + 0.03}{0.6} = 0.05
\]
\[
d_1 = d_2 + b = 0.65
\]
\[
c = 10000e^{0.03 + 0.65 \times 0.6} e^{-0.09} N(0.65) - 10000e^{-0.09} N(0.05)
\]
\[
= 8367.72 - 4751.89
\]
\[
c = 3616
\]

\#6.5

Since there are discrete dividends on Apple we must use the prepaid forward version of the Black-Scholes formula. The prepaid forward price of one share of Apple is
\[
115 - 5e^{-0.08} = 110.38,
\]
and the prepaid forward price of the strike is
\[
120e^{-0.08 \times 1.25} = 108.58.
\]

Using these two values we can compute \(d_1\) and \(d_2\).
\[
d_1 = \frac{\ln \left( \frac{110.38}{108.58} \right) + 0.5^2 \times 1.25}{0.5 \sqrt{1.25}}
\]
\[
= 0.31
\]
\[
d_2 = d_1 - 0.5 \sqrt{1.25}
\]
\[
= -0.25
\]

The value of our call is
\[
c = 110.38N(d_1) - 108.58N(d_2)
\]
\[
= 25.06
\]
(or 25.05 with a little rounding).
#6.6

This is an application of the $\Delta - \Gamma$ Approximation. In this case, $\epsilon = -2$.

$$V(t, 23) \approx V(t, 25) + (-0.32)(-2) + \left(\frac{0.13}{2}\right)(-2)^2$$

$$= 4.19$$