Research Statement

My research efforts focus on the analytic number theory involving the automorphic forms. In particular, I developed a strong interest in the shifted convolutions, non-vanishing of central values and the distinguishing problems. Additionally, I am also interested in the theory of automorphic representations.

First, I will give a quick summary of my publications and preprints. Then I will discuss those topics in the following section. Finally, I will introduce my ongoing and future projects in the last section.

(i) In my Ph. D thesis, I established some power saving theorems for the shifted convolution of the Fourier coefficients of cusp forms and the k-th powers. This can be considered as a generalization of the shifted convolution of Fourier coefficients of cusp forms and θ-series.

(ii) In [Wei21], I first investigate the linear relations of a one parameter family of Siegel Poincaré series. Then I apply is to study the non-vanishing of Fourier coefficients of Siegel cusp eigenforms and the central values.

(iii) In [WWYY22], we refine the Erdős-Kac Theorem and Loyd's recent result on Bergelson and Richter's dynamical generalizations of the Prime Number Theorem [BR22].

(iv) In [WY22], we establish several results on distinguishing Siegel cusp forms of degree two.

(v) In [WZ22], we showed that the generalized Möbius functions associated to a certain family of L-functions are weakly orthogonal to bounded sequences.

(vi) In [Wei22], I proved a zero density result for the Rankin-Selberg L-functions. As an application, I apply it to distinguish the holomorphic Hecke eigenforms for $\mathrm{SL}_2(\mathbb{Z})$ by (normalized) Fourier coefficients.

1 Main Results

1.1 Ph. D. Thesis: The Shifted Convolution Sum

The study of shifted convolution sums of arithmetic functions is a classical theme in analytic number theory. Since Selberg's paper [Sel65], the shifted convolution of GL_2 Fourier coefficients has received much attention. Indeed, nontrivial bounds of $\mathrm{GL}_2 \times \mathrm{GL}_2$ shifted convolutions have profound implications on subconvexity problems and quantum unique ergodicity. ([Blo04], [BH08] and [Hol09]).

My Ph. D. thesis investigates the shifted convolution of Fourier coefficients of cusp forms and k-th powers. Before my paper, several authors considered the shifted convolution of Fourier coefficients of cusp forms and θ-series, which is the case $k = 2$. One can refer to [Luo11], [Sun17] and [JL19]. My paper generalizes those results by considering higher k-th powers. We can formulate the problem as follows: let s and k be natural numbers. Denote by $r_{s,k}(n)$ the number of representations of a positive integer n as the sum of s positive integral k-th powers. Additionally, denote by $a_f(n)$ the n-th normalized Fourier coefficient of a holomorphic cusp form of weight l,

$$f(z)=\sum_{n=1}^{\infty}a_f(n)n^{\frac{l-1}{2}}e(nz)\in\mathcal{S}_l(\mathrm{SL}_2(\mathbb{Z}))$$

or a Maass cusp form

$$f(z)=y^{1/2}\sum_{n\neq 0}a_f(n)K_{it}(2\pi|n|y)e(nx)$$

with the Laplacian eigenvalue $\lambda=\frac{1}{4}+t^2$. Let $\phi(x)$ be a smooth function compactly supported in $[1/2,1]$. I am interested in the following (smooth) shifted convolution sum

$$\sum_{n=1}^{\infty}a_f(n+1)r_{s,k}(n)\phi\left(\frac{n}{X}\right)$$

By Deligne's seminal work [Del74] on the Fourier coefficients of holomorphic cusp forms, the trivial bound for the shifted convolution is $X^{\frac{s}{k}+\epsilon}$ for any $\epsilon>0$. On the other hand, by the cancellations in the sum of $a_f(n)$ twisted by additive characters, it is natural to ask whether we can establish power saving results for such shifted convolutions. For the non-holomorphic case, we have not proved the Ramanujan conjecture yet. However, it is also of great interest to establish similar power saving results for the Maass form case.

More generally, let $a_\pi(n_1,\dots,n_{m-1})$ denote a Whittaker-Fourier coefficient of a cusp form π on $\mathrm{SL}_m(\mathbb{Z})$. We can establish a shifted convolution of the form

$$\sum_{n=1}^{\infty}a_\pi(1,\dots,1,n+1)r_{s,k}(n)\phi\left(\frac{n}{X}\right).$$

and expect power saving results for such shifted convolutions.

To establish the power saving result, we define the following functions:

$$\mathcal{F}_k(\alpha)=\sum_{m\leq X^{1/k}}e(\alpha m^k)$$

$$\mathcal{G}(\alpha)=\sum_{n=1}^{\infty}a(n+1)\phi\left(\frac{n}{X}\right)e(-\alpha n)$$

(Here $\{a_n\}$ can be chosen to be any complex sequence.) Then:

$$\sum_{n=1}^{\infty}a(n+1)r_{s,k}(n)\phi\left(\frac{n}{X}\right)=\int_0^1\mathcal{G}(\alpha)\mathcal{F}_k^s(\alpha)\,d\alpha.$$

Later the sequence is taken to be the Fourier coefficients of automorphic forms. The idea of the circle method is to dissect the interval $[0,1]$ into the so-called major arc and the minor arc. For the major arc, we can use the asymptotic formula for $\mathcal{F}_k(\alpha)$ (Theorem 4.4 in [Vau97]) and apply the Voronoi formula ([Goo81], [Meu88] and [JL19]). An interesting phenomenon is, the major arc part becomes negligibly small. For the minor arc part, we need to apply Hua's inequality and Weyl's inequality which was improved by Bourgain [Bou17]. In fact, these improvements was closely related to the Vinogradov mean value conjecture, which predicts the asymptotic formula of the number of solutions, say $J_{s,k}(Y)$, of the following homogeneous equations:

$$x_1^i + \cdots + x_s^i = y_1^i + \cdots + x_s^i \qquad 1 \le i \le k$$

where $1 \le x_i, y_i \le Y$. This conjecture is proved by [Woo16] for $k = 3$ via efficient congruencing and [BDG16] for $k \ge 4$ via l^2-decoupling method.

For the GL(2) case, I establish the following theorem:

Theorem 1.1. *Let $k \ge 3$. Define $A(k)$ to be*

$$A(k) = \begin{cases} \frac{(k+1)^2}{4} & \text{if } k \text{ is odd} \\ \frac{k^2+k}{4} & \text{if } k \text{ is even} \end{cases}$$

Then for $s > A(k)$, we have

$$\sum_{n=1}^{\infty} a_f(n+1) r_{s,k}(n) \phi\left(\frac{n}{X}\right) \ll X^{\frac{s}{k}-\delta+\epsilon}$$

for some $\delta > 0$ depending on s, k and any $\epsilon > 0$.

In Luo's paper [Luo21], Theorem 1 established the power saving for $s \ge k^2 + O(k)$ when f is a GL(2) cusp form. Theorem 1.1 improves this result by reducing the number of variables to $\frac{1}{4}k^2 + O(k)$.

For the higher rank case, the theorem is stated as follows:

Theorem 1.2. *Denote by $\lceil t \rceil$ the smallest integer no smaller than t. Then we define the following functions*

$$f_1(k) = \frac{1}{2}\left\{k^2 + 1 - \max_{i<k, 2^i<k^2}\left\lceil \frac{ki - 2^i}{k-i+1}\right\rceil\right\} = \frac{k^2+1}{2} - \frac{1}{2}\frac{\log k}{\log 2} + O(1),$$

and

$$f_2(k) = \frac{1}{2}\left\{k^2 + 1 - \max_{i<k}\left\lceil i\frac{k-i-1}{k-i+1}\right\rceil\right\} = \frac{k^2-k}{2} + O(\sqrt{k})$$

for integers $k \ge 2$. Let π be an even Maass cusp form on $\mathrm{SL}_m(\mathbb{Z})$ with $m \ge 3$. Suppose that $k \ge 3$, $k \ne 4$ and

$$s > \min\{f_1(k), f_2(k)\},$$

we have

$$\sum_{n=1}^{\infty} a_\pi(1, \ldots, 1, n+1) r_{s,k}(n) \phi\left(\frac{n}{X}\right) \ll X^{\frac{s}{k}-\delta+\epsilon}$$

for some positive δ and any $\epsilon > 0$.

In Luo's paper [Luo21], Theorem 3 showed the power saving for $s \ge k(k+1)$ in the higher rank case. Theorem 1.2 improves his result by reducing the number of variables to around half of it. On thing that I need to point out is, for the case $k = 3$, I use a non-standard circle method.

Remark 1.3. *For the case $k = 4$, we can establish the power saving for $s > 8$. To achieve this, one needs to refine Wooley's Theorem 2.1 in [Woo12] for non-standard circle method. Then we need the idea proving Theorem 11 in [Bou17].*

For the case $k=3$, we can prove a stronger result than that in Theorem 1.2. That is, for $s \geq 5$ and π being an even Maass cusp form on $\mathrm{SL}_m(\mathbb{Z})$, one has, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} a_\pi(1,\ldots,1,n+1) r_{s,3}(n) \phi\left(\frac{n}{X}\right) \ll X^{\frac{s}{3}-\frac{1}{4m}+\epsilon}.$$

Additionally, we can save more when $m=4$ using the standard circle method, which is the following theorem:

Theorem 1.4. *Let π be an even Hecke-Maass cusp form on $\mathrm{SL}_4(\mathbb{Z})$. Then we have*

$$\sum_{n=1}^{\infty} a_\pi(1,1,n+1) \phi\left(\frac{n}{X}\right) r_{5,3}(n) \ll X^{\frac{5}{3}-\frac{1}{12}+\epsilon}$$

for any $\epsilon > 0$.

1.2 The Linear Relations of Siegel Poincaré Series and the Non-vanishing of the Central Values

In [Wei21], I studied the linear relations of a one parameter family of Siegel Poincaré series, which implied the non-vanishing of central values of the spinor L-functions.

Denote by Γ_∞ the group of translations in $\mathrm{Sp}_4(\mathbb{Z})$, that is,

$$\Gamma_\infty = \left\{ \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) \middle| X = {}^tX \right\}.$$

In the theory of Siegel cusp forms, the Poincaré series are defined by

$$\mathbb{P}_Q(Z) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{Sp}_4(\mathbb{Z})} \det(J(\gamma, Z))^{-k} e(\mathrm{Tr}(Q\gamma Z))$$

for Z in the Siegel upper half plane and any Q, a symmetric, positive definite and half-integral matrix. This will give Siegel cusp forms of weight k for $\mathrm{Sp}_4(\mathbb{Z})$.

In [Wei21], I investigated Siegel Poincare series $\mathbb{P}_Q$ with $Q = nI_2$ for n a positive integer and proved the following theorem:

Theorem 1.5. *Let $\epsilon > 0$. Denote by $\mathcal{V}$ the vector space spanned by $\mathbb{P}_{nI_2}$. Then for sufficiently large k, we can find a subspace $\mathcal{W} \subseteq \mathcal{V}$, such that*

$$\dim \mathcal{W} \geq k^{2/3-\epsilon},$$

and

$$\mathcal{W} \cap \mathcal{K} = \{0\}.$$

As an application, one can study the number of Siegel Hecke eigenforms F satisfying $A(F, I_2) \neq 0$. I proved the following result:

Corollary 1.6. *Let $\epsilon > 0$ be arbitrary. Then for sufficiently large k,*

$$\#\{F \in \mathcal{B}_k | F \text{ is not a Saito-Kurokawa lift and } A(F, I_2) \neq 0\} \geq k^{2/3-\epsilon}.$$

Here $\mathcal{B}_k$ is the set of Siegel Hecke eigenforms.

In [Böc86], Böcherer made a remarkable conjecture that $A(F, I_2)$ should be related to central L-values and that was proved by Furusawa and Morimoto in [FM21]. This can be regarded as a generalization of Waldspurger's theorem. A precise version of Böcherer's conjecture was given in [DPSS20]. In my case, the statement is, for a non-Saito-Kurokawa lift F and $F \in \mathcal{B}_k$,

$$\frac{\pi^{1/2}}{4}(4\pi)^{3-2k}\Gamma(k-3/2)\Gamma(k-2)\frac{A(F,I_2)^2}{||F||^2} = \frac{64\pi^6\Gamma(2k-4)}{\Gamma(2k-1)}\frac{L(1/2,\mathrm{spin},F)L(1/2,\mathrm{spin},F\times\chi_{-4})}{L(1,\pi_F,\mathrm{Ad})}$$

where $L(s,\pi_F,\mathrm{Ad})$ is a degree 10 L-function. Then Corollary 1.6 implies the following corollary:

Corollary 1.7. Let $\epsilon > 0$ be arbitrary. Then for sufficiently large k,

$$\#\{F \in \mathcal{B}_k | F \text{ is not a Saito-Kurokawa lift and } L(1/2,\mathrm{spin},F)L(1/2,\mathrm{spin},F\times\chi_{-4}) \neq 0.\} \geq k^{2/3-\epsilon}.$$

1.3 Generalizations of the Erdős-Kac Theorem and the prime number theorem (with Biao Wang, Pan Yan and Shaoyun Yi)

In [WWYY22], we study the linear independence between the distribution of the number of prime factors of integers and that of the largest prime factors of integers. Respectively, under a restriction on the largest prime factors of integers, we will refine the Erdős-Kac Theorem. As an application, we can refine Loyd's recent result [Loy21] on Bergelson and Richter's dynamical generalizations of the Prime Number Theorem [BR22].

At the end, we will show that the analogue of these results holds with respect to the Erdős-Pomerance Theorem as well.

1.4 On distinguishing Siegel cusp forms of degree two (with Shaoyun Yi)

In [WY22], we establish several results on distinguishing Siegel cusp forms of degree two.

For Siegel Hecke eigenforms of different weights, we can distinguish them by their eigenvalues. Furthermore, a Hecke eigenform of level one can be determined by its second Hecke eigenvalue under an analogue of Maeda's conjecture.

Moreover, we can distinguish two Hecke eigenforms of level one by using L-functions. For Saito-Kurokawa lifts, we can use the central value of the L-functions twisted by quadratic characters to distinguish them. Indeed,this can be reduced to the central value of GL_2 L-functions twisted by quadratic characters. Then we can apply [LR97]. For the non-Saito-Kurokawa lifts, we apply [GH93].

1.5 The weak orthogonality between generalized Möbius functions and bounded sequences (with Shifan Zhao)

Let f be a holomorphic cusp forms of $\mathrm{SL}_2(\mathbb{Z})$. Then a recent result by Newton and Thorne [NT21] shows that, for any positive integer n, the symmetric n-th power lifting $\mathrm{Sym}^n f$ is associated to an automorphic cuspidal representation of GL_{n+1}. Then for positive integers n_1, n_2, and distinct holomorphic cusp forms f, g, we can define the L-function $L(s, \mathrm{Sym}^{n_1} f \otimes \mathrm{Sym}^{n_2} g)$ and the Möbius function associated to $\mathrm{Sym}^{n_1} f \otimes \mathrm{Sym}^{n_2} g$ can be defined.

In [WZ22], we showed that the such Möbius functions are weakly orthogonal to bounded sequences. The strategy is first reducing the summation to a new summation over primes. Then we can apply some functoriality results to prove the weak orthogonality.

In this paper, we also show that the Möbius functions associated to $\mathrm{SL}_2(\mathbb{Z})$ Maass forms are weakly orthogonal to bounded sequences. The proof is similar but more technical.

1.6 The zero density theorem for the Rankin-Selberg L-function and its applications

In [Wei22], I proved a zero density result for the Rankin-Selberg L-functions based on [Mon71] and [LW06]. Let $L(s,\pi)$ be an L-function defined in [IK04, Chapter 5]. For fixed $\alpha > \frac{1}{2}$, denote by $N(\alpha, T, \pi)$ the number of zeros $\rho = \beta + i\gamma$ of $L(s,\pi)$ with $\beta \geq \alpha$ and $0 \leq \gamma \leq T$. Denote by H_k the holomorphic weight k Hecke eigenforms for $\mathrm{SL}_2(\mathbb{Z})$. The main result can be stated as follows:

Theorem 1.8. *Denote by $\mathcal{F}_k$ the set of pairs of distinct holomorphic weight k Hecke eigenforms for* $\mathrm{SL}_2(\mathbb{Z})$*, that is,*

$$\mathcal{F}_k = \{(f,g) \in H_k \times H_k | f \neq g\},$$

For any $\delta > 0$, we have

$$\sum_{(f,g)\in\mathcal{F}_k} N(\alpha, T, f \otimes g) \ll_\delta T^2 (\log T) k^{34(1-\alpha)/(3-2\alpha)} (\log k)^{25}$$

uniformly for $\frac{1}{2} + \delta \leq \alpha \leq 1$, $(\log k)^3 \leq T \leq k$ and $f, g \in H_k$. The constant is dependent on δ when δ is small. When $\frac{1}{2} + \delta$ is closed to 1, the implied constant is absolute.

The strategy is first establishing a large sieve inequality and then use molification functions to detect the zeros of families of Rankin-Selberg L-functions. Via a similar argument, I established a zero density theorem for the families of symmetric square L-functions. In a recent paper [BTZ22], Brumley, Thorner and Zaman established a zero density theorem of the Rankin-Selberg L-functions under certain Hypothesis. In their paper, they fixed one cuspidal representation π_0 and varied the other representation in a given family. In my work, I can vary two modular forms at the same time.

As an application, I applied the zero density theorems to distinguish $\mathrm{SL}_2(\mathbb{Z})$ holomorphic Hecke eigenforms by (normalized) Fourier coefficients. Let $f, g \in H_k$ be distinct Hecke eigenforms. We say that f and g are *distinguishable* if for every $\epsilon > 0$, we can find $n \ll_\epsilon k^\epsilon$ such that $\lambda_f(n) \neq \lambda_g(n)$. Then the mian theorem is as follows:

Theorem 1.9. *As $k \to \infty$, we can find a set of Hecke eigenforms H_k^* ($\subseteq H_k$) such that*

$$\lim_{k\to\infty} \frac{|H_k^*|}{|H_k|} = 1$$

and for any $f, g \in H_k^*$ and $f \neq g$, they are distinguishable.

This proof is based on [GH93]. The assumption of Riemann Hypothesis can be removed with a mild adjustment.

It should be pointed out that, I was later notified that Farrell Brumley considered a similar problem in his Ph. D. thesis.

2 Ongoing and Future Projects

2.1 The strong multiplicity one theorem for paramodular forms (with Pan Yan and Shaoyun Yi)

For GL_n, the strong multiplicity one theorem was established in [JPSS83]. This shows that, let π_1, π_2 be two (unitary) automorphic cuspidal representations and suppose that for almost all primes, the local representations are isomorphic, then two global representations are the same.

Indeed, we can consider a refined version of the strong multiplicity one: denoted by $L(s, \pi) = \sum_{n=1}^{\infty} \lambda_\pi(n)/n^s$ for $\mathrm{Re}(s) > 1$. Suppose that for almost all primes p, $\lambda_{\pi_1}(p) = \lambda_{\pi_2}(p)$, will this imply that $\pi_1 = \pi_2$? We have two motivations for such problems: the first is the Selberg's orthogonality conjecture [Sel91]. If we assume the conjecture, then this will be a direct corollary. On the other hand, if we assume that both of representations can be associated with Galois representations, then this is still valid via some standard argument. So it is natural to ask whether this is true for all automorphic cuspidal representations. One application is, such results will play an important role in determining cusp forms by using the central values of L-functions twisted by quadratic forms [LR97], which I will discuss in the next subsection. Now we have some results either for the low rank cases or under some hypothesis. This is still an ongoing project.

On the other hand, we can also consider the problem for other classical groups. Recently, Schmidt [Sch18] established the strong multiplicity one theorem for paramodular forms on $\mathrm{GSp}(4)$. Then via the Langlands transfer, we can prove the refined strong multiplicity one for paramodular forms (via an analytic method). Then it is natural to ask the same questions for cusp forms on other congruence subgroups. This will be based on the local newform theory for other congruence subgroups, which is widely open now.

2.2 Determination of cusp forms by central L-values

In 1997, Luo and Ramakrishnam used the critical L-values (by twisting characters) to determine the cusp forms of GL_2. One of the main theorem is, we can determine the Hecke eigenform by using the central L-valiues twisted by quadratic characters. In the following years, a lot of papers considered the problem in the GL_2 set up. Recently, Hua and Huang [HH22] considered the determination of SL_3 Maass forms by central L-values twisted by quadratic characters.

In my recent work, I am considering the problem for weight k Siegel cusp forms of $\mathrm{Sp}_4(\mathbb{Z})$. If we only consider the Saito-Kurokawa lifts, this will be a direct corollary of [LR97], which was mentioned in Section 1.4. For the non Saito-Kurokawa lifts, the problem is more subtle. One of the main questions is, we lack the strong multiplicity one used in [LR97]. However, this can be proved by combining [Sch18] and our refined strong multiplicity one discussed in Section 2.1. In this case, we can follow the work in [Iwa90] or [LR97]. It is worth to point out that, we can use the upper bound of the second moment which can be deduced by the large sieve inequality of quadratic characters. This was proved by Heath-Brown in [HB95].

2.3 The zero density theorem for Maass forms and injectivity of liftings

In [Wei22], I considered the zero density theorem for the Rankin-Selberg convolutions of holomorphic cusp forms. It is natural to extend such results to Maass forms. Indeed, one of the main difficulty is to show that the lifting $\mathcal{A}(\mathrm{GL}_2) \times \mathcal{A}\mathrm{GL}_2 \to \mathcal{A}(\mathrm{GL}_4)$, constructed by Ramakrishnan [Ram00], is "sufficiently" injective. More precisely, given two pairs of cusp forms (f_1, g_1) and (f_2, g_2) satisfying $f_1 \neq g_1$ and $f_2 \neq g_2$, then $f_i \times g_i$ can be lifted to GL_4 automophic representations Π_i. Then $\Pi_1 = \Pi_2$ if and only if either $f_1 = f_2, g_1 = g_2$ or $f_1 = g_2, f_2 = g_1$. This can be reduced to the following problem: to what extent, the lifting is cuspidal. This was answered in [Ram00]. For the holomorphic case with squarefree level, this was further refined in the appendix of [DK00] by Ramakrishnan. In this case, we can show that the liffting is "sufficiently injective" for non-CM holomorphic newforms. However, for the Maass form case, we can only show the injectivity for level 1 case.

Once we have such injectivity results, we can establish the zero density theorem for Maass forms. It should be pointed out that, the second main question is, we can not bound the L-functions at $s = 1$ properly due to the lack of the Ramanujan conjecture. This will cause a bad error term in distinguishing Maass forms. One way to solve it is to apply the new method in [ST19] and [BTZ22] via new zero detecting functions. In addition, reducing the exponential in their work is also an ongoing project.

On the other hand, it is interesting to study those "injectivity" results for the liftings. We also have the following lifting: $\mathcal{A}(\mathrm{GL}_2) \times \mathcal{A}\mathrm{GL}_3 \to \mathcal{A}(\mathrm{GL}_6)$, constructed by Kim and Shahidi [KS02]. It is natural to ask to what extent, the lifting is injective. This is also an ongoing project with Xiyuan Wang.

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