Irreducibility of Galois representations associated to low weight Siegel modular forms

Ariel Weiss

University of Sheffield

a.weiss@sheffield.ac.uk

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The classical case

- $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(N, \epsilon)$ normalised Hecke eigenform, $k \geq 2$
- Associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

  unramified for all $p \nmid \ell N$ with

$$\mathrm{Tr}\, \rho_\ell(\mathrm{Frob}_p) = a_p, \qquad \det \rho_\ell = \epsilon \chi_\ell^{k-1}$$

- Associated mod ℓ Galois representation

$$\overline{\rho}_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$$

When are ρ_ℓ and $\overline{\rho}_\ell$ irreducible?

Example: a reducible ℓ-adic Galois representation

$$G_{12}(z) = \frac{691}{65520} + \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \quad \rightsquigarrow \quad \rho_\ell \cong \mathbf{1} \oplus \chi_\ell^{11}$$

$$a_p = 1 + p^{11} \qquad \operatorname{Tr} \rho_\ell(\mathrm{Frob}_p) = 1 + p^{11}$$

Theorem (Ribet)

If f is cuspidal, then

1. *ρ_ℓ is irreducible for all ℓ;*
2. *$\overline{\rho}_\ell$ is irreducible for all but finitely many ℓ.*

Example: a reducible mod ℓ Galois representation

$$\Delta(z) = 1 + \sum_{n \geq 2} \tau(n) q^n \quad \rightsquigarrow \quad \overline{\rho}_{691} \cong \mathbf{1} \oplus \overline{\chi}_{691}^{11}$$

Genus 2 Siegel modular forms

"Cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_{\mathbf{Q}})$ + conditions at ∞"

- has weights (k_1, k_2), $k_1 \geq k_2 \geq 2$
- has a level N
- has a character ϵ
- has Hecke operators T_p and Hecke eigenvalues a_p

4 types of cuspidal Siegel modular form:

- General
- Theta lifts/Automorphic inductions
- Saito-Kurokawa/CAP
- Yoshida/endoscopic

} reducible Galois representations (Saito-Kurokawa/CAP and Yoshida/endoscopic)

High weight: $k_2 > 2$
Low weight: $k_2 = 2$

The high weight case: $k_2 > 2$

- Associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$$

unramified for all $p \nmid \ell N$ with

$$\mathrm{Tr}\, \rho_\ell(\mathrm{Frob}_p) = a_p, \qquad \mathrm{sim} \rho_\ell = \epsilon \chi_\ell^{k_1+k_2-3}$$

- Associated mod ℓ Galois representation $\overline{\rho}_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{F}}_\ell)$
- ρ_ℓ is always "kinda nice", and is "nice" if $\ell \nmid N$
- The Hecke eigenvalues satisfy the generalised Ramanujan conjecture

Theorem

1. *(Ramakrishnan) If ρ_ℓ is nice and $\ell > 2(k_1 + k_2 - 3) + 1$, then ρ_ℓ is irreducible;*
2. *(Dieulefait-Zenteno) $\overline{\rho}_\ell$ is irreducible for 100% of primes.*

The low weight case: $k_2 = 2$

- Associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$$

unramified for all $p \nmid \ell N$ with

$$\mathrm{Tr}\,\rho_\ell(\mathrm{Frob}_p) = a_p, \qquad \mathrm{sim}\,\rho_\ell = \epsilon\chi_\ell^{k_1-1}$$

- Associated mod ℓ Galois representation $\overline{\rho}_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{F}}_\ell)$

Theorem (W.)

1. *If ρ_ℓ is nice and $\ell > 2(k_1 - 1) + 1$, then ρ_ℓ is irreducible;*
2. *$\overline{\rho}_\ell$ is irreducible for all but finitely many such primes.*

Theorem (W.)

For 100% of primes ℓ, ρ_ℓ is nice.

Irreducibility and modularity

Sketch proof for modular forms.

If $f \in S_k(N,\epsilon) \leftrightarrow \rho_\ell$ and ρ_ℓ is reducible then

$$\text{“kinda nice”} \implies \rho_\ell \simeq \psi \oplus \varphi\chi_\ell^{k-1}$$

1. **CFT:** ψ, φ correspond to Hecke (in this case Dirichlet) characters.
2. Write down another modular form

$$E_k^{\psi,\varphi} = a_0 + \sum_{n=1}^{\infty} \left(\sum_{d|n} \psi(\frac{n}{d})\varphi(d)d^{k-1} \right) q^n$$

where $a_p(E_k^{\psi,\varphi}) = \psi(p) + \varphi(p)p^{k-1} = \operatorname{Tr}\rho_\ell(\mathrm{Frob}_p) = a_p(f)$.

Strong multiplicity one: $f = E_k^{\psi,\varphi}$. □

Idea: use the modularity of the subrepresentations of ρ_ℓ to get a contradiction on the automorphic side.

Irreducibility and modularity II

Idea: use the modularity of the subrepresentations of ρ_ℓ to get a contradiction on the automorphic side.

Key lemma (W.)

Either ρ_ℓ is irreducible, or it splits as a direct sum of two-dimensional representations which are irreducible, regular and odd.

Theorem (Taylor)

If ℓ is sufficiently large, and $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ is an irreducible, regular, odd and nice Galois representation, then ρ is potentially modular.

- If ρ_ℓ is reducible then $\rho_\ell \simeq \sigma_1 \oplus \sigma_2$.
- If ρ_ℓ is also nice, find automorphic representations π_1, π_2 of $\mathrm{GL}_2(\mathbf{A}_K)$ corresponding to $\sigma_1|_K$, $\sigma_2|_K$.
- Apply a standard L-functions argument.

Irreducibility in general

Conjecture

If π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ then ρ_ℓ is irreducible for all primes.

Known results:

- $n = 2$: Ribet
- $n = 3$: Blasius-Rogawski if K totally real, π essentially self dual

Partial results:

- (Barnet-Lamb–Gee–Geraghty–Taylor) if K is CM and π is "extremely regular", then ρ_ℓ is irreducible for 100% of primes.

Thank you for listening!