Images of Galois representations attached to low weight Siegel modular forms

Ariel Weiss

University of Sheffield

a.weiss@sheffield.ac.uk

Explicit and computational approaches to Galois representations
4th July 2018

The classical case

- $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(N, \epsilon)$ normalised Hecke eigenform, $k \geq 2$
- Associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

unramified for all $p \nmid \ell N$ with

$$\operatorname{Tr} \rho_\ell(\mathrm{Frob}_p) = a_p, \qquad \det \rho_\ell = \epsilon \chi_\ell^{k-1}$$

- Associated mod ℓ Galois representation

$$\overline{\rho}_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$$

When are ρ_ℓ and $\overline{\rho}_\ell$ irreducible?

Example: a reducible ℓ-adic Galois representation

$$G_{12}(z) = \frac{691}{65520} + \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \rightsquigarrow \rho_\ell \cong \mathbf{1} \oplus \chi_\ell^{11}$$

Theorem (Ribet)

If f is cuspidal, then:

1. *ρ_ℓ is irreducible for all ℓ;*
2. *$\overline{\rho}_\ell$ is irreducible for all but finitely many ℓ;*

Example: a reducible mod ℓ Galois representation

$$\Delta(z) = 1 + \sum_{n \geq 2} \tau(n) q^n \rightsquigarrow \overline{\rho}_{691} \cong \mathbf{1} \oplus \overline{\chi}_{691}^{11}$$

Theorem (Ribet, Momose)

If f is not CM, then the image of ρ_ℓ is as large as possible for all but finitely many ℓ.

Genus 2 Siegel modular forms

"Cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_\mathbf{Q})$ + conditions at ∞"

- has weights (k_1, k_2), $k_1 \geq k_2 \geq 2$
- has a level N
- has a character ϵ
- has Hecke operators T_p and Hecke eigenvalues a_p

4 types of cuspidal Siegel modular form:

- General
- Theta lifts/Automorphic inductions
- Saito-Kurokawa/CAP
- Yoshida/endoscopic

} reducible Galois representations (Saito-Kurokawa/CAP and Yoshida/endoscopic)

High weight: $k_2 > 2$
Low weight: $k_2 = 2$

The high weight case: $k_2 > 2$

- Associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$$

unramified for all $p \nmid \ell N$ with

$$\mathrm{Tr}\,\rho_\ell(\mathrm{Frob}_p) = a_p, \qquad \mathrm{sim}\ \rho_\ell = \epsilon\chi_\ell^{k_1+k_2-3}$$

- Associated mod ℓ Galois representation $\overline{\rho}_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{F}}_\ell)$
- ρ_ℓ is de Rhamkinda nice for all ℓ, and is crystallinenice if $\ell \nmid N$
- The Hecke eigenvalues satisfy the generalised Ramanujan conjecture

Theorem

1. *(Ramakrishnan) If ρ_ℓ is nice and $\ell > 2(k_1 + k_2 - 3) + 1$, then ρ_ℓ is irreducible;*
2. *(BLGGT) $\overline{\rho}_\ell$ is irreducible for 100% of primes.*
3. *(Dieulefait-Zenteno) The image of $\overline{\rho}_\ell$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ 100% of primes.*

The low weight case: $k_2 = 2$

- Associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$$

unramified for all $p \nmid \ell N$ with

$$\mathrm{Tr}\,\rho_\ell(\mathrm{Frob}_p) = a_p, \qquad \mathrm{sim}\,\rho_\ell = \epsilon\chi_\ell^{k_1-1}$$

- Associated mod ℓ Galois representation $\overline{\rho}_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{F}}_\ell)$

Theorem (W.)

1. *If ρ_ℓ is nice and $\ell > 2(k_1 - 1) + 1$, then ρ_ℓ is irreducible;*
2. *$\overline{\rho}_\ell$ is irreducible for all but finitely many such primes;*
3. *The image of $\overline{\rho}_\ell$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for all but finitely many such primes.*

A theoretically checkable condition

Theorem (Jorza)

If $\ell \nmid N$ and the ℓ-th Hecke polynomial has distinct roots, then ρ_ℓ is nice.

Corollary

If $\ell > (2k_1 - 1) + 1$, $\ell \nmid N$, and the ℓ-th Hecke polynomial has distinct roots, then ρ_ℓ is irreducible.

Theorem (W.)

The ℓ-th Hecke polynomial has distinct roots for 100% of primes. Hence, ρ_ℓ is nice for 100% of primes.

Irreducibility and modularity

Sketch proof for modular forms (Ribet).

If $f \in S_k(N, \epsilon) \leftrightarrow \rho_\ell$ and ρ_ℓ is reducible then

$$\text{Kinda nice} \implies \rho_\ell \simeq \psi \oplus \varphi\chi_\ell^{k-1}$$

1. **CFT:** ψ, φ correspond to Hecke (in this case Dirichlet) characters.
2. Get an equality of partial L-functions

$$L^*(f \otimes \psi^{-1}, s) = \zeta^*(s)L^*(\varphi\psi^{-1}, s + k - 1);$$

3. The RHS has a pole at $s = 1$, but the LHS is holomorphic.

□

Idea: use the modularity of the subrepresentations of ρ_ℓ to get a contradiction on the automorphic side.

Irreducibility and modularity II

Idea: use the modularity of the subrepresentations of ρ_ℓ to get a contradiction on the automorphic side.

Key lemma (W.)

Either ρ_ℓ is irreducible, or it splits as a direct sum of two-dimensional representations which are irreducible, regular and odd.

Theorem (Taylor)

If ℓ is sufficiently large, and $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ is irreducible, regular, crystalline and odd, then ρ is potentially modular.

- If ρ_ℓ is reducible then $\rho_\ell \simeq \sigma_1 \oplus \sigma_2$.
- If ρ_ℓ is also crystalline, find automorphic representations π_1, π_2 of $\mathrm{GL}_2(\mathbf{A}_K)$ corresponding to $\sigma_1|_K$, $\sigma_2|_K$.
- Apply a standard L-functions argument.

Irreducibility in general

Conjecture

If π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ then ρ_ℓ is irreducible for all primes.

Known results:

- $n = 2$: Ribet
- $n = 3$: Blasius-Rogawski if K totally real, π essentially self dual

Partial results:

- (Barnet-Lamb–Gee–Geraghty–Taylor) if K is CM and π is "extremely regular", then ρ_ℓ is irreducible for 100% of primes.

Thank you for listening!