Galois theory 2018–19: Example Sheet 1

1. Solve the following cubics
   (a) $x^3 + 3x + 3$
   (b) $x^3 + 3x^2 + 3x + 12$

2. Let $f(x) = (x-1)(x-2)(x+3) = x^3 - 7x + 6$. Using the method given in lectures for solving a cubic, find the roots of $f(x) = 0$. Compare these with what you get from simply factorising the cubic. Clearly you should get the same answers. Can you prove it numerically (possibly using Maple)?

3. Find the resolvent cubic for the following quartics:
   (a) $x^4 - 8x^2 + 14$
   (b) $x^4 - 8x + 14$

4. Solve $x^4 - 2x^2 + 8x - 3 = 0$. (Hint: the resolvent cubic factorises.)

5. Let $f(x) = x^4 + 2x^2 + 4x + 2$. Explain why this polynomial is irreducible over $\mathbb{Q}$. Write down the resolvent cubic and factorise it. Hence solve the quartic.

6. Show that $\mathbb{Q}(\sqrt{3}, \sqrt{5}) = \mathbb{Q}(\sqrt{3} + \sqrt{5})$.

7. Remind yourself how the Euclidean algorithm works for polynomials, and explain why the highest common factor of two polynomials f, $g \in K[x]$, where K is a field, also has coefficients in K. (Maple knows how to do highest common factors of polynomials – try typing `?gcdex` for more information.)

8. Let $f \in K[x]$ be an irreducible polynomial, where $K \supseteq \mathbb{Q}$. Suppose that f has at least one repeated root in some field $L \supseteq K$. Derive a contradiction by taking the highest common factor of f and its derivative, and using Q7. (This shows that irreducible polynomials over fields of characteristic zero have distinct roots.)

   Can you see what might go wrong with your proof if $K = \mathbb{F}_p (= \mathbb{Z}/p\mathbb{Z})$, the finite field with p elements?

9. Show that the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$ is $x^4 - 10x^2 + 1$. What is the splitting field of this polynomial? What is the splitting field of $(x^2-2)(x^2-3)$? (This shows that different polynomials may have the same splitting field, and that reducible polynomials may have the same splitting field as irreducible ones.)