Galois theory 2018–19: Example Sheet 2

1. Determine the degree over $\mathbb{Q}$ of the splitting fields of the following polynomials:

   (a) $x^4 + 1$;

   (b) $x^4 + x^2 + 1$ (note that this is reducible);

   (c) $x^6 + 1$ (and so is this);

   (d) $x^6 + x^3 + 1$.

2. Show that the only $\mathbb{Q}$-automorphisms of $\mathbb{Q}(\sqrt{3})$ are given by the identity and by:

$$\begin{aligned} \mathbb{Q}(\sqrt{3}) &\rightarrow \mathbb{Q}(\sqrt{3}) \\ a + b\sqrt{3} &\mapsto a - b\sqrt{3} \end{aligned}$$

3. Let $K = \mathbb{Q}(\sqrt{2})$. What are the K-automorphisms of $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = K(\sqrt{3})$?

4. Let n be an odd prime, and $a \in \mathbb{Q}$. Show that there are no non-trivial $\mathbb{Q}$-automorphisms of $\mathbb{Q}(\sqrt[n]{a})$.

5. Let $L = \mathbb{Q}(\sqrt[3]{3}, \sqrt{-3})$. Give a well-known group to which $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic. What would the answer be if L were $\mathbb{Q}(\sqrt[3]{3}, \sqrt{-1})$? How about $L = \mathbb{Q}(\sqrt[4]{3}, \sqrt{-1})$?

6. Explicitly compute a polynomial over $\mathbb{Q}$ of degree six with $e^{\frac{3\pi i}{7}}$ as a root. Prove directly that this polynomial is irreducible over $\mathbb{Q}$, using Eisenstein's criterion.

   [*Hint: try a substitution* $x = y - 1$.]

7. Let ζ be a primitive 15th root of unity.

   (a) Using APR, show that an element of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is determined by its effect on ζ, and that ζ must be mapped to ζ^r for some r.

   (b) Write down all the possible r which may occur.

   (c) Is $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ cyclic? If not, is it isomorphic to a direct product of cyclic groups, and if so, which?

8. Compute the cyclotomic polynomials λ_n for $n \leq 16$ using the recursive method.

9. What is $\lambda_{200}(x)$?

10. (a) Show that if $n > 1$ is odd, $\lambda_{2n}(x) = \lambda_n(-x)$.

    (b) Show that if $m|n$, then the degree of $\lambda_{mn}(x)$ is the same as the degree of $\lambda_n(x^m)$. Prove also that these two polynomials have the same roots, and deduce that $\lambda_{mn}(x) = \lambda_n(x^m)$.

    (c) (Use Maple.) What is the smallest n such that λ_n has a coefficient different from 0 or ± 1? Which powers of x have coefficients different from 0 or ± 1?

    [*Hint: in (a), show that* ζ *is a primitive* $2n$*th root of unity if and only if* $-\zeta$ *is a primitive* n*th root of unity; (b) may be done similarly. In (c), you will be able to save yourself some work by thinking about what is implied by (a) and (b).*]