Galois theory 2018–19: Example Sheet 3

1. Let $K = \mathbb{Q}(i)$. You are given that $[K(\sqrt[4]{2}) : K] = 4$. Prove that $\mathrm{Gal}(K(\sqrt[4]{2})/K) \cong C_4$, the cyclic group of 4 elements.

2. Which of the following extensions are Galois? When they are Galois, say whether the Galois groups are cyclic or not.

   (a) $K = \mathbb{Q}$, $L = K(e^{\frac{2\pi i}{5}})$

   (b) $K = \mathbb{Q}$, $L = K(\sqrt[5]{3})$

   (c) $K = \mathbb{Q}(e^{\frac{2\pi i}{5}})$, $L = K(\sqrt[5]{3})$

3. Let $\zeta = e^{\frac{2\pi i}{5}}$ and put $\beta = \zeta + \frac{1}{\zeta}$. Given that $\zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$, show that $\beta = \frac{-1+\sqrt{5}}{2}$, and deduce that $\sqrt{5} \in \mathbb{Q}(\zeta)$.

   Draw the subfield and subgroup lattices for the field extension $\mathbb{Q}(e^{\frac{2\pi i}{5}})/\mathbb{Q}$.

4. Let $\zeta = e^{\frac{2\pi i}{11}}$, and put $\beta = \zeta + \frac{1}{\zeta} = 2\cos\left(\frac{2\pi}{11}\right)$ (by de Moivre's Theorem). Put $\gamma = \zeta + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^9$, and recall that

   $$\lambda_{11}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

   has roots $\zeta, \zeta^2, \ldots, \zeta^{10} = \zeta^{-1}$.

   (a) Explain why β satisfies a quintic equation over $\mathbb{Q}$, and write it down.

   (b) Expand γ^2 in powers of ζ, and hence deduce that $\gamma^2 + \gamma + 3 = 0$. Show that $\mathbb{Q}(\sqrt{-11}) \subseteq \mathbb{Q}(\zeta)$.

   (c) Use a result in the course to show that $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is cyclic with 10 elements.

   (d) Thus $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = \{1, \theta, \ldots, \theta^9\}$, where θ is some automorphism of order 10. Recall that any subgroup of a cyclic group is again cyclic. Write down the subgroups of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, and draw the subgroup lattice.

   (e) Using the earlier parts of the question, draw the subfield lattice.

5. Let $f(x) = x^4 + x^2 + 4$. Let L be a splitting field for f over $\mathbb{Q}$. Let $\alpha = \sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{-15}}$. You are given that $[L : \mathbb{Q}] = 4$.

   (a) Show that the roots of f are $\pm\alpha$, $\pm\frac{2}{\alpha}$.

   (b) Show that $L = \mathbb{Q}(\alpha)$.

   (c) Compute $\mathrm{Gal}(L/\mathbb{Q})$. What well-known group is it?

6. Let $f = x^4 + 8x^2 - 2 \in \mathbb{Q}[x]$, and let M be the splitting field for f over $\mathbb{Q}$. Let $\alpha = \sqrt{3\sqrt{2} - 4}$. It is given that $M = \mathbb{Q}(\alpha, i\sqrt{2})$ and that $[M : \mathbb{Q}] = 8$.

   (a) Show that f has roots $\pm\alpha, \pm\frac{i\sqrt{2}}{\alpha}$.

   (b) Compute the elements of $\mathrm{Gal}(M/\mathbb{Q})$ and write down in a table their effect on α and $i\sqrt{2}$.

   (c) Show that there exist automorphisms $\phi, \psi \in \mathrm{Gal}(M/\mathbb{Q})$ such that ϕ has order 4, ψ has order 2, and $\mathrm{Gal}(M/\mathbb{Q}) = \langle \phi, \psi \rangle$.

   (d) Write $\psi\phi\psi^{-1}$ in the form $\phi^i\psi^j$. To what well-known group is $\mathrm{Gal}(M/\mathbb{Q})$ isomorphic?