**Galois theory 2018–19: Example Sheet 5**

1. Suppose $f(x) = x^3+ax+b$. If $f$ has roots $\alpha$, $\beta$ and $\gamma$, then recall that its discriminant $D(f)$ is $(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2$. Let $M$ denote the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{pmatrix}.$$

   (a) Define $\Delta(f) = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$. Show that $\Delta(f) = \det M$.

   (b) Thus $D(f) = \Delta(f)^2$. Given that $\det M = \det M^t$, deduce that $D(f) = \det(MM^t)$.

   (c) Write $S_i = \alpha^i + \beta^i + \gamma^i$. Show that

$$MM^t = \begin{pmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix}.$$

   (d) Clearly $S_0 = 3$ and $S_1 = 0$ (as $S_1$ is the sum of the roots, which is zero as the coefficient of $x^2$ in $f$ is zero). Show that $S_2 = -2a$ by an explicit computation.

   (e) As $\alpha$, $\beta$ and $\gamma$ are roots of $f$, we have

$$\begin{aligned} \alpha^3 + a\alpha + b &= 0 \\ \beta^3 + a\beta + b &= 0 \\ \gamma^3 + a\gamma + b &= 0 \end{aligned}$$

   By summing these three, find $S_3$ in terms of $S_0$ and $S_1$. Similarly, multiplying these equations by $\alpha$, $\beta$ and $\gamma$ respectively, find $S_4$ in terms of $S_1$ and $S_2$. Compute the values of $S_3$ and $S_4$ in terms of $a$ and $b$.

   (f) Combining all the above, show that $D(f) = -(4a^3 + 27b^2)$.

2. Use the method of Q1 to show that the discriminant of the polynomial $x^4+px+q = 0$ is $256q^3 - 27p^4$.

3. You are given that a quartic polynomial has the four roots $\sqrt{2}+\sqrt{5}$, $\sqrt{2}-\sqrt{5}$, $-\sqrt{2}+\sqrt{5}$ and $-\sqrt{2}-\sqrt{5}$. What is its discriminant? What is the Galois group of the polynomial?

4. What is the Galois group of the following cubics:

   (a) $x^3 - 3x + 1$

   (b) $x^3 + 3x + 1$

5. What is the Galois group of the following quartics:

   (a) $x^4 + 8x + 12$

   (b) $x^4 + 8x - 12$

6. Which of the following polynomials are soluble by radicals over $\mathbf{Q}$?

(a) $2x^5 - 10x^2 - 5x$

(b) $2x^5 - 10x - 5$

(c) $2x^6 - 10x^2 - 5$

(d) $5x^5 + 10x^4 - 2$

(e) $x^5 - 405x + 3$

(f) $4x^{10} - 40x^6 - 20x^5 + 100x^2 + 100x + 25$

Three of these polynomials have the same splitting field. Which are they?

7. Let $f(x) = 30x^7 - 70x^6 - 42x^5 + 105x^4 - 21$. Show that its Galois group over $\mathbf{Q}$ is $S_7$.

8. Find an irreducible polynomial of degree 6 over $\mathbf{Q}$ with 4 real roots, but whose Galois group over $\mathbf{Q}$ is not $S_6$.