SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2003–04 **2 hours 30 minutes**

Galois Theory

Answer ***Question 1*** *and three other questions. You are advised* ***not*** *to answer more than three of the questions 2 to 5: if you do, only your best three will be counted.*

1 (i) Let $K \subseteq L$ be a field extension. What does it mean for $\theta : L \longrightarrow L$ to be a K-*automorphism* of L? ***(3 marks)***

(ii) For each of the following polynomials, give their splitting fields over $\mathbb{Q}$, and the degrees over $\mathbb{Q}$ of these extensions:

(a) $x^4 - 2$; ***(3 marks)***

(b) $x^4 - 4$. ***(3 marks)***

(iii) If $K \subseteq L$ is a field extension, define its *Galois group*, and say what it means, in terms of this group, for $K \subseteq L$ to be a *Galois* extension. Give an equivalent formulation involving splitting fields. ***(3 marks)***

(iv) Let $K = \mathbb{Q}(\sqrt{2})$ and $L = \mathbb{Q}(\sqrt{2}, \sqrt{7})$.

(a) Prove that $L = \mathbb{Q}(\sqrt{2} + \sqrt{7})$. ***(2 marks)***

(b) It is given that $\mathrm{Gal}(L/\mathbb{Q})$ has order 4. Write down the effect of each of the four elements of the group on $\sqrt{2}$ and $\sqrt{7}$. ***(3 marks)***

(c) Which of these four elements lie in $\mathrm{Gal}(L/K)$ and why? ***(2 marks)***

(v) Let ζ be a primitive 20th root of unity.

(a) What is the order of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$? ***(1 mark)***

(b) Is $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ cyclic? ***(2 marks)***

(vi) Suppose that $K \subseteq M \subseteq L$ are field extensions and that the extension $K \subseteq L$ is Galois. When is L/M Galois? When is M/K Galois? Give an example of fields $K \subseteq M \subseteq L$ with L/K Galois, but with M/K not Galois. ***(3 marks)***

2 Let $f(x)$ denote the quartic $x^4 - 10x^2 + 12x - 2$ (regarded as a polynomial over $\mathbb{Q}$).

(i) Write down its resolvent cubic. Show that 6 is a root of this cubic. Find the remaining two roots. ***(4 marks)***

(ii) Briefly explain the algorithm to solve a quartic equation, and illustrate the method by finding the solutions to the equation $f(x) = 0$. ***(8 marks)***

(iii) What is the splitting field of f? If K denotes this splitting field, what is $[K : \mathbb{Q}]$? Why is $K/\mathbb{Q}$ Galois, and how many elements does $\mathrm{Gal}(K/\mathbb{Q})$ have? ***(4 marks)***

(iv) Let α_1, α_2, α_3 and α_4 denote the four roots of f. Write down the $\mathbb{Q}$-automorphisms of K, and determine the permutation of the roots corresponding to each element of the Galois group. What is the structure of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$? ***(5 marks)***

(v) Define the *discriminant* of a quartic equation in terms of the roots of the quartic. Using the expressions for the roots that you found above, compute the discriminant of the quartic f. Explain how this helps you to check your answer to the last part of (iv). ***(4 marks)***

3 (i) Let $n \geq 1$ be an integer. Define the nth *cyclotomic polynomial* $\lambda_n(x)$. What is its degree? ***(2 marks)***

(ii) Show that

$$x^n - 1 = \prod_{d|n} \lambda_d(x).$$

(2 marks)

(iii) Hence prove that $\lambda_n(x)$ is monic with integer coefficients. ***(7 marks)***

(iv) Write down $\lambda_5(x)$. Show that $\lambda_5(x+1)$ is irreducible. ***(2 marks)***

(v) Prove that $\lambda_{50}(x) = x^{20} - x^{15} + x^{10} - x^5 + 1$. ***(7 marks)***

(vi) Using Eisenstein's criterion with $p = 5$ or otherwise, show that $\lambda_{50}(x-1)$ is irreducible.

[*You may like to do some of your working modulo 5, and to use the relation* $(x-1)^5 \equiv x^5 - 1$ *(mod 5).*] ***(5 marks)***

4 Let $f = x^4 + 6x^2 - 2 \in \mathbb{Q}[x]$ and let M be the splitting field of f over $\mathbb{Q}$. Let $\alpha = \sqrt{\sqrt{11} - 3}$. It is given that $M = \mathbb{Q}(\alpha, i\sqrt{2})$ and that $[M : \mathbb{Q}] = 8$.

(i) Show that f has roots $\pm\alpha, \pm i\sqrt{2}/\alpha$. ***(2 marks)***

(ii) Compute the elements of $\mathrm{Gal}(M/\mathbb{Q})$ and write them down in a table showing the effect of each element on α and the effect of each element on $i\sqrt{2}$. ***(9 marks)***

(iii) Show that there exist automorphisms $\phi, \psi \in \mathrm{Gal}(M/\mathbb{Q})$ such that ϕ has order 4, ψ has order 2, and $\mathrm{Gal}(M/\mathbb{Q}) = \langle \phi, \psi \rangle$. ***(5 marks)***

(iv) Write $\psi\phi\psi^{-1}$ in the form $\phi^i\psi^j$. To what well known group is $\mathrm{Gal}(M/\mathbb{Q})$ isomorphic? ***(3 marks)***

(v) Using the Galois correspondence, find $[\mathbb{Q}(\alpha + \dfrac{i\sqrt{2}}{\alpha}) : \mathbb{Q}]$. ***(6 marks)***

5 (i) Let K be a field and let $f \in K[x]$ be a non-constant polynomial. What does it mean to say that the equation $f(x) = 0$ is *soluble by radicals over* K? ***(3 marks)***

(ii) Let p be a prime number. Let G be a transitive subgroup of S_p (i.e., for any $a, b \in \{1, \ldots, p\}$ there exists $\theta \in G$ with $\theta(a) = b$) which contains a transposition. Prove that $G = S_p$. ***(11 marks)***

(iii) Show that the Galois group of $x^5 - 4x + 1$ over $\mathbb{Q}$ is isomorphic to S_5, and briefly explain why this implies that the polynomial is not soluble by radicals over $\mathbb{Q}$.

Deduce that $x^5 - 4x^4 + 1 = 0$ is not soluble by radicals. ***(11 marks)***

End of Question Paper