These solutions are designed for the checker and the external.
They will be amplified before any distribution to students in future years.

1 (i) Standard type, unfamiliar case Writing $x = u + v$ we have

$$(u+v)^3 - 3(u+v) + 4 = 0, \tag{1}$$

or

$$u^3 + v^3 + 3uv(u+v) - 3(u+v) + 4 = 0.$$

Collecting terms in $u + v$ and those without, a solution to the following will give a solution to (**??**):

$$u^3 + v^3 + 4 = 0, \qquad 3uv - 3 = 0. \qquad \boxed{\mathbf{4}}$$

Substitute $v = \frac{1}{u}$ into the first equation, to get

$$u^3 + \frac{1}{u^3} + 4 = 0 \text{ and thus } u^6 + 4u^3 + 1 = 0.$$

So $u^3 = \frac{1}{2}(-4 \pm 2\sqrt{3}) = -2 \pm \sqrt{3}$. Choose the negative root. $\boxed{\mathbf{3}}$

Write $\alpha = \sqrt[3]{-2-\sqrt{3}}$ for the (negative) real cube root of $-2-\sqrt{3}$.

Taking $u = \alpha$ we get $x = u + \frac{1}{u} = \alpha + \frac{1}{\alpha} \in \mathbb{R}$.

Taking $u = \omega\alpha$ we get $x = \omega\alpha + \frac{\omega^2}{\alpha}$.

Taking $u = \omega^2\alpha$ we get $x = \omega^2\alpha + \frac{\omega}{\alpha}$.

3 for correct solutions, 3 for awareness of pattern and identifying real root

(ii) Variation on bookwork, easy

Case $p = 2$, $q = 3$. $\boxed{3}$ Marks still awarded if only general case is done (correctly).

Clearly $\sqrt{p} + \sqrt{q} \in \mathbb{Q}(\sqrt{p}, \sqrt{q})$ so $\mathbb{Q}(\sqrt{p} + \sqrt{q}) \subseteq \mathbb{Q}(\sqrt{p}, \sqrt{q})$. $\boxed{1}$

Write $\alpha = \sqrt{p} + \sqrt{q}$. Then

$$\alpha^3 = (p + 3q)\sqrt{p} + (3p + q)\sqrt{q}$$

so

$$\alpha^3 - (p + 3q)\alpha = 2(p - q)\sqrt{q}.$$

Dividing through by $2(p - q)$ we have $\sqrt{q} \in \mathbb{Q}(\alpha)$.$\boxed{3}$

Likewise

$$\alpha^3 - (3p + q)\alpha = 2(q - p)\sqrt{p}$$

and dividing through by $2(q - p)$ we have $\sqrt{p} \in \mathbb{Q}(\alpha)$.$\boxed{1}$

So $\mathbb{Q}(\sqrt{p}, \sqrt{q}) \subseteq \mathbb{Q}(\sqrt{p} + \sqrt{q})$ and this completes the proof.

For the minimal polynomial, we first find $\alpha^2 = p + q + 2\sqrt{pq}$ as above and from this,

$$(\alpha^2 - (p + q))^2 = 4pq. \qquad (2)$$

Simplifying, α is a root of

$$x^4 - 2(p + q)x^2 + (p - q)^2 = 0. \qquad \boxed{2}$$

To obtain the other roots, note that (**??**) is also the square of (replacing α by x)

$$x^2 - (p + q) = -2\sqrt{pq},$$

and this has roots $\pm(\sqrt{p} - \sqrt{q})$.

So the roots of $x^4 - 2(p + q)x^2 + (p - q)^2 = 0$ are

$$\sqrt{p} + \sqrt{q}, \quad -\sqrt{p} - \sqrt{q}, \quad \sqrt{p} - \sqrt{q}, \quad -\sqrt{p} + \sqrt{q}. \qquad \boxed{2}$$

2(i) Standard types; (b) and (c) fairly difficult

(a) The roots of x^4+1 are the primitive 8th roots of unity:

$$\frac{1+i}{\sqrt{2}}, \quad \frac{1-i}{\sqrt{2}}, \quad \frac{-1+i}{\sqrt{2}}, \quad \frac{-1-i}{\sqrt{2}}. \qquad \boxed{1}$$

So the splitting field is $\mathbb{Q}(\frac{\pm 1 \pm i}{\sqrt{2}})$. $\boxed{1}$

Now $\sqrt{2} = \frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}$, so $\mathbb{Q}(\frac{\pm 1 \pm i}{\sqrt{2}}) = \mathbb{Q}(i, \sqrt{2})$. $\boxed{1}$

Lastly, $[\mathbb{Q}(i,\sqrt{2}) : \mathbb{Q}] = 4$. $\boxed{1}$

(b) Note that x^6+1 divides $x^{12}-1$, so its roots are necessarily 12th roots of unity. The roots of x^6+1 are the 6th roots of -1, namely

$$e^{\frac{\pi i}{6}}, \quad e^{\frac{3\pi i}{6}} = i, \quad e^{\frac{5\pi i}{6}}, \quad e^{\frac{7\pi i}{6}}, \quad e^{\frac{9\pi i}{6}} = -i, \quad e^{\frac{11\pi i}{6}}. \qquad \boxed{2}$$

Stated briefly, these are $\pm i$ and $\frac{\pm\sqrt{3}\pm i}{2}$.

It follows that the splitting field is $\mathbb{Q}(\pm i, \frac{\pm\sqrt{3}\pm i}{2}) = \mathbb{Q}(i, \sqrt{3})$. $\boxed{2}$

As in (a), $[\mathbb{Q}(i,\sqrt{3}) : \mathbb{Q}] = 4$. $\boxed{1}$

(c) Note that $x^9 - 1 = (x^3-1)(x^6+x^3+1)$, so the roots of x^6+x^3+1 are the primitive 9th roots of unity. $\boxed{1}$

If ζ is a primitive 9th root of unity, all other primitive 9th roots of unity are powers of ζ, so that the splitting field is just $\mathbb{Q}(\zeta)$. $\boxed{1}$

Its degree over $\mathbb{Q}$ is just the degree of the minimal polynomial of ζ.

Now x^6+x^3+1 itself is irreducible by shifted Eisenstein with $p=3$). $\boxed{3}$

So $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 6$. $\boxed{1}$

Any correct method acceptable.

(ii) Bookwork

Take S_n as acting on the set $P = \{1, \ldots, n\}$.

Define a relation $\sim$ on P by $i \sim j$ if and only if $i = j$ or $(i\ j) \in G$. $\boxed{\mathbf{1}}$

This $\sim$ is clearly reflexive and symmetric. $\boxed{\mathbf{1}}$ Further, if $i \sim j$ and $j \sim k$, then either $i = j$, $i = k$ or $j = k$ (in which case it is easy to see that $i \sim k$) or $(i\ k) = (i\ j)(j\ k)(i\ j) \in G$. So $\sim$ is an equivalence relation. $\boxed{\mathbf{2}}$

If $a \in P$, denote its equivalence class by $\overline{a}$. Let $b \in P$. As G is transitive, there exists $\theta \in G$ with $\theta(a) = b$. $\boxed{\mathbf{1}}$

Let $c \in \overline{a}$. Either $c = a$ or $(a\ c) \in G$. Consider $\theta(c)$. Either $\theta(c) = \theta(a)$ or $(\theta(a)\ \theta(c)) = \theta(a\ c)\theta^{-1} \in G$. $\boxed{\mathbf{1}}$

In either case, $\theta(c) \sim b$. It follows that θ gives a bijection from the equivalence class of a to the equivalence class of b. $\boxed{\mathbf{1}}$ So $|\overline{a}| = |\overline{b}|$. But P is partitioned into equivalence classes, and $|S| = n$ is prime, so either all classes have 1 element each, or there is only one class with n elements. $\boxed{\mathbf{1}}$ The first case is ruled out because G contains a transposition. $\boxed{\mathbf{1}}$ Thus all transpositions $(i\ j)$ lie in G. But S_n is generated by the transpositions. $\boxed{\mathbf{1}}$

Q3(i) Standard type

(a) ξ is a primitive 11th root of 1 so $\sum_{k=0}^{10} \xi^k = 0$. Dividing through by ξ^5 we get

$$\xi^{-1} + \xi^{-2} + \xi^{-3} + \xi^{-4} + \xi^{-5} + \xi^5 + \xi^4 + \xi^3 + \xi^2 + \xi + 1 = 0.$$

Calculating powers of $\beta = \xi + \xi^{-1}$, we get $\beta^2 = \xi^2 + 2 + \xi^{-2}$ and $\beta^3 = \xi^3 + 3\xi + 3\xi^{-1} + \xi^{-3}$ and so on. Combining these, we deduce that

$$\beta^5 + \beta^4 - 4\beta^3 - 3\beta^2 + 3\beta + 1 = 0. \qquad \boxed{\mathbf{4}}$$

(b)

$$\begin{aligned}
\gamma^2 &= \xi^2 + \xi^8 + \xi^7 + \xi^{10} + \xi^6 + \cdots \\
&\qquad \cdots + 2(\xi^5 + \xi^{10} + \xi^6 + \xi^4 + \xi^2 + \xi^9 + \xi^7 + \xi^3 + \xi + \xi^8) \\
&= (-1 - \xi - \xi^3 - \xi^4 - \xi^5 - \xi^9) + 2(-1) \\
&= -3 - \gamma \qquad \boxed{\mathbf{3}}
\end{aligned}$$

so that $\gamma^2 + \gamma + 3 = 0$. Since γ is a root of $x^2 + x + 3 = 0$, $\gamma = \frac{-1 \pm \sqrt{-11}}{2}$. Also, $\gamma \in \mathbb{Q}(\xi)$. We conclude that $\sqrt{-11} \in \mathbb{Q}(\xi)$, and thus that $\mathbb{Q}(\sqrt{-11}) \subseteq \mathbb{Q}(\xi)$. $\boxed{\mathbf{1}}$

(c) Theorem: If ξ is a primitive nth root of unity, then

$$\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \cong U(\mathbb{Z}_n),$$

the multiplicative group of integers modulo n and prime to n. $\boxed{\mathbf{2}}$

Theorem: For n a prime, $U(\mathbb{Z}_n)$ is the cyclic group of order $n-1$. $\boxed{\mathbf{1}}$

(d) For example, take θ to be $\xi \mapsto \xi^2$. $\boxed{\mathbf{1}}$

Since $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ is cyclic, we can then write $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) = \{1, \theta, \ldots, \theta^9\}$.

The subgroups of a cyclic group are cyclic and for C_{10}, have orders 1, 2, 5 and 10. $\boxed{\mathbf{1}}$

For order 2 the subgroup is $\{1, \theta^5\}$. $\boxed{\mathbf{1}}$

For order 5 the subgroup is $\{1, \theta^2, \theta^4, \theta^6, \theta^8\}$. $\boxed{\mathbf{1}}$

(ii) Standard type

(a) As $\beta = \xi + \frac{1}{\xi}$, it follows that $\beta\xi = \xi^2 + 1$, so that

$$\xi^2 - \beta\xi + 1 = 0.$$

This is a quadratic equation with coefficients in $\mathbb{Q}(\beta)$. So the minimal polynomial for ξ over $\mathbb{Q}(\beta)$ has degree at most 2. $\boxed{\mathbf{1}}$

So $[\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] \leqslant 2$. Clearly $\mathbb{Q}(\beta, \xi) = \mathbb{Q}(\xi)$. $\boxed{\mathbf{1}}$

(b) As ξ is a root of unity, it has modulus 1. So $|\xi|^2 = \xi\overline{\xi} = 1$. Thus $\frac{1}{\xi} = \overline{\xi}$ and $\beta = \xi + \overline{\xi}$, the sum of a complex number and its conjugate. So $\beta \in \mathbb{R}$, $\boxed{\mathbf{2}}$ and $\mathbb{Q}(\beta) \subseteq \mathbb{R}$. But as $n \geqslant 3$, $\xi \notin \mathbb{R}$, $\boxed{\mathbf{1}}$ so that $\xi \notin \mathbb{Q}(\beta)$.

(c) Since $\xi \notin \mathbb{Q}(\beta)$, it follows that $[\mathbb{Q}(\xi) : \mathbb{Q}(\beta)] > 1$. $\boxed{\mathbf{1}}$

Combining this with the result of (b), we get that $[\mathbb{Q}(\xi) : \mathbb{Q}(\beta)] = 2$. $\boxed{\mathbf{1}}$ By the Tower of Fields result

$$[\mathbb{Q}(\xi) : \mathbb{Q}] = [\mathbb{Q}(\xi) : \mathbb{Q}(\beta)].[\mathbb{Q}(\beta) : \mathbb{Q}]. \qquad \boxed{\mathbf{2}}$$

But $[\mathbb{Q}(\xi) : \mathbb{Q}] = \varphi(n)$, so $[\mathbb{Q}(\beta) : \mathbb{Q}] = \frac{1}{2}\varphi(n)$. $\boxed{\mathbf{1}}$

4(i) Bookwork Let L be a Galois extension of K, and let $G = \mathrm{Gal}(L/K)$. There is a bijection from

$$\mathscr{S} := \{\text{subgroups of } G\}$$

to

$$\mathscr{F} := \{\text{intermediate fields } K \subseteq M \subseteq L\}$$

given by $H \mapsto L^H$ with inverse $M \mapsto \mathrm{Gal}(L/M)$. $\boxed{2}$

Moreover, the correspondence is inclusion reversing, that is,

$$H_1 \supseteq H_2 \iff L^{H_1} \subseteq L^{H_2}, \qquad \boxed{1}$$

and indexes equal degrees, that is,

$$\frac{|H_1|}{|H_2|} = [L^{H_2} : L^{H_1}]. \qquad \boxed{2}$$

Finally, normal subgroups of G correspond to intermediate fields $K \subseteq M \subseteq L$ such that M/K is Galois. $\boxed{1}$

4(ii) Bookwork A group G is *soluble* if it has a chain of subgroups

$$G = G_0 > G_1 > \cdots > G_n = \{1\}$$

with each $G_{i+1} \triangleleft G_i$ and each G_i/G_{i+1} abelian. $\boxed{3}$

4(iii)(a) Standard type; seen in different context

See next page for diagram: $\boxed{\textbf{4 for subgroups, 4 for inclusions}}$

Notice that there is a chain of normal subgroups:

$$G > \langle R \rangle > \{1, R^2\} > \{1\}. \qquad \boxed{1}$$

To check that $\langle R \rangle$ is normal in G, it is only necessary to observe that $FR^kF = (FRF)^k = R^{-k}$. $\boxed{1}$

Clearly $\{1, R^2\}$ is normal in $\langle R \rangle$. $\boxed{1}$

The quotients are all of order two, and are therefore abelian. So G is soluble. $\boxed{1}$

$\boxed{\textbf{At most 2 marks if solubility is deduced from that of } S_4}$

4(iii)(b) Slightly nonstandard

The subgroups that are not normal are

$$\{1, F\}, \quad \{1, RF\}, \quad \{1, R^2F\}, \quad \{1, R^3F\}. \qquad \boxed{2}$$

For the first, $R^{-1}FR = R^3FR = R^3(FRF)F = R^2F$ is not in the subgroup.

For the second, $F(RF)F = R^3F$ is not in the subgroup.

For the third, $F(R^2F)F = FR^2$ is not in the subgroup.

For the last, $F(R^3F)F = FR^3$ is not in the subgroup.

$\boxed{\tfrac{1}{2}\textbf{ each. (Any correct method acceptable.)}}$