Irreducibility of Galois representations attached to low weight Siegel modular forms

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p-adic modular forms and Galois representations
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The classical case

- $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(N, \epsilon)$ normalised Hecke eigenform, $k \geq 2$
- Associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

  unramified for all $p \nmid \ell N$ with

$$\mathrm{Tr}\, \rho_\ell(\mathrm{Frob}_p) = a_p, \qquad \det \rho_\ell = \epsilon \chi_\ell^{k-1}$$

- Associated mod ℓ Galois representation

$$\overline{\rho}_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$$

When are ρ_ℓ and $\overline{\rho}_\ell$ irreducible?

Example: a reducible ℓ-adic Galois representation

$$G_{12}(z) = \frac{691}{65520} + \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \rightsquigarrow \rho_\ell \cong \mathbf{1} \oplus \chi_\ell^{11}$$

Theorem (Ribet, '70s)

If f is cuspidal, then:

1. ρ_ℓ *is irreducible for all* ℓ*;*
2. $\overline{\rho}_\ell$ *is irreducible for all but finitely many* ℓ*;*

Example: a reducible mod ℓ Galois representation

$$\Delta(z) = 1 + \sum_{n \geq 2} \tau(n) q^n \rightsquigarrow \overline{\rho}_{691} \cong \mathbf{1} \oplus \overline{\chi}_{691}^{11}$$

Genus 2 Siegel modular forms

Cuspidal automorphic representation π of $\mathrm{GSp}_4(\mathbf{A}_{\mathbf{Q}})$, such that π_∞ a holomorphic (limit of) discrete series.

- has weights (k_1, k_2), $k_1 \geq k_2 \geq 2$
- has a level N
- has a character ϵ
- has Hecke operators T_p and Hecke eigenvalues a_p

High weight: $k_2 > 2$ Low weight: $k_2 = 2$

Arthur's classification: 5 classes of cuspidal Siegel modular form:

- General
- Yoshida
- Saito-Kurokawa
- Soudry
- Howe–Piatetski-Shapiro

} reducible Galois representations (Yoshida, Saito-Kurokawa, Soudry, Howe–Piatetski-Shapiro)

The high weight case: $k_2 > 2$

- Associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$$

unramified for all $p \nmid \ell N$ with

$$\mathrm{Tr}\,\rho_\ell(\mathrm{Frob}_p) = a_p, \qquad \mathrm{sim}\ \rho_\ell = \epsilon\chi_\ell^{k_1+k_2-3}$$

- Associated mod ℓ Galois representation $\overline{\rho}_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{F}}_\ell)$
- ρ_ℓ is de Rham for all ℓ and crystalline if $\ell \nmid N$
- Hodge–Tate weights $\{0, k_2 - 2, k_1 - 1, k_1 + k_2 - 3\}$
- The Hecke eigenvalues satisfy the generalised Ramanujan conjecture

Theorem

1. *(Ramakrishnan 2013) If ρ_ℓ is crystalline and if $\ell > 2(k_1 + k_2 - 3) + 1$, then ρ_ℓ is irreducible.*
2. *(BLGGT 2014) $\overline{\rho}_\ell$ is irreducible for 100% of primes.*

The low weight case: $k_2 = 2$

- Associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$$

unramified for all $p \nmid \ell N$ with

$$\mathrm{Tr}\,\rho_\ell(\mathrm{Frob}_p) = a_p, \qquad \mathrm{sim}\,\rho_\ell = \epsilon\chi_\ell^{k_1-1}$$

- Associated mod ℓ Galois representation $\overline{\rho}_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{F}}_\ell)$
- Hodge–Tate–Sen weights $\{0, 0, k_1 - 1, k_1 - 1\}$

Theorem (W.)

1. *If ρ_ℓ is crystalline and $\ell > 2(k_1 - 1) + 1$, then ρ_ℓ is irreducible.*
2. *$\overline{\rho}_\ell$ is irreducible for all but finitely many such primes.*

Theorem (W.)

For 100% of primes ℓ, ρ_ℓ is crystalline.

Irreducibility and modularity

Proof for elliptic modular forms (Ribet).

1. Suppose that $f \in S_k(N, \epsilon) \leftrightarrow \rho_\ell$ and that ρ_ℓ is reducible. Then

$$\rho_\ell \text{ has Hodge–Tate weights } \{0, k-1\} \implies \rho_\ell \simeq \psi_1 \oplus \psi_2 \chi_\ell^{k-1},$$

with ψ_1, ψ_2 finite order characters.
Decompose ρ_ℓ into 'nice' subrepresentations.

2. By class field theory, ψ_1, ψ_2 correspond to Dirichlet characters.
Apply a modularity theorem.

3. Get an equality of partial L-functions

$$L^*(f \otimes \psi_1^{-1}, s) = \zeta^*(s) L^*(\psi_2 \psi_1^{-1}, s + k - 1);$$

The RHS has a pole at $s = 1$, but the LHS is holomorphic.
Use automorphic arguments to reach a contradiction.

□

Irreducibility and modularity II

Theorem (W.)

If ρ_ℓ is crystalline and $\ell > 2(k_1 - 1) + 1$, then ρ_ℓ is irreducible.

1. Decompose ρ_ℓ into 'nice' subrepresentations.

Lemma (W.)

For *all* ℓ, either ρ_ℓ is irreducible, or it splits as a direct sum of distinct two-dimensional representations that are irreducible, regular and odd.

2. Apply a modularity theorem.

Theorem (Taylor 2006)

If $\ell > 2(k_1 - 1) + 1$ and $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ is irreducible, regular, crystalline and odd, then ρ is potentially modular.

3. Use automorphic arguments to reach a contradiction.
   Apply a standard L-functions argument using Brauer induction.

Crystallinity

Theorem (Jorza 2012)

If $\ell \nmid N$ and the roots of the ℓ^{th} Hecke polynomial are distinct, then ρ_ℓ is crystalline.

Theorem (W.)

If π is not CM, then the roots of the ℓ^{th} Hecke polynomial are distinct for 100% of primes ℓ.

Proof:

- Decomposition lemma + density argument $\implies$ roots of the ℓ^{th} Hecke polynomial are distinct for positive density of primes.
- Irreducibility Theorem $\implies \rho_\ell$ is irreducible for a positive density of primes.
- Density argument again $\implies$ roots of the ℓ^{th} Hecke polynomial are distinct for 100% of primes.

Irreducibility in general

Conjecture

If π is an algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$, then ρ_ℓ is irreducible for all primes.

Known cases:

- $n = 2$: Ribet if K totally real
- $n = 3$: Blasius–Rogawski if K totally real, π polarisable

Partial results:

- BLGGT (2014): if K is CM and π is "extremely regular" and polarisable, then ρ_ℓ is irreducible for 100% of primes.
- Patrikis–Taylor (2015): if K is CM and π is regular and polarisable, then ρ_ℓ is irreducible for a positive density of primes.
- Xia (2018): if $n \leq 6$, K is CM and π is regular and polarisable, then ρ_ℓ is irreducible for 100% of primes.

Thank you for listening!