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Superconvergence of the local discontinuous Galerkin method for the linearized Korteweg–de Vries equation

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ABSTRACT

We study the superconvergence property of the local discontinuous Galerkin (LDG) method for solving the linearized Korteweg–de Vries (KdV) equation. We prove that, if the piecewise P^k polynomials with $k \geq 1$ are used, the LDG solution converges to a particular projection of the exact solution with the order $k + 3/2$, when the upwind flux is used for the convection term and the alternating flux is used for the dispersive term. Numerical examples are provided at the end to support the theoretical results.

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1. Introduction

In this paper, we consider the linearized Korteweg–de Vries (KdV) equation given by

$$u_t + \alpha u_x + \beta u_{xxx} = 0, \tag{1.1}$$

where α, β are both constants. We study the superconvergence of the local discontinuous Galerkin (LDG) solutions toward a particular projection of the exact solution.

The nonlinear KdV equation

$$u_t + \alpha u_x + \gamma u u_x + \beta u_{xxx} = 0, \tag{1.2}$$

with constants α, β, and γ, was first introduced by Dutch mathematicians Diederik Korteweg and Gustav de Vries. It was originally proposed with the purpose of modeling shallow water waves [1], and later has been found to model numerous other wave-like phenomena in nature, such as ion-acoustic waves and collisionless-plasma waves [2]. The Eq. (1.1) considered in this paper is a linearized model which is obtained by assuming $\gamma = 0$ in (1.2).

The numerical methods discussed here are the discontinuous Galerkin (DG) methods. They belong to a class of finite element methods using piecewise polynomial spaces for both the numerical solution and the test functions, and were originally devised to solve hyperbolic conservation laws with only first order spatial derivatives, e.g. [3–7]. They allow arbitrarily unstructured meshes, and have a compact stencil; moreover, they easily accommodate arbitrary *h-p* adaptivity. The DG methods were later generalized to the LDG methods by Cockburn and Shu to solve the convection–diffusion equation [8], motivated by successful numerical experiments from Bassi and Rebay for the compressible Navier–Stokes equations [9]. As a result, the LDG methods have been successfully applied to solve various partial differential equations (PDEs) containing higher-order derivatives. For KdV-type equations, generalized by (1.2), an LDG method was first developed in [10], in which a sub-optimal error estimate was provided for the linear problem (1.1). In [11], Xu and Shu proved the

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$k + 1/2$-th order convergence rate for the LDG method applied to the fully nonlinear KdV equation. Later, an optimal L^2 error estimate was derived in [12] for the linearized equation. Recently, there has been a different approach in solving the KdV equations by using the DG method directly without introducing any auxiliary variables, nor rewriting the original equation into a larger system. Cheng and Shu proposed such DG methods in [13] for PDEs involving high-order derivatives, and an energy-conserving DG method for the KdV equation was developed by Bona et al. in [14]. Another class of numerical methods based on discontinuous piecewise polynomials, the spectral volume (SV) methods, was recently developed by Wang et al. [15–17]. Kannan and Wang [18] also proposed the LDG2 method, which is a variant of the LDG method but reduces its unsymmetrical nature. The generalization of the SV and LDG2 methods to the partial differential equations with high order spatial derivative, including the KdV equation, is studied in [19,20].

The superconvergence property of the DG and LDG methods has been abundantly studied by many researchers in the literature. Postprocessing techniques, by using a specially designed convolution kernel, have been studied in [21–23] to obtain superconvergence for the DG methods. In [24,25], Adjerid et al. proved superconvergence of the LDG solutions at Radau points for solving convection- or diffusion-dominant time-dependent equations. For convection–diffusion equations, Celiker and Cockburn [26] found superconvergence of order $2k + 1$ for the numerical fluxes for a large class of DG methods applied to the steady-state solution of convection–diffusion equations. Their results were extended by Zhang, Xie and Zhang [27] to relate the leading term of the discretization error with the Legendre polynomial. Based on Fourier analysis, Cheng and Shu [13] proved the superconvergence of the DG solutions toward a particular projection of the exact solution in the case of piecewise linear polynomials on uniform meshes for the linear conservation law. Extensive numerical results demonstrate that the superconvergence property also holds for very general cases, including nonlinear equations, systems, and high dimensions. The results were later improved by using a different framework introduced in [28] which does not rely on Fourier analysis and is more general. They proved the superconvergence results for nonuniform meshes and schemes of any order for the linear conservation law and convection–diffusion equation. The same technique was later used in [29] to prove the superconvergence property of the LDG methods for a class of fourth-order problems.

Deriving an error estimate or superconvergence for the LDG methods for the linearized KdV equation (1.1) is more difficult than doing so for the diffusion equations since there is no control on the derivatives from the initial condition itself. In a recent paper [12], Xu and Shu provided a new technique to prove the optimal L^2 error estimate for this problem. In this paper, we extend these results in [12] and the approach in [28] to obtain the superconvergence property of the LDG methods for the linearized KdV equation (1.1). This generalization is not straightforward, and it involves several difficulties, including the non-trivial design of a special projection of the initial condition to guarantee the superconvergence property, as well as analysis for different numerical fluxes. This paper is organized as follows: in Section 2, we introduce the linearized KdV equation, and present the semi-discrete LDG methods; in Section 3, the superconvergence of LDG methods toward a particular projection of the exact solution is provided; Section 4 contains numerical experiments that support the superconvergence results; the concluding remarks are provided in Section 5; lastly, an Appendix is attached that details the more technical proof of one lemma utilized in this paper.

2. Local discontinuous Galerkin discretization

We are interested in the linearized KdV equation with periodic boundary conditions given by

$$
\begin{aligned}
&u_t + \alpha u_x + \beta u_{xxx} = 0 \quad \text{in } [a, b] \times [0, T], \qquad (2.1)\\
&u(x, 0) = u_0(x),\\
&u(a, t) = u(b, t),
\end{aligned}
$$

where α, β are constants and $u_0(x)$ is a smooth 2π-periodic function. The periodic boundary condition is assumed for the sake of simplicity only and is not essential.

2.1. Notations

We divide the interval $I = [a, b]$ into N subintervals and denote the cells by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j = 1, \ldots, N$. The center of each cell is $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$, and the mesh size is denoted by $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, with $h = \max_{1\le j\le N} h_j$ being the maximal mesh size. We assume that the mesh is regular, namely, the ratio between the maximal and the minimal mesh sizes stays bounded during mesh refinement, and we denote $\lambda \ge \max_j \Delta x_j / \min_j \Delta x_j$. The piecewise polynomial space V_h^k is defined as the space of polynomials of degree up to k in each cell I_j, that is,

$$V_h^k = \{v : v|_{I_j} \in P^k(I_j),\ j = 1, 2, \ldots, N\}. \qquad (2.2)$$

Note that functions in V_h^k are allowed to have discontinuities across element interfaces.

The solution of the numerical scheme is denoted by u_h, which belongs to the finite element space V_h^k. We denote by $(u_h)^+_{j+\frac{1}{2}}$ and $(u_h)^-_{j+\frac{1}{2}}$ the limit values of u_h at $x_{j+\frac{1}{2}}$ from the right cell I_{j+1} and from the left cell I_j, respectively. We use the usual notations $[u_h] = u_h^+ - u_h^-$ and $\bar{u}_h = \frac{1}{2}(u_h^+ + u_h^-)$ to represent the jump and the mean of the function u_h at the element interfaces, respectively. For any element $K = I_j$, we define the inner product as

$$(w, v)_K = \int_K wv dx$$

for the scalar variables w, v. The L^2 norm of v over the element K is denoted by $\|v\|_K = \sqrt{(v, v)_K}$.

2.2. The LDG method

In this subsection, we define the semi-discrete LDG method for the linearized KdV equation (2.1) by discretizing the space with the LDG method and leaving the time dependence continuous.

We write the KdV equation into a first-order system by substituting u_x with variable v and u_{xx} with variable w:

$$u_t + \alpha u_x + \beta w_x = 0, \qquad w = v_x, \qquad v = u_x. \tag{2.3}$$

The LDG method for (2.3) is then formulated as follows: find $u_h, v_h, w_h \in V_h^k$ such that

$$((u_h)_t, \phi)_{I_j} - (\alpha u_h, \phi_x)_{I_j} + (\alpha \tilde{u}_h \phi^-)_{j+\frac{1}{2}} - (\alpha \tilde{u}_h \phi^+)_{j-\frac{1}{2}} - (\beta w_h, \phi_x)_{I_j} + (\beta \hat{w}_h \phi^-)_{j+\frac{1}{2}} - (\beta \hat{w}_h \phi^+)_{j-\frac{1}{2}} = 0, \tag{2.4}$$

$$(w_h, \varphi)_{I_j} + (v_h, \varphi_x)_{I_j} - (\hat{v}_h \varphi^-)_{j+\frac{1}{2}} + (\hat{v}_h \varphi^+)_{j-\frac{1}{2}} = 0, \tag{2.5}$$

$$(v_h, \psi)_{I_j} + (u_h, \psi_x)_{I_j} - (\hat{u}_h \psi^-)_{j+\frac{1}{2}} + (\hat{u}_h \psi^+)_{j-\frac{1}{2}} = 0 \tag{2.6}$$

for all test functions $\phi, \varphi, \psi \in V_h^k$. The tilde and hatted terms, $\tilde{u}_h$, $\hat{w}_h$, $\hat{v}_h$, and $\hat{u}_h$ in (2.4)–(2.6) are the cell boundary terms obtained from integration by parts, and are referred to as the numerical fluxes. These numerical fluxes are single-valued functions defined on the cell boundaries and should be designed according to guiding principles for different PDEs to ensure numerical stability. Based on the upwinding idea, $\tilde{u}_h$ should be picked as u_h^- if $\alpha > 0$ or u_h^+ if $\alpha < 0$, and $\hat{v}_h$ should be picked as v_h^+ if $\beta > 0$ or v_h^- if $\beta < 0$. Also, $\hat{u}_h$ and $\hat{w}_h$ should be chosen alternatively from the left and the right. Without loss of generality, we assume both $\alpha, \beta > 0$. Therefore, we can use the simple fluxes:

$$\tilde{u}_h = u_h^-, \qquad \hat{w}_h = w_h^+, \qquad \hat{v}_h = v_h^+, \qquad \hat{u}_h = u_h^-, \tag{2.7}$$

or

$$\tilde{u}_h = u_h^-, \qquad \hat{w}_h = w_h^-, \qquad \hat{v}_h = v_h^+, \qquad \hat{u}_h = u_h^+. \tag{2.8}$$

The scheme presented here is a special case of the LDG methods in [10] when applied to the simple linearized KdV equation (2.1).

2.3. Projections

Next, we introduce the projections to be used throughout this paper. The standard L^2 projection of a function $\omega(x)$ with $k + 1$ continuous derivatives into the space V_h^k is denoted by P_h, i.e.,

$$(P_h\omega, \phi)_{I_j} = (\omega, \phi)_{I_j} \quad \forall \phi \in P^k(I_j).$$

In addition, we define $P_h^-\omega$ to be a projection of ω into V_h^k such that

$$(P_h^-\omega, \phi)_{I_j} = (\omega, \phi)_{I_j} \quad \forall \phi \in P^{k-1}(I_j) \quad \text{and} \quad (P_h^-\omega)^-(x_{j+\frac{1}{2}}) = \omega^-(x_{j+\frac{1}{2}}).$$

Similarly, the projection $P_h^+\omega$ is defined as the projection of ω into V_h^k such that

$$(P_h^+\omega, \phi)_{I_j} = (\omega, \phi)_{I_j} \quad \forall \phi \in P^{k-1}(I_j) \quad \text{and} \quad (P_h^+\omega)^+(x_{j-\frac{1}{2}}) = \omega^+(x_{j-\frac{1}{2}}).$$

For these projections, it can be shown (see [30]) that

$$\|\omega^e\|_{L^2} + h\|\omega^e\|_{L^\infty} + h^{\frac{1}{2}}\|\omega^e\|_{\Gamma_h} \leq Ch^{k+1}, \tag{2.9}$$

where $\omega^e = \omega - P_h\omega$ or $\omega^e = \omega - P_h^\pm\omega$, and Γ_h denotes the set of boundary points of all cells. The constant C depends on the function ω, but is independent of the mesh size h.

Assume the exact solution is ω and the numerical solution is ω_h (ω can be u, v or w). Let us denote the errors by

$$e_\omega = \omega - \omega_h = \eta_\omega + \zeta_\omega, \qquad \eta_\omega = \omega - P_h^\pm\omega, \qquad \zeta_\omega = P_h^\pm\omega - \omega_h, \tag{2.10}$$

which, from left to right, respectively, represent the error between the exact solution and the numerical solution, the projection error, and the error between the numerical solution and the particular projection of the exact solution. Here, the projection $P_h^\pm$ can be P_h^- or P_h^+ depending on the choice of numerical flux and will be specified later for each variable u, v, and w.

2.4. Initial condition

To obtain the superconvergence property of the proposed LDG method, the projections of the initial conditions for the numerical scheme need to be carefully chosen. We need to define the initial conditions according to which set of numerical fluxes is being used in the LDG method.

If the numerical fluxes (2.7) are used in the LDG method, we define the projection P_h^1 as follows: for any function u, choose $P_h^1 u \in V_h^k$ such that, if $v_h, w_h \in V_h^k$ are the solutions to (with given $P_h^1 u$)

$$(w_h, \varphi)_{I_j} + (v_h, \varphi_x)_{I_j} - (v_h^+ \varphi^-)_{j+\frac{1}{2}} + (v_h^+ \varphi^+)_{j-\frac{1}{2}} = 0, \tag{2.11}$$

$$(v_h, \psi)_{I_j} + (P_h^1 u, \psi_x)_{I_j} - ((P_h^1 u)^- \psi^-)_{j+\frac{1}{2}} + ((P_h^1 u)^- \psi^+)_{j-\frac{1}{2}} = 0 \tag{2.12}$$

for any $\psi, \varphi \in V_h^k$, then we require

$$((P_h^- u - P_h^1 u) - (P_h^+ v - v_h - P_h^+ w + w_h), \phi)_{I_j} = 0, \quad \forall \phi \in P^{k-1}(I_j), \tag{2.13}$$

$$(P_h^- u - P_h^1 u)^-(x_{j+\frac{1}{2}}) = (P_h^+ v - v_h - P_h^+ w + w_h)^+(x_{j+\frac{1}{2}}). \tag{2.14}$$

If the numerical fluxes (2.8) are used in the LDG method, we introduce a new variable $z = u + \alpha^{-1}\beta w$, which will also be used later in the proof of superconvergence in Section 3. The projection P_h^2 is defined as follows: for any function u, choose $P_h^2 u \in V_h^k$ such that, if $v_h, z_h \in V_h^k$ are the solutions to (with given $P_h^2 u$)

$$(\alpha\beta^{-1}(z_h - P_h^2 u), \varphi)_{I_j} + (v_h, \varphi_x)_{I_j} - (v_h^+ \varphi^-)_{j+\frac{1}{2}} + (v_h^+ \varphi^+)_{j-\frac{1}{2}} = 0, \tag{2.15}$$

$$(v_h, \psi)_{I_j} + (P_h^2 u, \psi_x)_{I_j} - ((P_h^2 u)^+ \psi^-)_{j+\frac{1}{2}} + ((P_h^2 u)^+ \psi^+)_{j-\frac{1}{2}} = 0 \tag{2.16}$$

for any $\psi, \varphi \in V_h^k$, then we require

$$P_h^- z - z_h = P_h^+ u - P_h^2 u + \alpha^{-1}\beta(P_h^+ v - v_h - P_h^+ u + P_h^2 u). \tag{2.17}$$

Lemma 2.1. *The projections $P_h^1 u$ and $P_h^2 u$ both exist and are unique. Moreover, there holds the error estimate*

$$\|P_h^- u - P_h^1 u\|_I + \|P_h^+ v - v_h\|_I + \|P_h^+ w - w_h\|_I \le C(\alpha, \beta, \lambda, \|u\|_{k+3}) h^{k+3/2}, \tag{2.18}$$

$$\|(u - P_h^1 u)_t\|_I \le C(\alpha, \beta, \lambda, \|u\|_{k+3}, \|u_t\|_{k+1}) h^{k+1} \tag{2.19}$$

for the projection $P_h^1 u$ and v_h, w_h defined in (2.11) and (2.12), and

$$\|P_h^+ u - P_h^2 u\|_I + \|P_h^+ v - v_h\|_I + \|P_h^- z - z_h\|_I \le C(\alpha, \beta, \lambda, \|u\|_{k+3}) h^{k+3/2}, \tag{2.20}$$

$$\|(u - P_h^2 u)_t\|_I \le C(\alpha, \beta, \lambda, \|u\|_{k+3}, \|u_t\|_{k+1}) h^{k+1} \tag{2.21}$$

for the projection $P_h^2 u$ and v_h, z_h defined in (2.15) and (2.16).

The proof of this lemma is provided in the Appendix. We would like to remark that the operators P_h^1 and P_h^2 are introduced only for the purpose of technical proof of superconvergence. In actual numerical computation, included in Section 4, we have found that superconvergence can still be observed if the usual L^2 projection of u is used as the initial condition. In that case, the superconvergence result does not hold at $t = 0$ and for very small t, but – for a later time – the numerical scheme seems to help recover the superconvergence performance.

2.5. Preliminary

In a recent paper by Xu and Shu [12], the optimal error estimate of the LDG method has been studied for the third- and fifth-order wave equations. Following the same technique, we have the following error estimate for the KdV equation (see the proof of Theorem 2.5 in [12]):

Lemma 2.2. *Let u, v, and w be the exact solutions of the linearized KdV equation (2.3). Also, let u_h, v_h, and w_h be the numerical solutions of the semi-discrete LDG method (2.4)–(2.6) with the numerical fluxes defined in (2.7) and the initial condition $u_h(\cdot, 0) = P_h^1 u_0(x)$, or the numerical fluxes defined in (2.8) and the initial condition $u_h(\cdot, 0) = P_h^2 u_0(x)$. Then there holds the following error estimate:*

$$\|e_u\|_I + \|e_v\|_I + \|e_w\|_I + \|(e_u)_t\|_I \le Ch^{k+1}, \tag{2.22}$$

where $C = C(t, \lambda, \|u\|_{L^\infty((0,t);H^{k+3}(I))}, \|u_t\|_{L^\infty((0,t);H^{k+2}(I))})$ is a constant independent of h.

At the end of this section, the following two functionals related to the L^2 norm of a function $f(x)$ on I_j, as defined in [28] to obtain the superconvergence property of the method, are needed:

$$\mathcal{B}_j^-(f) = \int_{I_j} f(x)\frac{x-x_{j-1/2}}{h_j}\frac{d}{dx}\Big(f(x)\frac{x-x_j}{h_j}\Big)dx,$$

$$\mathcal{B}_j^+(f) = \int_{I_j} f(x)\frac{x-x_{j+1/2}}{h_j}\frac{d}{dx}\Big(f(x)\frac{x-x_j}{h_j}\Big)dx.$$

The properties of these functionals in the following lemma are essential to the proof of superconvergence.

Lemma 2.3. *For any function $f(x) \in C^1$ on I_j, we have*

$$\mathcal{B}_j^-(f) = \frac{1}{4h_j}\int_{I_j} f^2(x)dx + \frac{f^2(x_{j+1/2})}{4}, \tag{2.23}$$

$$\mathcal{B}_j^+(f) = -\frac{1}{4h_j}\int_{I_j} f^2(x)dx - \frac{f^2(x_{j-1/2})}{4}. \tag{2.24}$$

The proof of this lemma can be found in [28] and is therefore omitted.

3. Superconvergence

The superconvergence property of the LDG method is studied in this section. We will prove superconvergence of order $k+3/2$ toward a particular projection of the exact solution. Since the proofs of the superconvergence property for different numerical fluxes are slightly different, we start by presenting the following theorem for when the numerical fluxes (2.7) are used:

Proposition 3.1. *Let u, v, and w be the exact solutions of the linearized KdV equation (2.3) when $\alpha, \beta > 0$. Also, let u_h, v_h, and w_h be the numerical solutions of the semi-discrete LDG method (2.4)–(2.6) with the numerical fluxes defined in (2.7) and the initial condition $u_h(\cdot, 0) = P_h^1 u_0(x)$. The particular projections of the exact solutions are defined as $P_h^- u$, $P_h^+ v$, and $P_h^+ w$, and the corresponding errors (2.10) are given by*

$$\zeta_u = P_h^- u - u_h, \qquad \zeta_v = P_h^+ v - v_h, \qquad \zeta_w = P_h^+ w - w_h \tag{3.1}$$

to be consistent with the choice of numerical fluxes. For regular triangulations of $I = [a, b]$, if the finite element space V_h^k with $k \geq 1$ is used, then there holds the following error estimate:

$$\|\zeta_u(\cdot, t)\|_I \leq Ch^{k+\frac{3}{2}}, \tag{3.2}$$

where $C = C(t, \alpha, \beta, \lambda, \|u\|_{L^\infty((0,t);H^{k+3}(I))}, \|u_t\|_{L^\infty((0,t);H^{k+2}(I))}, \|u_{tt}\|_{L^\infty((0,t);H^{k+1}(I))})$.

Proof. Without loss of generality, we will only show the proof for the case $\alpha = 1$. Since the proof is long, we divide the process into three parts.

Part 1. By subtracting the LDG method (2.4)–(2.6) with the fluxes (2.7) from the weak formulation satisfied by the exact solutions u, v, and w, we can derive the error equations

$$((e_u)_t, \phi)_{I_j} - (e_u, \phi_x)_{I_j} + (e_u^-\phi^-)_{j+\frac{1}{2}} - (e_u^-\phi^+)_{j-\frac{1}{2}} - (\beta e_w, \phi_x)_{I_j} + (\beta e_w^+\phi^-)_{j+\frac{1}{2}} - (\beta e_w^+\phi^+)_{j-\frac{1}{2}} = 0, \tag{3.3}$$

$$(e_w, \varphi)_{I_j} + (e_v, \varphi_x)_{I_j} - (e_v^+\varphi^-)_{j+\frac{1}{2}} + (e_v^+\varphi^+)_{j-\frac{1}{2}} = 0, \tag{3.4}$$

$$(e_v, \psi)_{I_j} + (e_u, \psi_x)_{I_j} - (e_u^-\psi^-)_{j+\frac{1}{2}} + (e_u^-\psi^+)_{j-\frac{1}{2}} = 0 \tag{3.5}$$

for all test functions $\phi, \varphi, \psi \in V_h^k$. Using the properties of the projections $P_h^\pm$, the error equations are equivalent to

$$((e_u)_t, \phi)_{I_j} - (\zeta_u, \phi_x)_{I_j} + (\zeta_u^-\phi^-)_{j+\frac{1}{2}} - (\zeta_u^-\phi^+)_{j-\frac{1}{2}} - (\beta \zeta_w, \phi_x)_{I_j} + (\beta \zeta_w^+\phi^-)_{j+\frac{1}{2}} - (\beta \zeta_w^+\phi^+)_{j-\frac{1}{2}} = 0, \tag{3.6}$$

$$(e_w, \varphi)_{I_j} + (\zeta_v, \varphi_x)_{I_j} - (\zeta_v^+\varphi^-)_{j+\frac{1}{2}} + (\zeta_v^+\varphi^+)_{j-\frac{1}{2}} = 0, \tag{3.7}$$

$$(e_v, \psi)_{I_j} + (\zeta_u, \psi_x)_{I_j} - (\zeta_u^-\psi^-)_{j+\frac{1}{2}} + (\zeta_u^-\psi^+)_{j-\frac{1}{2}} = 0. \tag{3.8}$$

Choosing the test functions $\phi = \zeta_u, \varphi = \beta\zeta_v$, and $\psi = -\beta\zeta_w$, and summing up the above three equations over all cells, one obtains

$$((e_u)_t, \zeta_u)_I + \frac{1}{2}\sum_j [\zeta_u]^2_{j+\frac{1}{2}} + \sum_j \beta \left[-(\zeta_w, (\zeta_u)_x)_{I_j} + (\zeta_w^+\zeta_u^-)_{j+\frac{1}{2}} - (\zeta_w^+\zeta_u^+)_{j-\frac{1}{2}}\right] = 0, \tag{3.9}$$

$$(e_w, \beta\zeta_v)_I + \frac{1}{2}\sum_j \beta[\zeta_v]^2_{j+\frac{1}{2}} = 0, \tag{3.10}$$

$$(e_v, -\beta\zeta_w)_I + \sum_j \beta \left[-(\zeta_u, (\zeta_w)_x)_{I_j} + (\zeta_u^-\zeta_w^-)_{j+\frac{1}{2}} - (\zeta_u^-\zeta_w^+)_{j-\frac{1}{2}}\right] = 0. \tag{3.11}$$

By summing up these three equations and using the periodic boundary conditions, we have

$$((e_u)_t, \zeta_u)_I + \beta(\eta_w, \zeta_v)_I - \beta(\eta_v, \zeta_w)_I + \frac{1}{2}\sum_j [\zeta_u]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j \beta[\zeta_v]^2_{j+\frac{1}{2}} = 0. \tag{3.12}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\zeta_u\|_I^2 = ((\zeta_u)_t, \zeta_u)_I \leq |((\eta_u)_t, \zeta_u)_I| + \beta\,|(\eta_w, \zeta_v)_I| + \beta\,|(\eta_v, \zeta_w)_I|\,. \tag{3.13}$$

Part 2. Now, we return to the error equation (3.7). Rewrite it as

$$(e_w, \varphi)_{I_j} - ((\zeta_v)_x, \varphi)_{I_j} - ([\zeta_u]\varphi^-)_{j+\frac{1}{2}} = 0 \tag{3.14}$$

by performing integration by parts on the term $(\zeta_v, \varphi_x)_{I_j}$. Define $\zeta_v = s_j + b_j(x)(x - x_j)/h_j$ on the cell I_j, where s_j is a constant and $b_j(x) \in P^{k-1}$. By choosing the test function φ in (3.14) to be $b_j(x)(x - x_{j+\frac{1}{2}})/h_j$ on I_j, the last term $\varphi^-_{j+\frac{1}{2}}$ will be reduced to 0. Using the definition of $\mathcal{B}_j^+(f)$, one has

$$\int_{I_j} e_w b_j(x)(x - x_{j+\frac{1}{2}})/h_j dx - \mathcal{B}_j^+(b_j(x)) = 0.$$

By Lemma 2.3, this is equivalent to

$$\int_{I_j} e_w b_j(x)\frac{x - x_{j+\frac{1}{2}}}{h_j}dx + \frac{1}{4h_j}\int_{I_j} b_j^2(x)dx + \frac{b_j^2(x_{j-1/2})}{4} = 0,$$

and hence,

$$\int_{I_j} b_j^2(x)dx \leq 4\left|\int_{I_j} e_w b_j(x)(x - x_{j+\frac{1}{2}})dx\right|. \tag{3.15}$$

We define piecewise polynomials $b(x)$ and $\phi_1(x)$ such that $b(x) = b_j(x)$ and $\phi_1(x) = x - x_{j+\frac{1}{2}}$ on I_j. Clearly, $\|\phi_1\|_{L^\infty} = \max_j h_j = h$, and Eq. (3.15) leads to

$$\|b\|_I^2 \leq 4\|e_w\|_I\|b\|_I\|\phi_1\|_{L^\infty} = 4h\|e_w\|_I\|b\|_I.$$

Therefore, by Lemma 2.2, we have

$$\|b\|_I \leq 4h\|e_w\|_I \leq Ch^{k+2}.$$

Secondly, rewrite the error equation (3.8) as

$$(e_v, \psi)_{I_j} - ((\zeta_u)_x, \psi)_{I_j} - ([\zeta_u]\psi^+)_{j-\frac{1}{2}} = 0$$

by performing integration by parts on the term $(\zeta_u, \psi_x)_{I_j}$. Define $\zeta_u = t_j + a_j(x)(x - x_j)/h_j$ on the cell I_j, where t_j is a constant and $a_j(x) \in P^{k-1}$. Also, define piecewise polynomials $a(x)$ and $\phi_2(x)$ such that $a(x) = a_j(x)$ and $\phi_2(x) = x - x_{j-\frac{1}{2}}$ on I_j. Similar to the previous analysis, we conclude that

$$\|a\|_I \leq 4h\|e_v\|_I \leq Ch^{k+2}.$$

Lastly, combine (3.6) and (3.8), which – after using integration by parts on the term $(\beta\zeta_w, \phi_x)_{I_j}$ – yields

$$((e_u)_t + e_v, \phi)_{I_j} + (\beta(\zeta_w)_x, \phi)_{I_j} + (\beta[\zeta_w]\phi^-)_{j+\frac{1}{2}} = 0.$$

Define $\zeta_w = r_j + d_j(x)(x - x_j)/h_j$ on the cell I_j, where r_j is a constant and $d_j(x) \in P^{k-1}$. Also, define a piecewise polynomial $d(x)$ such that $d(x) = d_j(x)$ on I_j. Similar to the previous analysis, we ultimately conclude that

$$\|d\|_I \le 4h\beta^{-1}\|(e_u)_t + e_v\|_I \le Ch^{k+2}.$$

Part 3. In this last part of the proof, we will use the previous results to bound the right-hand side of Eq. (3.13). By the definition of projections, $(\eta_u)_t$, η_w, and η_v are all orthogonal to piecewise constant functions, hence

$$\begin{aligned}
&((\eta_u)_t, \zeta_u)_I = ((\eta_u)_t, t_j + a_j(x)(x - x_j)/h_j)_I = ((\eta_u)_t, a_j(x)(x - x_j)/h_j)_I,\\
&(\eta_w, \zeta_v)_I = (\eta_w, b_j(x)(x - x_j)/h_j)_I,\\
&(\eta_v, \zeta_w)_I = (\eta_v, d_j(x)(x - x_j)/h_j)_I.
\end{aligned}$$

Define yet another piecewise polynomial $\phi_3(x)$ such that $\phi_3(x) = (x - x_j)/h_j$ on I_j, so $\|\phi_3\|_{L^\infty} = \frac{1}{2}$. Therefore, Eq. (3.13) leads to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\zeta_u\|_I^2 &= ((\zeta_u)_t, \zeta_u)_I \le \|(\eta_u)_t\|_I\|\phi_3\|_{L^\infty}\|a\|_I + \beta\|\eta_w\|_I\|\phi_3\|_{L^\infty}\|b\|_I + \beta\|\eta_v\|_I\|\phi_3\|_{L^\infty}\|d\|_I\\
&\le C_1h^{k+1}\frac{1}{2}C_2h^{k+2} = Ch^{2k+3}.
\end{aligned}$$

Integrating with respect to t and combining this inequality with the initial condition (2.18), we have

$$\|\zeta_u(t)\| \le Ch^{k+\frac{3}{2}},$$

where the constant C depends on the constants shown below (3.2). □

We now consider the case when the other choice of numerical fluxes (2.8) is used in the LDG method. The new variable z, defined as $u + \alpha^{-1}\beta w$, is introduced and used in the proof. Similarly, we have $z_h = u_h + \alpha^{-1}\beta w_h$.

Proposition 3.2. *Let u, v, and w be the exact solutions of the linearized KdV equation (2.3) when $\alpha, \beta > 0$. Also, let u_h, v_h, and w_h be the numerical solutions of the semi-discrete LDG method (2.4)–(2.6), with the numerical fluxes defined in (2.8) and the initial condition $u_h(\cdot, 0) = P_h^2u_0(x)$. The particular projections of the exact solutions are defined as P_h^+u, P_h^+v, and P_h^-z, and the corresponding errors (2.10) are given by*

$$\zeta_u = P_h^+u - u_h, \qquad \zeta_v = P_h^+v - v_h, \qquad \zeta_z = P_h^-z - z_h \tag{3.16}$$

to be consistent with the choice of numerical fluxes. For regular triangulations of $I = [a, b]$, if the finite element space V_h^k with $k \ge 1$ is used, then there holds the following error estimate:

$$\|\zeta_u(\cdot, t)\|_I \le Ch^{k+\frac{3}{2}}, \tag{3.17}$$

where $C = C(t, \alpha, \beta, \lambda, \|u\|_{L^\infty((0,t);H^{k+3}(I))}, \|u_t\|_{L^\infty((0,t);H^{k+2}(I))}, \|u_{tt}\|_{L^\infty((0,t);H^{k+1}(I))})$.

Proof. Without loss of generality, we will only show the proof for the case $\alpha = 1$. Most of the proof is similar to that of Proposition 3.1. Here we briefly mention the proof by highlighting the differences. To be consistent, we again divide the process into three parts.

Part 1. By subtracting the LDG method (2.4)–(2.6) with the fluxes (2.8) from the weak formulation satisfied by the exact solutions u, v, and z, and using the properties of the projections $P_h^\pm$, we have the error equations

$$((e_u)_t, \phi)_{I_j} - (\zeta_z, \phi_x)_{I_j} + (\zeta_z^-\phi^-)_{j+\frac{1}{2}} - (\zeta_z^-\phi^+)_{j-\frac{1}{2}} = 0, \tag{3.18}$$

$$(\beta^{-1}(e_z - e_u), \varphi)_{I_j} + (\zeta_v, \varphi_x)_{I_j} - (\zeta_v^+\varphi^-)_{j+\frac{1}{2}} + (\zeta_v^+\varphi^+)_{j-\frac{1}{2}} = 0, \tag{3.19}$$

$$(e_v, \psi)_{I_j} + (\zeta_u, \psi_x)_{I_j} - (\zeta_u^+\psi^-)_{j+\frac{1}{2}} + (\zeta_u^+\psi^+)_{j-\frac{1}{2}} = 0 \tag{3.20}$$

for all test functions $\phi, \varphi, \psi \in V_h^k$. Choosing the test functions $\phi = \zeta_u$, $\varphi = \beta\zeta_v$, and $\psi = -(\zeta_z - \zeta_u)$, summing up the previous three equations over all cells, and using the periodic boundary conditions, we have

$$((e_u)_t, \zeta_u)_I + (\eta_z - \eta_u, \zeta_v)_I - (\eta_v, \zeta_z - \zeta_u)_I + \frac{1}{2}\sum_j[\zeta_u]_{j+\frac{1}{2}}^2 + \frac{1}{2}\sum_j\beta[\zeta_v]_{j+\frac{1}{2}}^2 = 0.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\zeta_u\|_I^2 \le |((\eta_u)_t, \zeta_u)_I| + |(\eta_z, \zeta_v)_I| + |(\eta_v, \zeta_z)_I| + |(\eta_u, \zeta_v)_I| + |(\eta_v, \zeta_u)_I|\,. \tag{3.21}$$

Part 2. We define $\zeta_u = t_j + a_j(x)(x - x_j)/h_j$, $\zeta_v = s_j + b_j(x)(x - x_j)/h_j$, and $\zeta_z = p_j + q_j(x)(x - x_j)/h_j$ on the cell I_j, where t_j, s_j, p_j are constant and $a_j(x), b_j(x), q_j(x) \in P^{k-1}$. We also define a piecewise polynomial $a(x)$ such that $a(x) = a_j(x)$ on I_j. Similarly, piecewise polynomials $b(x)$ and $q(x)$ are defined.

Following similar analysis to that in Part 2 of Proposition 3.1, we conclude that

$$\begin{aligned}
&\|a\|_I \le 4h\|e_v\|_I \le Ch^{k+2},\\
&\|b\|_I \le 4\beta^{-1}h\|e_z - e_u\|_I \le Ch^{k+2},\\
&\|q\|_I \le 4h\|(e_u)_t\|_I \le Ch^{k+2}.
\end{aligned}$$

Part 3. In this last part of the proof, we will use the previous results to bound the right-hand side of Eq. (3.21). Therefore,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\zeta_u\|_I^2 \le\ & \|(\eta_u)_t\|_I\|\phi_3\|_{L^\infty}\|a\|_I + \|\eta_z\|_I\|\phi_3\|_{L^\infty}\|b\|_I\\
&+ \|\eta_v\|_I\|\phi_3\|_{L^\infty}\|q\|_I + \|\eta_u\|_I\|\phi_3\|_{L^\infty}\|b\|_I + \|\eta_v\|_I\|\phi_3\|_{L^\infty}\|a\|_I\\
\le\ & C_1h^{k+1}\frac{1}{2}C_2h^{k+2} = Ch^{2k+3}.
\end{aligned}$$

Integrating with respect to t and combining this inequality with the initial condition (2.19), we have

$$\|\zeta_u(t)\| \le Ch^{k+\frac{3}{2}},$$

where the constant C depends on the constants shown below (3.17). □

Propositions 3.1 and 3.2 can be generalized to the cases when $\alpha, \beta \le 0$. The case of $\alpha < 0$ or $\beta < 0$ is similar and is therefore omitted. If $\beta = 0$, the linearized KdV equation (2.1) becomes the simple first-order wave equation, which has been carefully studied in [28] for the superconvergence property. For the case $\alpha = 0$, the equation reduces to the third-order wave equation

$$u_t + \beta u_{xxx} = 0, \tag{3.22}$$

and we have the following result:

Proposition 3.3. *Let u, $v = u_x$, and $w = u_{xx}$ be the exact solutions of the third-order wave equation (3.22) when $\beta > 0$. Also, let u_h, v_h, and w_h be the numerical solutions of the semi-discrete LDG method (2.4)–(2.6) when $\alpha = 0$, with the numerical fluxes defined in (2.7) and the initial condition $u_h(\cdot, 0) = P_h^1 u_0(x)$. The particular projections of the exact solutions are defined as $P_h^- u$, $P_h^+ v$, and $P_h^+ w$, and the corresponding errors (2.10) are given by*

$$\zeta_u = P_h^- u - u_h, \qquad \zeta_v = P_h^+ v - v_h, \qquad \zeta_w = P_h^+ w - w_h$$

to be consistent with the choice of numerical fluxes. For regular triangulations of $I = [a, b]$, if the finite element space V_h^k with $k \ge 1$ is used, then there holds the following error estimate:

$$\|\zeta_u(\cdot, t)\|_I \le Ch^{k+\frac{3}{2}}, \tag{3.23}$$

where $C = C(t, \beta, \lambda, \|u\|_{L^\infty((0,t);H^{k+3}(I))}, \|u_t\|_{L^\infty((0,t);H^{k+2}(I))}, \|u_{tt}\|_{L^\infty((0,t);H^{k+1}(I))})$.

The proof of this theorem is similar to those for the previous two propositions and is therefore omitted. The situation with the other choice of numerical fluxes (2.8), as well as the case when $\beta < 0$, is similar to that with the fluxes (2.7).

4. Numerical experiments

In this section, we provide some numerical examples to demonstrate the superconvergence property of the proposed LDG scheme. Since the explicit high-order TVD Runge–Kutta methods are known to suffer from small time-step restrictions due to the stiffness of the LDG spatial discretization for the equations containing high-order derivatives, we use the second-order implicit Crank–Nicholson time discretization in the following numerical examples. As our interest is in the effect of the spatial discretization, we determine the time-step by the relation $\Delta t = Ch^2$, so that the temporal error is of order h^4. Since the spatial error is of order h^{k+1}, where $k = 1, 2, 3$ is tested in this paper, this choice of Δt guarantees that the error will be dominated by the spatial discretization.

We consider the linearized KdV equation

$$u_t + u_x + u_{xxx} = 0, \quad x \in [0, \pi],$$

with the initial condition

$$u(x, 0) = \sin(2x),$$

Table 4.1
Numerical errors and orders of LDG method for the linearized KdV equation with uniform meshes and space P^1.

N	e_u		e_v		e_w	
	L^2 error	Order	L^2 error	Order	L^2 error	Order
10	2.9074E−02		5.7647E−02		1.1535E−01	
20	5.2101E−03	2.4803	1.0386E−02	2.4725	2.0795E−02	2.4717
40	1.1273E−03	2.2084	2.2526E−03	2.2049	4.5075E−03	2.2058
80	2.6964E−04	2.0637	5.3917E−04	2.0627	1.0785E−03	2.0632
	ζ_u		ζ_v		ζ_w	
10	2.2682E−02		4.7869E−02		9.5650E−02	
20	3.0391E−03	2.9620	6.0962E−03	2.9731	1.2207E−02	2.9699
40	3.8209E−04	2.9916	7.6341E−04	2.9973	1.5306E−03	2.9956
80	4.7841E−05	2.9975	9.5398E−05	3.0004	1.9138E−04	2.9995

Table 4.2
Numerical errors and orders of LDG method for the linearized KdV equation with uniform meshes and space P^2.

N	e_u		e_v		e_w	
	L^2 error	Order	L^2 error	Order	L^2 error	Order
10	8.5104E−04		1.7164E−03		3.4359E−03	
20	1.0670E−04	2.9955	2.1387E−04	3.0045	4.2785E−04	3.0055
40	1.3363E−05	2.9973	2.6740E−05	2.9996	5.3484E−05	2.9999
80	1.6712E−06	2.9992	3.3430E−06	2.9998	6.6861E−06	2.9998
	ζ_u		ζ_v		ζ_w	
10	1.1511E−04		2.3320E−04		4.0525E−04	
20	5.1307E−06	4.4877	1.0311E−05	4.4992	1.4684E−05	4.7864
40	2.7790E−07	4.2064	5.5658E−07	4.2115	6.4564E−07	4.5073
80	1.6525E−08	4.0718	3.3065E−08	4.0732	3.4009E−08	4.2467

Table 4.3
Numerical errors and orders of LDG method for the linearized KdV equation with uniform meshes and space P^3.

N	e_u		e_v		e_w	
	L^2 error	Order	L^2 error	Order	L^2 error	Order
5	5.1900E−04		1.0565E−03		2.1109E−03	
10	3.2830E−05	3.9826	6.5944E−05	4.0019	1.3188E−04	4.0005
20	2.0623E−06	3.9926	4.1290E−06	3.9973	8.2581E−05	3.9972
40	1.2907E−07	3.9979	2.5821E−07	3.9991	5.1641E−06	3.9992
	ζ_u		ζ_v		ζ_w	
5	9.7665E−05		1.9239E−04		3.2249E−04	
10	2.1235E−06	5.5232	4.2348E−06	5.5056	5.9058E−05	5.7709
20	5.7184E−08	5.2147	1.1435E−07	5.2107	1.3145E−07	5.4894
40	1.7540E−09	5.0268	3.5077E−09	5.0268	3.9109E−08	5.0709

and a periodic boundary condition $u(0, t) = u(\pi, t)$ for all $t \geq 0$. This problem has the exact solution

$$u(x, t) = \sin(2x + 6t).$$

We implemented the LDG method (2.4)–(2.6) with the numerical fluxes (2.7) and took the time-step $\Delta t = 0.1h^2$. We have tried both the special projection $P_h^1 u$ and the standard L^2 projection as the initial condition, obtaining similar convergence rates for each. To save space, we only report the results when the standard L^2 projection is used as the initial condition. The numerical fluxes (2.8) have also been implemented, again yielding similar results.

In the first numerical example, we consider the case of uniform meshes, in which the domain is uniformly divided into N cells. Table 4.1 lists the numerical errors and the orders of convergence for P^1 spaces. The L^2-norm of the errors e_u, ζ_u, e_v, ζ_v, e_w and ζ_w at final time $T = 1$ are presented. Also, the third-order convergence rate for ζ_u, ζ_v and ζ_w can clearly be observed. This indicates that the $k + 3/2$ superconvergence rate, proved in Section 3, is not optimal, and we will work on how to derive the optimal superconvergence later. Note that the same phenomenon has been observed in [28,29]. For the case of $k = 2, 3$, the results in Tables 4.2 and 4.3 list the numerical errors and the order of convergence for P^2 and P^3 spaces, respectively, which demonstrate the superconvergence property of ζ_u, ζ_v and ζ_w.

Next are reported simulations made when the mesh is far from uniform. Indeed, the mesh is taken to be of size $2h, h, \ldots, 2h, h$. We use the same setup as was previously described, and then repeat the numerical experiment for P^1, P^2, and P^3 spaces. Similar results are observed: the order of convergence is $k + 2$ for P^k spaces. To save space, we show

Table 4.4
Numerical errors and orders of LDG method for the linearized KdV equation with non-uniform meshes of type $2h, h, \ldots, 2h, h$ and space P^2.

N	e_u		e_v		e_w	
	L^2 error	Order	L^2 error	Order	L^2 error	Order
10	2.4934E−03		5.0790E−03		1.0172E−02	
20	3.1114E−04	3.0025	6.2531E−04	3.0219	1.2512E−03	3.0231
40	3.9033E−05	2.9947	7.8162E−05	3.0000	1.5634E−04	3.0005
80	4.8848E−06	2.9983	9.7727E−06	2.9996	1.9546E−05	2.9997
N	ζ_u		ζ_v		ζ_w	
10	6.0115E−04		1.2293E−03		2.2343E−03	
20	2.4887E−05	4.5942	5.0270E−05	4.6120	7.7807E−05	4.8437
40	1.2642E−06	4.2990	2.5363E−06	4.3088	3.1474E−06	4.6276
80	7.3762E−08	4.0992	1.4765E−07	4.1024	1.5929E−07	4.3044

only the results for P^2 spaces, as Table 4.4 lists the numerical errors and the orders of convergence, where the fourth-order convergence rate for ζ_u, ζ_v, and ζ_w can clearly be observed.

5. Concluding remarks

In this paper, we studied the superconvergence property of the LDG method for solving the linearized KdV equation. When polynomials of degree k are used, we have proved that the error between a particular projection of the exact solution and the numerical solution achieves superconvergence of order $k + 3/2$. Numerical examples have also been provided to verify these results. Future work includes superconvergence analysis of the LDG method for the nonlinear KdV equation. Also, because we observed superconvergence of order $k + 2$ in the numerical examples, future work will include investigating how to improve our proofs to derive this better convergence rate.

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Appendix A. The proof of Lemma 2.1

In this Appendix, we provide the proof for Lemma 2.1, which includes the error estimate of the initial condition. Without loss of generality, we will only show the proof for the case $\alpha = 1$. For ease of presentation, we separate the process into six parts.

Part 1. We will first prove the existence and uniqueness of $P_h^1 u$. Assume $v_h, w_h \in V_h^k$ are computed in (2.11) and (2.12). As in (2.10), we denote the error by

$$\begin{aligned} u - P_h^1 u &= u - P_h^- u + P_h^- u - P_h^1 u = \eta_u + \zeta_u, \\ v - v_h &= v - P_h^+ v + P_h^+ v - v_h = \eta_v + \zeta_v, \\ w - w_h &= w - P_h^+ w + P_h^+ w - w_h = \eta_w + \zeta_w. \end{aligned}$$

By subtracting the LDG method (2.5)–(2.6) with the fluxes (2.7) from the weak formulation satisfied by the exact solutions v and w, and using the properties of the projections $P_h^{\pm}$, we can obtain the error equations (3.7) and (3.8), which are copied here:

$$(\eta_w + \zeta_w, \varphi)_{I_j} + (\zeta_v, \varphi_x)_{I_j} - (\zeta_v^+ \varphi^-)_{j+\frac{1}{2}} + (\zeta_v^+ \varphi^+)_{j-\frac{1}{2}} = 0, \tag{A.1}$$

$$(\eta_v + \zeta_v, \psi)_{I_j} + (\zeta_u, \psi_x)_{I_j} - (\zeta_u^- \psi^-)_{j+\frac{1}{2}} + (\zeta_u^- \psi^+)_{j-\frac{1}{2}} = 0, \tag{A.2}$$

for any $\varphi, \psi \in V_h^k$. Coupled with the initial conditions (2.13) and (2.14), they become

$$(\zeta_w, \varphi)_{I_j} + (\zeta_v, \varphi_x)_{I_j} - (\zeta_v^+ \varphi^-)_{j+\frac{1}{2}} + (\zeta_v^+ \varphi^+)_{j-\frac{1}{2}} = -(\eta_w, \varphi)_{I_j}, \tag{A.3}$$

$$(\zeta_v, \psi)_{I_j} + (\zeta_v - \zeta_w, \psi_x)_{I_j} - ((\zeta_v^+ - \zeta_w^+)\psi^-)_{j+\frac{1}{2}} + ((\zeta_v^+ - \zeta_w^+)\psi^+)_{j-\frac{1}{2}} = -(\eta_v, \psi)_{I_j} \tag{A.4}$$

for any $\varphi, \psi \in V_h^k$. Note that Eqs. (A.3) and (A.4) are a linear system, hence the existence of (ζ_v, ζ_w) follows by their uniqueness, so we now seek to prove the solutions are unique. Suppose there are two solutions (ζ_v^1, ζ_w^1) and (ζ_v^2, ζ_w^2) to

Eqs. (A.3)–(A.4). Define $g_v = \zeta_v^1 - \zeta_v^2$ and $g_w = \zeta_w^1 - \zeta_w^2$, and we have

$$(g_w, \varphi)_{I_j} + (g_v, \varphi_x)_{I_j} - (g_v^+\varphi^-)_{j+\frac{1}{2}} + (g_v^+\varphi^+)_{j-\frac{1}{2}} = 0, \tag{A.5}$$

$$(g_v, \psi)_{I_j} + (g_v - g_w, \psi_x)_{I_j} - ((g_v^+ - g_w^+)\psi^-)_{j+\frac{1}{2}} + ((g_v^+ - g_w^+)\psi^+)_{j-\frac{1}{2}} = 0 \tag{A.6}$$

for any $\varphi, \psi \in V_h^k$. On one hand, let $\varphi = g_v$ and $\psi = g_v - g_w$, so summing over all cells I_j, we obtain

$$(g_v, g_v)_I + \frac{1}{2}\sum_j [g_v]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j [g_v - g_w]^2_{j+\frac{1}{2}} = 0,$$

which leads to $g_v = 0$. On the other hand, let $\varphi = g_v + g_w$ and $\psi = -g_w$, so summing over all cells I_j, we obtain

$$(g_w, g_w)_I + \frac{1}{2}\sum_j [g_v]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j [g_w]^2_{j+\frac{1}{2}} = 0,$$

which leads to $g_w = 0$. Thus, we have proved the uniqueness, and hence the existence, of ζ_v and ζ_w. This implies the uniqueness and existence of ζ_u through the initial conditions (2.13)–(2.14), and therefore $P_h^1 u = P_h^- u - \zeta_u$.

Part 2. Next, we prove the error estimate of the initial projection $P_h^1 u$. We define $\zeta_v = s_j + b_j(x)(x - x_j)/h_j$, $\zeta_w = r_j + d_j(x)(x - x_j)/h_j$ on the cell I_j, where s_j, r_j are constant and $b_j(x), d_j(x) \in P^{k-1}$. We define a piecewise polynomial $b(x)$ such that $b(x) = b_j(x)$ on I_j and define $d(x)$ similarly. From Eqs. (A.3) and (A.4), following similar analysis to that used in Part 2 of the proof of Proposition 3.1, we conclude that

$$\begin{aligned}
&\|b\|_I \le 4h\|\eta_w + \zeta_w\|_I \le Ch^{k+2} + Ch\|\zeta_w\|_I, \\
&\|d\|_I \le 4h(\|\eta_w + \zeta_w\|_I + \|\eta_v + \zeta_v\|_I) \le Ch^{k+2} + Ch\|\zeta_w\|_I + Ch\|\zeta_v\|_I.
\end{aligned}$$

On one hand, let $\varphi = \zeta_v$ and $\psi = \zeta_v - \zeta_w$ in Eqs. (A.3)–(A.4) and sum over all cells I_j, which yields

$$(\zeta_v, \zeta_v)_I + \frac{1}{2}\sum_j [\zeta_v]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j [\zeta_v - \zeta_w]^2_{j+\frac{1}{2}} = -(\eta_w, \zeta_v)_I - (\eta_v, \zeta_v - \zeta_w)_I.$$

Following similar analysis to that used in Part 3 of the proof of Proposition 3.1, we have

$$\begin{aligned}
\|\zeta_v\|_I^2 &= (\zeta_v, \zeta_v)_I \le |(\eta_w, \zeta_v)_I| + |(\eta_v, \zeta_v)_I| + |(\eta_v, \zeta_w)_I| \\
&\le \|\eta_w\|_I\|\phi_3\|_{L^\infty}\|b\|_I + \|\eta_v\|_I\|\phi_3\|_{L^\infty}\|b\|_I + \|\eta_v\|_I\|\phi_3\|_{L^\infty}\|d\|_I \\
&\le C_1 h^{k+1}\left(C_2 h^{k+2} + C_3 h\|\zeta_w\|_I + C_4 h\|\zeta_v\|_I\right).
\end{aligned} \tag{A.7}$$

On the other hand, let $\varphi = \zeta_v + \zeta_w$ and $\psi = -\zeta_w$ in Eqs. (A.3)–(A.4) and sum over all cells I_j, which yields

$$(\zeta_w, \zeta_w)_I + \frac{1}{2}\sum_j [\zeta_v]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j [\zeta_w]^2_{j+\frac{1}{2}} = -(\eta_w, \zeta_v + \zeta_w)_I + (\eta_v, \zeta_w)_I.$$

Similarly, we have

$$\begin{aligned}
\|\zeta_w\|_I^2 &= (\zeta_w, \zeta_w)_I \le |(\eta_w, \zeta_v)_I| + |(\eta_w, \zeta_w)_I| + |(\eta_v, \zeta_w)_I| \\
&\le \|\eta_w\|_I\|\phi_3\|_{L^\infty}\|b\|_I + \|\eta_w\|_I\|\phi_3\|_{L^\infty}\|d\|_I + \|\eta_v\|_I\|\phi_3\|_{L^\infty}\|d\|_I \\
&\le C_1 h^{k+1}\left(C_2 h^{k+2} + C_3 h\|\zeta_w\|_I + C_4 h\|\zeta_v\|_I\right).
\end{aligned} \tag{A.8}$$

Combining (A.7) and (A.8), one obtains

$$\|\zeta_v\|_I^2 + \|\zeta_w\|_I^2 \le C_1 h^{k+1}\left(C_2 h^{k+2} + C_3 h\|\zeta_w\|_I + C_4 h\|\zeta_v\|_I\right),$$

which leads to

$$\|\zeta_w\|_I^2 + \|\zeta_v\|_I^2 \le Ch^{2k+3}. \tag{A.9}$$

The error estimate of $\|\zeta_u\|_I$ is obtained by the conditions (2.13) and (2.14). Suppose that

$$\zeta_v - \zeta_w = \sum_{n=0}^{k} a_n^j P_n\left(\frac{2(x - x_j)}{h_j}\right), \qquad \zeta_u = \sum_{n=0}^{k} b_n^j P_n\left(\frac{2(x - x_j)}{h_j}\right),$$

on I_j, where $P_n(\cdot)$ denotes the nth-order Legendre polynomial. The relation (2.13) leads to

$$a_n^j = b_n^j \quad \text{for } n = 0, 1, \ldots, k - 1, \tag{A.10}$$

due to the orthogonality of the Legendre polynomial. Note that (2.14) implies

$$\sum_{n=0}^{k} b_n^j = \sum_{n=0}^{k} (-1)^n a_n^{j+1},$$

and therefore,

$$b_k^j = \sum_{n=0}^{k} (-1)^n a_n^{j+1} - \sum_{n=0}^{k-1} b_n^j = \sum_{n=0}^{k} (-1)^n a_n^{j+1} - \sum_{n=0}^{k-1} a_n^j. \tag{A.11}$$

Since,

$$\begin{aligned} h_j (b_k^j)^2 &= h_j \left(\sum_{n=0}^{k} (-1)^n a_n^{j+1} - \sum_{n=0}^{k-1} a_n^j \right)^2 \\ &\le 2h_j \left[\left(\sum_{n=0}^{k} (-1)^n a_n^{j+1} \right)^2 + \left(\sum_{n=0}^{k-1} a_n^j \right)^2 \right] \\ &\le 2(k+1) h_j \left(\sum_{n=0}^{k} (a_n^{j+1})^2 + \sum_{n=0}^{k} (a_n^j)^2 \right) \\ &\le 2(k+1)(2k+1) \left(\sum_{n=0}^{k} (a_n^{j+1})^2 \frac{h_j}{2n+1} + \sum_{n=0}^{k} (a_n^j)^2 \frac{h_j}{2n+1} \right) \\ &\le 2(k+1)(2k+1) \left(\frac{1}{\lambda} \|\zeta_w\|_{I_{j+1}}^2 + \|\zeta_w\|_{I_j}^2 \right), \end{aligned}$$

where λ is the maximum ratio of two different mesh sizes. Thus, we have

$$\begin{aligned} \|\zeta_u\|_I^2 &= \sum_{j=1}^{N} \|\zeta_u\|_{I_j}^2 = \sum_{j=1}^{N} \left(\sum_{n=0}^{k} (b_n^j)^2 \frac{h_j}{2n+1} \right) \\ &\le \sum_{j=1}^{N} \left(\sum_{n=0}^{k-1} (a_n^j)^2 \frac{h_j}{2n+1} + (b_k^j)^2 \frac{h_j}{2k+1} \right) \\ &\le \sum_{j=1}^{N} \left(\|\zeta_w\|_{I_j}^2 + 2(k+1) \left(\frac{1}{\lambda} \|\zeta_w\|_{I_{j+1}}^2 + \|\zeta_u\|_{I_j}^2 \right) \right) \\ &\le \left(1 + 2(k+1) + 2\frac{k+1}{\lambda} \right) \|\zeta_w\|_I^2, \end{aligned}$$

yielding

$$\|\zeta_u\|_I \le C(\lambda, \|u\|_{k+3}) h^{k+3/2}.$$

Part 3. Finally, we consider the error term $\|(e_u)_t(\cdot, 0)\|_I$. Coupled with the initial conditions (2.13)–(2.14), the error equation (3.6) becomes

$$\begin{aligned} &((e_u)_t, \phi)_{I_j} - (1-\beta)(\zeta_u, \phi_x)_{I_j} + (1-\beta)(\zeta_u^- \phi^-)_{j+\frac{1}{2}} - (1-\beta)(\zeta_u^- \phi^+)_{j-\frac{1}{2}} \\ &\quad - \beta(\zeta_v, \phi_x)_{I_j} + \beta(\zeta_v^+ \phi^-)_{j+\frac{1}{2}} - \beta(\zeta_v^+ \phi^+)_{j-\frac{1}{2}} = 0. \end{aligned}$$

It follows from (A.1) and (A.2) that, at time $t = 0$,

$$((e_u)_t, \phi)_{I_j} + (1-\beta)(\eta_v + \zeta_v, \phi)_{I_j} + \beta(\eta_w + \zeta_w, \phi)_{I_j} = 0$$

for any $\phi \in V_h^k$. Taking $\phi = (\zeta_u)_t(\cdot, 0)$ and summing the above equality over all cells I_j, we obtain, at time $t = 0$,

$$\|(\zeta_u)_t(\cdot, 0)\|_I \le \|(\eta_u)_t(\cdot, 0)\|_I + |1-\beta| \|\eta_v + \zeta_v\|_I + \beta \|\eta_w + \zeta_w\|_I \le Ch^{k+1}, \tag{A.12}$$

where the constant C depends on α, β, λ, $\|u\|_{k+3}$ and $\|u_t\|_{k+1}$.

Part 4. Now, we prove the existence and uniqueness of the projection $P_h^2 u$. Assume $v_h, z_h \in V_h^k$ are computed in (2.15) and (2.16). As in (2.10), we denote the error by

$$\begin{aligned} u - P_h^2 u &= u - P_h^+ u + P_h^+ u - P_h^1 u = \eta_u + \zeta_u, \\ v - v_h &= v - P_h^+ v + P_h^+ v - v_h = \eta_v + \zeta_v, \\ z - z_h &= z - P_h^- z + P_h^- z - z_h = \eta_z + \zeta_z, \end{aligned}$$

and the initial condition (2.17) becomes

$$\zeta_z - \zeta_u = \beta(\zeta_v - \zeta_u). \tag{A.13}$$

By subtracting the LDG method (2.5)–(2.6) with the fluxes (2.8) from the weak formulation satisfied by the exact solutions v and $w = \beta^{-1}(z - u)$, and using the properties of the projections $P_h^{\pm}$, we can obtain the error equations

$$\beta^{-1}(\eta_z + \zeta_z - \eta_u - \zeta_u, \varphi)_{I_j} + (\zeta_v, \varphi_x)_{I_j} - (\zeta_v^+\varphi^-)_{j+\frac{1}{2}} + (\zeta_v^+\varphi^+)_{j-\frac{1}{2}} = 0, \tag{A.14}$$

$$(\eta_v + \zeta_v, \psi)_{I_j} + (\zeta_u, \psi_x)_{I_j} - (\zeta_u^+\psi^-)_{j+\frac{1}{2}} + (\zeta_u^+\psi^+)_{j-\frac{1}{2}} = 0 \tag{A.15}$$

for any $\varphi, \psi \in V_h^k$. Coupled with the initial condition (A.13), they become

$$(\zeta_v - \zeta_u, \varphi)_{I_j} + (\zeta_v, \varphi_x)_{I_j} - (\zeta_v^+\varphi^-)_{j+\frac{1}{2}} + (\zeta_v^+\varphi^+)_{j-\frac{1}{2}} = -\beta^{-1}(\eta_z - \eta_u, \varphi)_{I_j}, \tag{A.16}$$

$$(\zeta_v, \psi)_{I_j} + (\zeta_u, \psi_x)_{I_j} - (\zeta_u^+\psi^-)_{j+\frac{1}{2}} + (\zeta_u^+\psi^+)_{j-\frac{1}{2}} = -(\eta_v, \psi)_{I_j} \tag{A.17}$$

for any $\varphi, \psi \in V_h^k$. Note that Eqs. (A.16) and (A.17) are a linear system, hence the existence of (ζ_u, ζ_v) follows by their uniqueness, so we now seek to prove the solutions are unique. Suppose there are two solutions (ζ_u^1, ζ_v^1) and (ζ_u^2, ζ_v^2) to Eqs. (A.16)–(A.17). Define $g_u = \zeta_u^1 - \zeta_u^2$ and $g_v = \zeta_v^1 - \zeta_v^2$, so we have

$$(g_v - g_u, \varphi)_{I_j} + (g_v, \varphi_x)_{I_j} - (g_v^+\varphi^-)_{j+\frac{1}{2}} + (g_v^+\varphi^+)_{j-\frac{1}{2}} = 0, \tag{A.18}$$

$$(g_v, \psi)_{I_j} + (g_u, \psi_x)_{I_j} - (g_u^+\psi^-)_{j+\frac{1}{2}} + (g_u^+\psi^+)_{j-\frac{1}{2}} = 0 \tag{A.19}$$

for any $\varphi, \psi \in V_h^k$. On one hand, let $\varphi = g_v$ and $\psi = g_u$, so summing over all cells I_j, we obtain

$$(g_v, g_v)_I + \frac{1}{2}\sum_j [g_v]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j [g_u]^2_{j+\frac{1}{2}} = 0,$$

which leads to $g_v = 0$. On the other hand, let $\varphi = g_v - g_u$ and $\psi = 2g_u - g_v$, so summing over all cells I_j, we obtain

$$(g_u, g_u)_I + \frac{1}{2}\sum_j [g_u]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j [g_u - g_v]^2_{j+\frac{1}{2}} = 0,$$

which leads to $g_u = 0$. Thus, we have proved the uniqueness, and hence the existence of ζ_u and ζ_v. Therefore, $P_h^2 u = P_h^+ u - \zeta_u$ exists and is unique.

Part 5. Next, we prove the error estimate of the initial projection $P_h^2 u$. We define $\zeta_u = t_j + a_j(x)(x - x_j)/h_j$ and $\zeta_v = s_j + b_j(x)(x - x_j)/h_j$ on the cell I_j, where $t_j s_j$ are constant and $a_j(x), b_j(x) \in P^{k-1}$. We define a piecewise polynomial $a(x)$ such that $a(x) = a_j(x)$ on I_j and define $b(x)$ similarly. From Eqs. (A.16) and (A.17), following similar analysis to that used in Part 2 of Proposition 3.1, we conclude that

$$\|a\|_I \le 4h\|\eta_v + \zeta_v\|_I \le Ch^{k+2} + Ch\|\zeta_v\|_I,$$
$$\|b\|_I \le 4h(\|\zeta_v - \zeta_u + \beta^{-1}(\eta_z - \eta_u)\|_I) \le Ch^{k+2} + Ch\|\zeta_v\|_I + Ch\|\zeta_u\|_I.$$

On one hand, let $\varphi = \zeta_v$ and $\psi = \zeta_u$ in Eqs. Eqs. (A.16)–(A.17) and sum over all cells I_j, which yields

$$(\zeta_v, \zeta_v)_I + \frac{1}{2}\sum_j [\zeta_v]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j [\zeta_u]^2_{j+\frac{1}{2}} = -\beta^{-1}(\eta_z - \eta_u, \zeta_v)_I - (\eta_v, \zeta_u)_I.$$

On the other hand, let $\varphi = \zeta_v - \zeta_u$ and $\psi = 2\zeta_u - \zeta_v$ in Eqs. (A.16)–(A.17) and sum over all cells I_j, which yields

$$(\zeta_u, \zeta_u)_I + \frac{1}{2}\sum_j [\zeta_u]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j [\zeta_u - \zeta_v]^2_{j+\frac{1}{2}} = -\beta^{-1}(\eta_z - \eta_u, \zeta_v - \zeta_u)_I - (\eta_v, 2\zeta_u - \zeta_v)_I.$$

Combining the above two equations, and following similar analysis to that used in Part 3 of the proof of Proposition 3.1, we have

$$\begin{aligned}
&\|\zeta_v\|_I^2 + \|\zeta_u\|_I^2 + \frac{1}{2}\sum_j [\zeta_v]^2_{j+\frac{1}{2}} + \sum_j [\zeta_u]^2_{j+\frac{1}{2}} + \frac{1}{2}\sum_j [\zeta_u - \zeta_v]^2_{j+\frac{1}{2}} \\
&\quad \le C\left(|(\eta_z, \zeta_v)_I| + |(\eta_u, \zeta_v)_I| + |(\eta_v, \zeta_v)_I| + |(\eta_z, \zeta_u)_I| + |(\eta_u, \zeta_u)_I| + |(\eta_v, \zeta_u)_I|\right) \\
&\quad \le C\left((\|\eta_z\|_I + \|\eta_u\|_I + \|\eta_v\|_I)\|\phi_3\|_{L^\infty}\|b\|_I + (\|\eta_z\|_I + \|\eta_u\|_I + \|\eta_v\|_I)\|\phi_3\|_{L^\infty}\|a\|_I\right) \\
&\quad \le C_1 h^{k+1}\left(C_2 h^{k+2} + C_3 h\|\zeta_u\|_I + C_4 h\|\zeta_v\|_I\right),
\end{aligned}$$

which leads to

$$\|\zeta_v\|_I^2 + \|\zeta_u\|_I^2 + \frac{1}{2}\sum_j [\zeta_v]_{j+\frac{1}{2}}^2 + \sum_j [\zeta_u]_{j+\frac{1}{2}}^2 \le Ch^{2k+3}. \tag{A.20}$$

By the initial condition (A.13), we derive the error estimate of $\|\zeta_z\|_I$:

$$\|\zeta_z\|_I^2 \le Ch^{2k+3}. \tag{A.21}$$

Part 6. Finally, we consider the error term $\|(e_u)_t(\cdot, 0)\|_I$. Coupled with the initial condition (A.13), the error equation (3.18) becomes

$$\begin{aligned}
&((e_u)_t, \phi)_{I_j} - (1-\beta)(\zeta_u, \phi_x)_{I_j} + (1-\beta)(\zeta_u^- \phi^-)_{j+\frac{1}{2}} - (1-\beta)(\zeta_u^- \phi^+)_{j-\frac{1}{2}} \\
&\quad - \beta(\zeta_v, \phi_x)_{I_j} + \beta(\zeta_v^- \phi^-)_{j+\frac{1}{2}} - \beta(\zeta_v^- \phi^+)_{j-\frac{1}{2}} = 0.
\end{aligned}$$

Summing the above equality over all I_j and applying (A.14) and (A.15), we have, at time $t = 0$,

$$\begin{aligned}
&((e_u)_t, \phi)_I + (1-\beta)(\eta_v + \zeta_v, \phi)_I + (\eta_z + \zeta_z - \eta_u - \zeta_u, \phi)_I \\
&\quad + (1-\beta)\sum_j [\zeta_u]_{j+\frac{1}{2}}[\phi]_{j+\frac{1}{2}} + \beta \sum_j [\zeta_v]_{j+\frac{1}{2}}[\phi]_{j+\frac{1}{2}} = 0
\end{aligned}$$

for any $\phi \in V_h^k$. Taking $\phi = (\zeta_u)_t(\cdot, 0)$, one obtains, at time $t = 0$,

$$\begin{aligned}
\|(\zeta_u)_t\|_I^2 &= -((\eta_u)_t, (\zeta_u)_t)_I - (1-\beta)(\eta_v + \zeta_v, (\zeta_u)_t)_I - (\eta_z + \zeta_z - \eta_u - \zeta_u, (\zeta_u)_t)_I \\
&\quad - (1-\beta)\sum_j [\zeta_u]_{j+\frac{1}{2}}[(\zeta_u)_t]_{j+\frac{1}{2}} - \beta \sum_j [\zeta_v]_{j+\frac{1}{2}}[(\zeta_u)_t]_{j+\frac{1}{2}} \\
&\le C\|(\eta_u)_t\|_I^2 + C\|\eta_v + \zeta_v\|_I^2 + C\|\eta_z + \zeta_z\|_I^2 + C\|\eta_u + \zeta_u\|_I^2 + \frac{1}{4}\|(\zeta_u)_t\|_I^2 \\
&\quad + C\sum_j h^{-1}[\zeta_u]_{j+\frac{1}{2}}^2 + C\sum_j h^{-1}[\zeta_v]_{j+\frac{1}{2}}^2 + C_1 \sum_j h[(\zeta_u)_t]_{j+\frac{1}{2}}^2 \\
&\le Ch^{2k+2} + \frac{1}{4}\|(\zeta_u)_t\|_I^2 + C_1 \sum_j h[(\zeta_u)_t]_{j+\frac{1}{2}}^2 \\
&\le Ch^{2k+2} + \frac{1}{2}\|(\zeta_u)_t\|_I^2,
\end{aligned}$$

where the last inequality is due to the trace inequality. Therefore, we conclude that

$$\|(\zeta_u)_t(\cdot, 0)\|_I \le Ch^{k+1}, \tag{A.22}$$

where the constant C depends on α, β, λ, $\|u\|_{k+3}$, and $\|u_t\|_{k+1}$.

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