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Finite Element Approximations of a Class of Nonlinear Stochastic Wave Equations with Multiplicative Noise

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Abstract
Wave propagation problems have many applications in physics and engineering, and the stochastic effects are important in accurately modeling them due to the uncertainty of the media. This paper considers and analyzes a fully discrete finite element method for a class of nonlinear stochastic wave equations, where the diffusion term is globally Lipschitz continuous while the drift term is only assumed to satisfy weaker conditions as in Chow (Ann Appl Probab 12(1):361–381, 2002). The novelties of this paper are threefold. First, the error estimates cannot be directly obtained if the numerical scheme in primal form is used. An equivalent numerical scheme in mixed form is therefore utilized and several Hölder continuity results of the strong solution are proved, which are used to establish the error estimates in both L^2 norm and energy norms. Second, two types of discretization of the nonlinear term are proposed to establish the L^2 stability and energy stability results of the discrete solutions. These two types of discretization and proper test functions are designed to overcome the challenges arising from the stochastic scaling in time issues and the nonlinear interaction. These stability results play key roles in proving the probability of the set on which the error estimates hold approaches one. Third, higher moment stability results of the discrete solu-

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tions are proved based on an energy argument and the underlying energy decaying property of the method. Numerical experiments are also presented to show the stability results of the discrete solutions and the convergence rates in various norms.

Keywords Stochastic wave equation · Multiplicative noise · Finite element method · Higher moment · Error estimate · Stability analysis

Mathematics Subject Classification 65N12 · 65N15 · 65N30

1 Introduction

In this paper, we consider the nonlinear stochastic wave equation with Neumann boundary condition and functional-type multiplicative noise, taking the form of

$$du_t = \Delta u dt + f(u)dt + g(u)dW(t), \qquad \text{in } \mathcal{D}\times(0,T], \tag{1.1}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{in } \partial\mathcal{D}\times(0,T], \tag{1.2}$$

$$u(0) = h_1(x) \quad u_t(0) = h_2(x). \qquad \text{in } \mathcal{D}. \tag{1.3}$$

Here, $\mathcal{D}\subset\mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded domain, $W : \Omega\times(0,T]\to\mathbb{R}$ is a standard Wiener process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$, $h_1(x)$ and $h_2(x)$ are the initial data whose regularity assumptions will be given later. The nonlinear drift term $f(u)$ is assumed to satisfy the conditions specified in [12], which discusses the existence and uniqueness for local and global solutions of (1.1) in Sobolev space. More specifically, we assume

$$f(u) = \sum_{j=1}^{q} a_j(x)u^j, \tag{1.4}$$

where q is an odd integer with $1 \leq q \leq 3$ for $d = 3$ and $q \geq 1$ for $d = 1, 2$, $a_j(x)$ are bounded and continuous for any j, and there exist positive constants $\alpha \geq 0$ and $\lambda > 0$ such that

$$F(u) := -\int_0^u f(s)ds = -\sum_{j=1}^{q}\frac{1}{j+1}a_j(x)u^{j+1} \geq \left(\frac{\alpha}{2} + \frac{\lambda}{2}u^{q-1}\right)u^2. \tag{1.5}$$

One example that satisfies these conditions is $f(u) = -u - u^3$. Furthermore, we assume that $g(u)$ is continuously differentiable, globally Lipschitz continuous, $|g''(u)|$ is bounded, and satisfies the growth condition, i.e., there exists a constant C such that

$$g \in C^1, \tag{1.6}$$

$$|g(a) - g(b)| \leq C|a - b|, \tag{1.7}$$

$$|g(a)|^2 \leq C(1 + a^2), \tag{1.8}$$

$$|g''(u)| \leq C. \tag{1.9}$$

Throughout this paper, C denotes a generic positive constant, which may have different values at different occasions. Under these assumptions on the drift term and the diffusion term, it is proved in [12] that, for a bounded domain $\mathcal{D}$ with C^2 boundary, there exists a unique continuous solution $u(t,\cdot) \in H^1(\mathcal{D})$ and $u_t(t,\cdot) \in L^2(\mathcal{D})$ in $[0, T]$ such that

$$\mathbb{E}\left[\sup_{t\leq T}\left\{\|u_t(t,x)\|_{L^2}^2 + \|\nabla u(t,x)\|_{L^2}^2 + \|u(t,x)\|_{L^{q+1}}^{q+1}\right\}\right] < \infty. \tag{1.10}$$

Wave propagation problem arises in many fields of scientific and engineering applications [25], with examples including the geoscience, petroleum engineering, telecommunication, and the defense industry etc. The deterministic wave equations have been extensively investigated in the last few decades. For the second-order wave equation with nonlinear drift terms, we refer the readers to some partial differential equation (PDE) papers in [32, 45], which discuss their well-posedness or blow-up properties based on different polynomial nonlinear drift terms. A large variety of numerical methods have been proposed for the numerical approximations of the second-order wave equation, including finite difference, finite element, finite volume, spectral methods and integral equation based methods. We refer to [1, 3, 4, 11, 15, 16, 24, 26, 33, 34, 42, 46–48, 50, 51], and the references therein for various numerical methods based on the Galerkin approach.

In the wave propagation applications, the stochastic effects are important in accurately modeling them due to the uncertainty of the media. The stochastic wave equation is a hyperbolic type stochastic partial differential equation (SPDE), and its solution behavior is very different from that of the stochastic heat equation. Numerical methods for the wave equation with various forms of stochasticity have been studied in the literature. For example, the wave equations with random coefficients or random initial/boundary conditions were studied in [10, 31, 43], and numerical methods for the stochastic wave equation with additive noise were presented in [18, 20, 35, 37, 39]. For the multiplicative noise, the well-posedness or the regularity of the solutions of the stochastic wave equation was considered in [21, 22, 40, 41], where the nonlinear drift term is Lipschitz continuous, and the noise is white in time and correlated in space. Later, in [12, 14], some blow-up solutions were presented for a class of nonlinear stochastic wave equations with white in time and correlated in space noise. In [12], the theorems on well-posedness of the local and global solutions were given for the stochastic wave equation with nonlinear drift term in the form of (1.4). The long-time asymptotic bounds of the solutions were proved in [13]. There have also been some studies on numerical methods for the stochastic wave equation with multiplicative noise. For example, in [49], a difference scheme was presented for the model when both the nonlinear drift term f and diffusion term g are Lipschitz continuous, and the optimal rates of convergence were established for such method. The stochastic wave equations with Lipschitz continuous nonlinear drift term and diffusion term were studied using the semi-group approach in [2, 17, 19, 38, 44]. As a comparison, the multiplicative noise is considered in this paper based on the variational approach (see [27–30]), and the nonlinear drift term is not Lipschitz continuous.

In this paper, we present the fully discrete methods for the stochastic wave equation with multiplicative noise, by utilizing the finite element approximation in space, and the implicit Euler/modified Crank–Nicolson approximation in time. There are two main objectives in this paper. First, we want to establish the second moment and higher moment stability results of the discrete solutions in L^2 norm and various energy norms. This is motivated by the conjecture that the numerical solutions will inherit the stability properties of the strong solutions. The goal is achieved by designing the proper numerical scheme, choosing the test function, and using an energy argument. Second, we want to provide the convergence rates of the error estimates in both L^2 and energy norms for the proposed method when the nonlinear drift term is not Lipschitz continuous and the diffusion term is multiplicative noise, and this would be the first work using the variational approach, to our best knowledge.

To achieve this objective, we rewrite the numerical scheme in the mixed form, utilize the above stability results, establish several Hölder continuity results, and construct a subset with nearly probability one, to handle complex interaction between the multiplicative noise and super-linear nonlinear term, such that the error estimates hold on this subset. This concept is proposed based on the bounds of the solutions and is motivated by the idea in [9].

The rest of the paper is organized as follows. In Sect. 2, we prove the Hölder continuity results and some technical lemmas that can be used to establish the error estimates. In Sect. 3, we present the fully discrete finite element methods for the stochastic wave equation, and prove the discrete stability in L^2 norm, discrete stability in various energy norm, as well as the discrete higher moment stability in L^2 norm and energy norms. In Sect. 4, the error estimates in L^2 norm and energy norms are established on a subset of which the probability approaches 1 as spatial mesh size decreases to 0. Numerical experiments are provided in Sect. 5 to validate the theoretical results, including discrete stability results and convergence rates in different norms.

2 Preliminaries and Properties of SPDE Solution

In this section, we first present some basic lemmas that will be used in the stability and error estimate analysis in Sects. 3 and 4. We then present some more useful lemmas about the Hölder continuity in time for the strong solution u in various norms, whose proofs are given in Appendix.

The standard Sobolev notations are adopted in the paper. We use $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k}$ to denote the L^p and H^k norms in the whole domain $\mathcal{D}$, respectively. The notation $(\cdot,\cdot)$ represents the standard inner product on $\mathcal{D}$.

Lemma 1 *In the 1D and 2D settings, for each individual term u^q of the nonlinear function $f(\cdot)$ defined in (1.4), we have for any integer $0 \le j \le q$,*

$$\begin{aligned}&\|u^j v^{q-j} - \tilde{u}^j \tilde{v}^{q-j}\|_{L^2}^2\\&\quad \le C\left(\|u\|_{H^1}^{2(q-1)} + \|v\|_{H^1}^{2(q-1)} + \|\tilde{u}\|_{H^1}^{2(q-1)} + \|\tilde{v}\|_{H^1}^{2(q-1)}\right)\left(\|u-\tilde{u}\|_{H^1}^2 + \|v-\tilde{v}\|_{H^1}^2\right),\end{aligned}$$

where $q > 0$ is an integer, and $u, v, \tilde{u}, \tilde{v}$ are real value functions belonging to $H^1(\Omega)$. The same result holds for $q = 1, 2, 3$ in the 3D setting.

Proof A direct calculation shows that

$$u^j v^{q-j} - \tilde{u}^j \tilde{v}^{q-j} = v^{q-j}(u^j - \tilde{u}^j) + \tilde{u}^j(v^{q-j} - \tilde{v}^{q-j}),$$

whence by applying Young's inequality and the embedding theorem,

$$\begin{aligned}\|v^{q-j}(u^j - \tilde{u}^j)\|_{L^2}^2 &= \|v^{q-j}(u^{j-1} + u^{j-2}\tilde{u} + \cdots + \tilde{u}^{j-1})(u-\tilde{u})\|_{L^2}^2\\&\le C\left(\|v^{q-1}(u-\tilde{u})\|_{L^2}^2 + \|u^{q-1}(u-\tilde{u})\|_{L^2}^2 + \|\tilde{u}^{q-1}(u-\tilde{u})\|_{L^2}^2\right)\\&\le C\left(\|u\|_{H^1}^{2(q-1)} + \|v\|_{H^1}^{2(q-1)} + \|\tilde{u}\|_{H^1}^{2(q-1)}\right)\|u-\tilde{u}\|_{H^1}^2, \qquad (2.1)\end{aligned}$$

and

$$\begin{aligned}\|\tilde{u}^j(v^{q-j} - \tilde{v}^{q-j})\|_{L^2}^2 &= \|\tilde{u}^j(v^{q-j-1} + v^{q-j-2}\tilde{v} + \cdots + \tilde{v}^{q-j-1})(v-\tilde{v})\|_{L^2}^2\\&\le C\left(\|v^{q-1}(v-\tilde{v})\|_{L^2}^2 + \|\tilde{u}^{q-1}(v-\tilde{v})\|_{L^2}^2 + \|\tilde{v}^{q-1}(v-\tilde{v})\|_{L^2}^2\right)\end{aligned}$$

$$\leq C\left(\|v\|_{H^1}^{2(q-1)}+\|\tilde{u}\|_{H^1}^{2(q-1)}+\|\tilde{v}\|_{H^1}^{2(q-1)}\right)\|v-\tilde{v}\|_{H^1}^2. \tag{2.2}$$

Combining (2.1) and (2.2) leads to the desired result. □

The following lemma on the discrete summation-by-parts property is also provided. The proof is straightforward and skipped here.

Lemma 2 (summation-by-parts) *Suppose* $\{a_n\}_{n=0}^{\ell}$ *and* $\{b_n\}_{n=0}^{\ell}$ *are two sequences of functions. Then*

$$\sum_{n=1}^{\ell}(a^n-a^{n-1},b^n)=(a^\ell,b^\ell)-(a^0,b^0)-\sum_{n=1}^{\ell}(a^{n-1},b^n-b^{n-1}).$$

Next, we present some results on the Hölder continuity in time for the strong solution u in various norms, which are useful in the error estimates. Note that the regularity of the solution is given in (1.10), and the requirements of the spatial regularity of the solution in these Lemmas are more stringent. The proofs of these Lemmas are postponed to Appendix for the reader's convenience.

Define a new variable v as $v:=u_t$. We first give the Hölder continuity in time for the strong solution u and v with respect to the spatial L^2 norm.

Lemma 3 (Hölder continuity in time for u in L^2 norm) *Let u be the strong solution to problem* (1.1)–(1.3). *Then for any $s,t\in[0,T]$ with $s<t$, we have*

$$\mathbb{E}\big[\|u(t)-u(s)\|_{L^2}^2\big]\leq C(t-s)^2, \tag{2.3}$$

where

$$C=C\mathbb{E}\big[\|h_2\|_{L^2}^2\big]+C\mathbb{E}\Big[\int_0^t\|\Delta u(\zeta)\|_{L^2}^2d\zeta\Big]+C\mathbb{E}\Big[\int_0^t\|u(\zeta)\|_{L^{2q}}^{2q}d\zeta\Big]+C.$$

Lemma 4 (Hölder continuity in time for v in L^2 norm) *For any $s,t\in[0,T]$ with $s<t$, we have*

$$\mathbb{E}\big[\|v(t)-v(s)\|_{L^2}^2\big]\leq C(t-s), \tag{2.4}$$

where

$$C=C\mathbb{E}\Big[\int_s^t\|\Delta u(\zeta)\|_{L^2}^2d\zeta\Big]+C\mathbb{E}\Big[\int_s^t\|u(\zeta)\|_{L^{2q}}^{2q}d\zeta\Big]+C\sup_{s\leq\zeta\leq t}\mathbb{E}\big[\|u(\zeta)\|_{L^2}^2\big]+C.$$

Next, we give the Hölder continuity in time for the strong solution u and v with respect to the spatial H^1-seminorm.

Lemma 5 (Hölder continuity in time for u in H^1-seminorm) *Let u be the strong solution to problem* (1.1)–(1.3). *Under the assumptions* (1.6)–(1.8), *for any $s,t\in[0,T]$ with $s<t$, we have*

$$\mathbb{E}\big[\|\nabla(u(t)-u(s))\|_{L^2}^2\big]\leq C(t-s)^2, \tag{2.5}$$

where

$$C=C\mathbb{E}\big[\|\nabla h_2\|_{L^2}^2\big]+C\mathbb{E}\Big[\int_0^t\|\nabla\Delta u(\zeta)\|_{L^2}^2d\zeta\Big]+C\mathbb{E}\Big[\int_0^t\|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)}d\zeta\Big]$$
$$+C\mathbb{E}\Big[\int_0^t\|\nabla u(\zeta)\|_{L^4}^4d\zeta\Big]+C.$$

Lemma 6 (Hölder continuity in time for v in H^1-seminorm) *Let u be the strong solution to problem (1.1)–(1.3). Under the assumptions (1.6)–(1.8), for any $s, t \in [0, T]$ with $s < t$, we have*

$$\mathbb{E}\big[\|\nabla(v(t) - v(s))\|_{L^2}^2\big] \leq C(t - s), \tag{2.6}$$

where

$$C = C\mathbb{E}\big[\int_s^t \|\nabla\Delta u(\zeta)\|_{L^2}^2 d\zeta\big] + C\mathbb{E}\big[\int_s^t \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)} d\zeta\big] + C \sup_{s\leq\zeta\leq t} \mathbb{E}\big[\|\nabla u(\zeta)\|_{L^4}^4\big] + C.$$

At the end of this section, we give the Hölder continuity in time for the strong solution u with respect to the spatial H^2-seminorm.

Lemma 7 (Hölder continuity in time for u in H^2-seminorm) *Let u be the strong solution to problem (1.1)–(1.3). Under the assumptions (1.6)–(1.9), for any $s, t \in [0, T]$ with $s < t$, we have*

$$\mathbb{E}\big[\|\nabla^2(u(t) - u(s))\|_{L^2}^2\big] \leq C(t - s)^2, \tag{2.7}$$

where

$$C = C\mathbb{E}\big[\|\nabla^2 h_2\|_{L^2}^2\big] + C\mathbb{E}\big[\int_0^t \|\nabla^2\Delta u(\zeta)\|_{L^2}^2 d\zeta\big] + C\mathbb{E}\big[\int_0^t \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)} d\zeta\big] + C\mathbb{E}\big[\int_0^t \|\nabla^2 u(\zeta)\|_{L^4}^4 d\zeta\big] + C\mathbb{E}\big[\int_0^t \|\nabla u(\zeta)\|_{L^8}^8 d\zeta\big] + C.$$

Remark 1 We note that the constants in the above Lemmas hinge only on the regularity of u in space. More precisely, under the regularity assumption that

$$\begin{aligned}\mathbb{E}\Big[\|h_2\|_{H^2}^2 + \int_0^T & \|\Delta u(\zeta)\|_{H^2}^2 + \|\nabla^2 u(\zeta)\|_{L^4}^4 + \|\nabla u(\zeta)\|_{L^8}^8 \\ & + \|u(\zeta)\|_{L^{2q}}^{2q} + \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)}\, d\zeta\Big] \\ & + \sup_{0\leq\zeta\leq T} \mathbb{E}\big[\|\nabla u(\zeta)\|_{L^4}^4 + \|u(\zeta)\|_{L^2}^2\big] < \infty,\end{aligned} \tag{2.8}$$

all the constants in Lemmas 3–7 are bounded.

Remark 2 Following the proofs of Lemmas 3–7, we could derive the Hölder continutiy in time for u in H^σ norm and for v in $H^{\sigma-1}$ norm ($\sigma \geq 2$), namely, for any $s, t \in [0, T]$ with $s < t$,

$$\begin{aligned}\mathbb{E}[\|u(t) - u(s)\|_{H^\sigma}^2] &\leq C(\sigma, u)(s - t)^2, \\ \mathbb{E}[\|v(t) - v(s)\|_{H^{\sigma-1}}^2] &\leq C(\sigma, u)(s - t),\end{aligned} \tag{2.9}$$

where the constant $C(\sigma, u)$ depends on the proper spatial derivatives of the solutions for the space-smooth noise. In particular, the explicit dependence when $\sigma = 2$ is given in (2.8).

3 Fully Discrete Finite Element Methods and Stability Estimates

In this section, we start by presenting the fully discrete finite element methods for the stochastic wave equations (1.1)–(1.4), and then establish several stability estimates of the numerical solutions. In addition to the second moment stability in L^2 norm and energy norms of the discrete numerical solutions, the stability of higher moments is also provided.

3.1 Notations and the Finite Element Methods

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the domain $\mathcal{D}$. We consider the $\mathcal{P}_r$-Lagrangian finite element space

$$V_h = \left\{ v_h \in C(\bar{\mathcal{D}}) : v_h|_K \in \mathcal{P}_r(K),\ \ \forall K \in \mathcal{T}_h \right\}, \tag{3.1}$$

where $\bar{\mathcal{D}}$ is the closure of the domain $\mathcal{D}$, $\mathcal{P}_r(K)$ denotes the space of all polynomials of degrees up to r on K, and $r \geq 1$ is an integer. Consider a uniform partition of the time domain $[0,\ T]$ with $\tau = T/N$, and denote $t_n = n\tau$ for $n = 0, 1, \cdots N$.

The fully discretized numerical methods for (1.1) is to seek an $\mathcal{F}_{t_n}$ adapted V_h-valued process $\{u_h^n\}_{n=0}^N$ such that it holds $\mathbb{P}$-almost surely that:

$$\begin{aligned} &\left(\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\tau}, w_h \right) + \tau(\nabla u_h^{n+1}, \nabla w_h) \\ &\quad = \tau(f_h^{n+1}, w_h) + (g(u_h^n), w_h)\, \bar{\Delta} W_{n+1} \quad \forall\, w_h \in V_h, \end{aligned} \tag{3.2}$$

where the notation $\bar{\Delta} W_{n+1}$ is defined by

$$\bar{\Delta} W_{n+1} := W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, \tau), \tag{3.3}$$

and there are two choices for the discretization of the nonlinear drift term:

1. Fully implicit discretization:

$$f_h^{n+1} := f(u_h^{n+1}). \tag{3.4}$$

2. Modified Crank–Nicolson discretization:

$$f_h^{n+1} := \hat{f}(u_h^{n+1}, u_h^n) = \begin{cases} -\frac{F(u_h^{n+1}) - F(u_h^n)}{u_h^{n+1} - u_h^n} & \text{if } u_h^{n+1} \neq u_h^n, \\ f(u_h^{n+1}) & \text{if } u_h^{n+1} = u_h^n, \end{cases} \tag{3.5}$$

   where $F(\cdot)$ is defined in (1.5).

The finite element method (3.2) involves a two-step implicit temporal discretization, and would need two initial conditions u_h^0 and u_h^{-1} to start. The initial condition $u_h^0 := P_h u(x, 0)$ is obtained via a standard L^2-projection operator defined as $P_h : L^2(\mathcal{D}) \longrightarrow V_h$ satisfying

$$(P_h u, w_h) = (u, w_h) \quad \forall w_h \in V_h,$$

and $u_h^{-1} = u_h^0 - \tau P_h u_t(x, 0)$, namely, the backward Euler method is used for the initial step. The discrete Laplace operator $\Delta_h : V_h \mapsto V_h$ is defined as follows: given $z_h \in V_h$, $\Delta_h z_h \in V_h$ is chosen such that

$$(\Delta_h z_h, w_h) = -(\nabla z_h, \nabla w_h) \quad \forall\, w_h \in V_h. \tag{3.6}$$

3.2 Stability in L^2 Norm and Energy Norms

For the deterministic wave equation (i.e., $g = 0$ in (1.1)), it is well-known that this model preserves the Hamiltonian, defined as

$$\mathcal{H}(u):=\frac{1}{2}\|u_t\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2 + (F(u), 1),$$

where $F(u)$ satisfies the condition (1.5). The discrete analogue of the Hamiltonian is defined as

$$\tilde{\mathcal{H}}(u_h^n):=\frac{1}{2}\|d_t u_h^n\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2 + (F(u_h^n), 1), \tag{3.7}$$

where d_t denotes the temporal difference operator defined by

$$d_t u_h^n = \frac{u_h^n - u_h^{n-1}}{\tau}. \tag{3.8}$$

Before starting the stability estimate, we summarize the assumption on the nonlinear drift term below.

Assumption 1 The nonlinear drift term $f(u)$ given in (1.4) satisfies (1.5). Furthermore, $F(\cdot)$ is convex if the fully implicit discretization (3.4) of the nonlinear term is utilized.

Theorem 1 (stability of discrete Hamiltonian) *Let $\{u_h^\ell\}_{\ell=0}^N$ denote the numerical solutions of the finite element methods* (3.2). *Under the Assumption 1, the following inequality holds for any integer $\ell \in [1, N]$,*

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\|d_t u_h^\ell\|_{L^2}^2\right] + \frac{1}{2}\mathbb{E}\left[\|\nabla u_h^\ell\|_{L^2}^2\right] + (F(u_h^\ell), 1)\\
&\quad + \frac{1}{4}\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|d_t u_h^{n+1} - d_t u_h^n\|_{L^2}^2\right] + \frac{1}{2}\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2\right]\\
&= \mathbb{E}\left[\tilde{\mathcal{H}}(u_h^\ell)\right] + \frac{1}{4}\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|d_t u_h^{n+1} - d_t u_h^n\|_{L^2}^2\right] + \frac{1}{2}\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2\right] \le C.
\end{aligned}$$

Proof Taking the test function $w_h = d_t u_h^{n+1}$ in (3.2), we have

$$\begin{aligned}
&\left(\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\tau}, d_t u_h^{n+1}\right) + \tau\left(\nabla u_h^{n+1}, \nabla d_t u_h^{n+1}\right)\\
&= \tau\left(f_h^{n+1}, d_t u_h^{n+1}\right) + \left(g(u_h^n), d_t u_h^{n+1}\right)\bar{\Delta} W_{n+1}.
\end{aligned} \tag{3.9}$$

The two terms on the left can be rewritten as:

$$\begin{aligned}
&\left(\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\tau}, d_t u_h^{n+1}\right) = \left(d_t u_h^{n+1} - d_t u_h^n, d_t u_h^{n+1}\right)\\
&= \frac{1}{2}\|d_t u_h^{n+1}\|_{L^2}^2 - \frac{1}{2}\|d_t u_h^n\|_{L^2}^2 + \frac{1}{2}\|d_t u_h^{n+1} - d_t u_h^n\|_{L^2}^2,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\tau\left(\nabla u_h^{n+1}, \nabla d_t u_h^{n+1}\right) = \left(\nabla u_h^{n+1}, \nabla u_h^{n+1} - \nabla u_h^n\right)\\
&= \frac{1}{2}\|\nabla u_h^{n+1}\|_{L^2}^2 - \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2 + \frac{1}{2}\|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2.
\end{aligned} \tag{3.11}$$

Taking the expectation on the last term yields

$$\mathbb{E}\left[\left(g(u_h^n), d_t u_h^{n+1}\right) \bar{\Delta} W_{n+1}\right] = \mathbb{E}\left[\left(g(u_h^n), d_t u_h^{n+1} - d_t u_h^n\right) \bar{\Delta} W_{n+1}\right]$$
$$\leq C\tau \mathbb{E}\left[1 + \|u_h^n\|_{L^2}^2\right] + \frac{1}{4}\mathbb{E}\left[\|d_t u_h^{n+1} - d_t u_h^n\|_{L^2}^2\right], \quad (3.12)$$

where the first equality comes from the fact that

$$\mathbb{E}\left[\left(g(u_h^n), d_t u_h^n\right) \bar{\Delta} W_{n+1}\right] = \mathbb{E}\left[\left(g(u_h^n), d_t u_h^n\right)\right] \mathbb{E}\left[\bar{\Delta} W_{n+1}\right] = 0,$$

and the second inequality is a result of the Cauchy-Schwarz inequality, the growth condition of $g(u)$ in (1.8) and the property of $\bar{\Delta} W_{n+1}$ in (3.3).

The bound of the first term on the right-hand side is discussed case by case:

1. For the fully implicit discretization (3.4), use Taylor's formula to derive

$$F(u_h^n) = F(u_h^{n+1}) + f(u_h^{n+1})(u_h^{n+1} - u_h^n) + \frac{1}{2}F''(\xi)(u_h^{n+1} - u_h^n)^2,$$

   where ξ locates between u_h^n and u_h^{n+1}. Notice that $\frac{1}{2}F''(\xi)(u_h^{n+1} - u_h^n)^2 \geq 0$ under the Assumption 1.
2. For the modified Crank–Nicolson discretization (3.5),

$$F(u_h^n) = F(u_h^{n+1}) + \hat{f}(u_h^{n+1}, u_h^n)(u_h^{n+1} - u_h^n).$$

Therefore, one can conclude that, for both fully implicit (3.4) and modified Crank–Nicolson (3.5) discretizations,

$$\tau(-f_h^{n+1}, d_t u_h^{n+1}) \geq (F(u_h^{n+1}) - F(u_h^n), 1). \quad (3.13)$$

Summing the Eq. (3.9) over n from 0 to $\ell - 1$, taking the expectation on both sides and using the results (3.10)–(3.13), we have

$$\mathbb{E}\left[\tilde{\mathcal{H}}(u_h^\ell)\right] + \frac{1}{4}\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|d_t u_h^{n+1} - d_t u_h^n\|_{L^2}^2\right] + \frac{1}{2}\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2\right]$$
$$\leq \mathbb{E}\left[\tilde{\mathcal{H}}(u_h^0)\right] + C\tau\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|u_h^n\|_{L^2}^2\right] + C. \quad (3.14)$$

Applying the Gronwall's inequality yields

$$\mathbb{E}\left[\tilde{\mathcal{H}}(u_h^\ell)\right] + \frac{1}{4}\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|d_t u_h^{n+1} - d_t u_h^n\|_{L^2}^2\right] + \frac{1}{2}\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2\right] \leq C.$$

This gives the desired stability in L^2 norm and energy norms. □

3.3 Stability of the Higher Moments

The following stability of the higher moments can be established based on the stability results in Theorem 1.

Theorem 2 (higher monent stability) *Let $\{u_h^\ell\}_{\ell=0}^N$ denote the numerical solutions of the finite element methods (3.2). Under the Assumption 1, for any integer $p \geq 2$, it holds for any integer $\ell \in [1, N]$ that*

$$\mathbb{E}\left[\|\nabla u_h^\ell\|_{L^2}^p + \|d_t u_h^\ell\|_{L^2}^p + (F(u_h^\ell), 1)^p\right] \leq C. \quad (3.15)$$

Proof To ease the presentation, the proof is divided into three steps.

Step 1 Following the results (3.9)–(3.13) in the proof of Theorem 1, we have

$$\begin{aligned}&\frac{1}{2}\|d_tu_h^{n+1}\|_{L^2}^2-\frac{1}{2}\|d_tu_h^n\|_{L^2}^2+\frac{1}{2}\|d_tu_h^{n+1}-d_tu_h^n\|_{L^2}^2\\&\quad+\frac{1}{2}\|\nabla u_h^{n+1}\|_{L^2}^2-\frac{1}{2}\|\nabla u_h^n\|_{L^2}^2+\frac{1}{2}\|\nabla u_h^{n+1}-\nabla u_h^n\|_{L^2}^2\\&\quad+(F(u_h^{n+1})-F(u_h^n),1)\le(g(u_h^n),d_tu_h^{n+1})\bar{\Delta}W_{n+1},\end{aligned}\tag{3.16}$$

which can be recast as follows thanks to the definition (3.7),

$$\begin{aligned}&\tilde{\mathcal{H}}(u_h^{n+1})-\tilde{\mathcal{H}}(u_h^n)+\frac{1}{2}\|d_tu_h^{n+1}-d_tu_h^n\|_{L^2}^2+\frac{1}{2}\|\nabla u_h^{n+1}-\nabla u_h^n\|_{L^2}^2\\&\le\left(g(u_h^n),d_tu_h^{n+1}\right)\bar{\Delta}W_{n+1}.\end{aligned}\tag{3.17}$$

Utilizing the following identity

$$\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)=\frac{3}{4}\left(\tilde{\mathcal{H}}(u_h^{n+1})+\tilde{\mathcal{H}}(u_h^n)\right)+\frac{1}{4}\left(\tilde{\mathcal{H}}(u_h^{n+1})-\tilde{\mathcal{H}}(u_h^n)\right),$$

and multiplying (3.17) by the term $\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)$, we obtain

$$\begin{aligned}&\frac{3}{4}\left(\tilde{\mathcal{H}}(u_h^{n+1})^2-\tilde{\mathcal{H}}(u_h^n)^2\right)+\frac{1}{4}\left(\tilde{\mathcal{H}}(u_h^{n+1})-\tilde{\mathcal{H}}(u_h^n)\right)^2\\&\quad+\frac{1}{2}\left(\|d_tu_h^{n+1}-d_tu_h^n\|_{L^2}^2+\|\nabla u_h^{n+1}-\nabla u_h^n\|_{L^2}^2\right)\left(\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)\right)\\&\le\left(g(u_h^n),d_tu_h^{n+1}\right)\bar{\Delta}W_{n+1}\left(\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)\right).\end{aligned}\tag{3.18}$$

The right-hand side of (3.18) can be rewritten as

$$\begin{aligned}&\left(g(u_h^n),d_tu_h^{n+1}\right)\bar{\Delta}W_{n+1}\left(\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)\right)\\&=\left(g(u_h^n),d_tu_h^{n+1}-d_tu_h^n\right)\bar{\Delta}W_{n+1}\left(\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)\right)\\&\quad+\left(g(u_h^n),d_tu_h^n\right)\bar{\Delta}W_{n+1}\left(\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)\right)\\&\le\Bigg(\frac{1}{4}\|d_tu_h^{n+1}-d_tu_h^n\|_{L^2}^2+C\left(\|u_h^n\|_{L^2}^2+1\right)(\bar{\Delta}W_{n+1})^2\\&\quad+\left(g(u_h^n),d_tu_h^n\right)\bar{\Delta}W_{n+1}\Bigg)\left(\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)\right),\end{aligned}\tag{3.19}$$

by applying the Cauchy-Schwarz inequality and the growth condition of $g(u)$ in (1.8). The last two terms can be bounded as

$$
\begin{aligned}
&\left(C(\|u_h^n\|_{L^2}^2+1)(\bar{\Delta}W_{n+1})^2+\left(g(u_h^n),d_tu_h^n\right)\bar{\Delta}W_{n+1}\right)\\
&\quad\times\left(\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)\right)\\
&\quad\le\frac{1}{8}\left(\tilde{\mathcal{H}}(u_h^{n+1})-\tilde{\mathcal{H}}(u_h^n)\right)^2+C(\|u_h^n\|_{L^2}^4+1)(\bar{\Delta}W_{n+1})^4\\
&\quad+C\|d_tu_h^n\|_{L^2}^2(\|u_h^n\|_{L^2}^2+1)(\bar{\Delta}W_{n+1})^2+C\tilde{\mathcal{H}}(u_h^n)(\|u_h^n\|_{L^2}^2+1)(\bar{\Delta}W_{n+1})^2\\
&\quad+\frac{3}{2}\tilde{\mathcal{H}}(u_h^n)(g(u_h^n),d_tu_h^n)\bar{\Delta}W_{n+1}.
\end{aligned}\tag{3.20}
$$

Combining the Eq. (3.18) with the results in (3.19)–(3.20) yields

$$
\begin{aligned}
&\frac{3}{4}\left(\tilde{\mathcal{H}}(u_h^{n+1})^2-\tilde{\mathcal{H}}(u_h^n)^2\right)+\frac{1}{8}\left(\tilde{\mathcal{H}}(u_h^{n+1})-\tilde{\mathcal{H}}(u_h^n)\right)^2\\
&\quad+\left(\frac{1}{4}\|d_tu_h^{n+1}-d_tu_h^n\|_{L^2}^2+\frac{1}{2}\|\nabla u_h^{n+1}-\nabla u_h^n\|_{L^2}^2\right)\left(\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)\right)\\
&\le C(\|u_h^n\|_{L^2}^4+1)(\bar{\Delta}W_{n+1})^4+C\|d_tu_h^n\|_{L^2}^2(\|u_h^n\|_{L^2}^2+1)(\bar{\Delta}W_{n+1})^2\\
&\quad+C\tilde{\mathcal{H}}(u_h^n)(\|u_h^n\|_{L^2}^2+1)(\bar{\Delta}W_{n+1})^2+\frac{3}{2}\tilde{\mathcal{H}}(u_h^n)(g(u_h^n),d_tu_h^n)\bar{\Delta}W_{n+1}.
\end{aligned}\tag{3.21}
$$

Summing the Eq. (3.21) over n from 0 to $\ell-1$ and taking expectation on both sides, we obtain

$$
\begin{aligned}
&\frac{3}{4}\mathbb{E}\left[\tilde{\mathcal{H}}(u_h^\ell)^2\right]+\frac{1}{8}\sum_{n=0}^{\ell-1}\mathbb{E}\left[[\tilde{\mathcal{H}}(u_h^{n+1})-\tilde{\mathcal{H}}(u_h^n)]^2\right]+\sum_{n=0}^{\ell-1}\mathbb{E}\Bigg[\Bigg(\frac{1}{4}\|d_tu_h^{n+1}\\
&-d_tu_h^n\|_{L^2}^2+\frac{1}{2}\|\nabla u_h^{n+1}-\nabla u_h^n\|_{L^2}^2\Bigg)\left(\tilde{\mathcal{H}}(u_h^{n+1})+\frac{1}{2}\tilde{\mathcal{H}}(u_h^n)\right)\Bigg]\\
&\le\frac{3}{4}\mathbb{E}\left[\tilde{\mathcal{H}}(u_h^0)^2\right]+C\tau^2\sum_{n=0}^{\ell-1}\mathbb{E}\left[(\|u_h^n\|_{L^2}^4+1)\right]\\
&+C\tau\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|d_tu_h^n\|_{L^2}^2(\|u_h^n\|_{L^2}^2+1)\right]+C\tau\sum_{n=0}^{\ell-1}\mathbb{E}\left[\tilde{\mathcal{H}}(u_h^n)(\|u_h^n\|_{L^2}^2+1)\right].
\end{aligned}\tag{3.22}
$$

Following the definition of $\tilde{\mathcal{H}}(u_h^n)$ in (3.7), one has $\tilde{\mathcal{H}}(u_h^n)\ge\frac{1}{2}\max\left(\|u_h^n\|_{L^2}^2,\|d_tu_h^n\|_{L^2}^2\right)$, which implies that

$$
\begin{aligned}
&\|u_h^n\|_{L^2}^4+1\le C\tilde{\mathcal{H}}(u_h^n)^2+1,\\
&\|d_tu_h^n\|_{L^2}^2(\|u_h^n\|_{L^2}^2+1)\le C\|d_tu_h^n\|_{L^2}^4+\left(\|u_h^n\|_{L^2}^4+1\right)\le C\tilde{\mathcal{H}}(u_h^n)^2+1,\\
&\tilde{\mathcal{H}}(u_h^n)(\|u_h^n\|_{L^2}^2+1)\le C\tilde{\mathcal{H}}(u_h^n)^2+\left(\|u_h^n\|_{L^2}^4+1\right)\le C\tilde{\mathcal{H}}(u_h^n)^2+1.
\end{aligned}
$$

Therefore, the following result can be obtained by applying Gronwall's inequality:

$$
\begin{aligned}
&\mathbb{E}\left[\tilde{\mathcal{H}}(u_h^\ell)^2\right]+\sum_{n=0}^{\ell-1}\mathbb{E}\left[\left(\tilde{\mathcal{H}}(u_h^{n+1})-\tilde{\mathcal{H}}(u_h^n)\right)^2\right]\\
&\quad+\sum_{n=0}^{\ell-1}\mathbb{E}\left[\left(\|d_tu_h^{n+1}-d_tu_h^n\|_{L^2}^2+\|\nabla u_h^{n+1}-\nabla u_h^n\|_{L^2}^2\right)\left(\tilde{\mathcal{H}}(u_h^{n+1})+\tilde{\mathcal{H}}(u_h^n)\right)\right]\\
&\le C,
\end{aligned}\tag{3.23}
$$

which gives us the fourth moment stability (i.e., (3.15) when $p = 4$).

Step 2 Next, the higher moment stability (3.15) can be established for $p = 2^m$ with any positive integer m. Similar to the derivation in Step 1, we start from the equation (3.21) and multiply it by $\tilde{\mathcal{H}}(u_h^{n+1})^2 + \frac{1}{2}\tilde{\mathcal{H}}(u_h^n)^2$. After some simple algebra, we can obtain the fourth moment stability of the numerical solution u_h^ℓ below

$$\mathbb{E}\left[\tilde{\mathcal{H}}(u_h^\ell)^4\right] + \sum_{n=0}^{\ell-1}\mathbb{E}\left[\left(\left(\tilde{\mathcal{H}}(u_h^{n+1})\right)^2 - \left(\tilde{\mathcal{H}}(u_h^n)\right)^2\right)^2\right] \le C. \tag{3.24}$$

Applying this process repeatedly, the 2^m-th moment stability of the numerical solution u_h^ℓ can be obtained for any positive integer m.

Step 3 For arbitrary positive integer p, suppose $2^{m-1} \le p \le 2^m$ for some m, and one can apply the Young's inequality to obtain

$$\mathbb{E}\left[\tilde{\mathcal{H}}(u_h^\ell)^p\right] \le \mathbb{E}\left[\tilde{\mathcal{H}}(u_h^\ell)^{2^m}\right] + C \le C, \tag{3.25}$$

where the second inequality follows from the results in *Step 2*. The proof is therefore complete. □

4 Error Estimates

In this section, we present the error estimates of the proposed finite element methods. The stability estimates studied in the previous section are crucial in the analysis.

4.1 Error Equations in Mixed Form

Denoting $v = u_t$, we can rewrite the SPDE (1.1) as

$$\begin{cases} du = vdt, \\ dv = \Delta udt + f(u)dt + g(u)dW(t). \end{cases} \tag{4.1}$$

Define v_h^n by $v_h^n := \frac{u_h^n - u_h^{n-1}}{\tau}$, and define the numerical errors by $e_u^n := u(t_n) - u_h^n := \eta_u^n + \xi_u^n$, $e_v^n := v(t_n) - v_h^n := \eta_v^n + \xi_v^n$, where

$$\begin{aligned} \eta_u^n &:= u(t_n) - P_h u(t_n) \quad \text{and} \quad \xi_u^n := P_h u(t_n) - u_h^n, \quad n = 0, 1, 2, ..., N, \\ \eta_v^n &:= v(t_n) - P_h v(t_n) \quad \text{and} \quad \xi_v^n := P_h v(t_n) - v_h^n, \quad n = 0, 1, 2, ..., N, \end{aligned}$$

represent the errors of the L^2-projection and the errors between the numerical solution and projected strong solution, respectively.

It follows from (4.1) that for all t_n, there holds $\mathbb{P}$-almost surely

$$(u(t_{n+1}) - u(t_n), w_h) = \int_{t_n}^{t_{n+1}} (v(s), w_h)ds \quad \forall w_h \in V_h, \tag{4.2}$$

$$\begin{aligned} &(v(t_{n+1}) - v(t_n), z_h) + \int_{t_n}^{t_{n+1}} (\nabla u(s), \nabla z_h)ds \\ &\quad = \int_{t_n}^{t_{n+1}} (f(u(s)), z_h)ds + \int_{t_n}^{t_{n+1}} (g(u(s)), z_h)dW(s) \quad \forall z_h \in V_h. \end{aligned} \tag{4.3}$$

Also, the numerical scheme (3.2) can be rewritten in the equivalent mixed form as

$$(u_h^{n+1} - u_h^n, w_h) = \tau(v_h^{n+1}, w_h) \quad \forall w_h \in V_h, \tag{4.4}$$

$$\begin{aligned}&(v_h^{n+1} - v_h^n, z_h) + \tau(\nabla u_h^{n+1}, \nabla z_h) \\ &\quad = \tau(f_h^{n+1}, z_h) + (g(u_h^n), z_h)\,\bar{\Delta} W_{n+1} \quad \forall z_h \in V_h.\end{aligned} \tag{4.5}$$

From the properties of the L^2-projection, we have $(\eta_u^n, w_h) = 0$ and $(\eta_v^n, z_h) = 0$. Combining with the equations (4.2)–(4.5) leads to

$$(\xi_u^{n+1} - \xi_u^n, w_h) = \int_{t_n}^{t_{n+1}} (v(s) - v_h^{n+1}, w_h)ds \quad \forall w_h \in V_h, \tag{4.6}$$

$$\begin{aligned}&(\xi_v^{n+1} - \xi_v^n, z_h) + \int_{t_n}^{t_{n+1}} (\nabla u(s) - \nabla u_h^{n+1}, \nabla z_h)ds \\ &= \int_{t_n}^{t_{n+1}} (f(u(s)) - f_h^{n+1}, z_h)ds \\ &\quad + \int_{t_n}^{t_{n+1}} (g(u(s)) - g(u_h^n), z_h)dW(s) \quad \forall z_h \in V_h.\end{aligned} \tag{4.7}$$

4.2 Error Estimates

To handle the nonlinearity, we define a sequence of subsets as

$$\tilde{\Omega}_{\kappa,m} = \Big\{\omega \in \Omega : \max_{1\le n\le m} \|u_h^n\|_{H^1}^2 + \max_{s\le t_m} \|u(s)\|_{H^1}^2 \le \kappa\Big\}. \tag{4.8}$$

Here $u(s)$ is the strong solution of (1.1)–(1.3), u_h^n is the numerical solution of (3.2), and $\kappa \ge \kappa_0 > 0$ will be specified. Clearly, it holds that $\tilde{\Omega}_{\kappa,0} \supset \tilde{\Omega}_{\kappa,1} \supset \cdots \supset \tilde{\Omega}_{\kappa,\ell}$.

The following lemma about the nonlinear term is needed in the proof of the error estimates.

Lemma 8 (Hölder continuity in time for nonlinear term on subsets) *Let u be the strong solution to problem (1.1)–(1.3). Under the assumptions (1.6)–(1.8), for any $s, t, t' \in [0, T]$ with $t' < s < t \le t_{m+1}$, we have*

$$\mathbb{E}\big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}} \|f(u(s)) - f(u(t))\|_{L^2}^2\big] \le C\kappa^{q-1}(t-s)^2, \tag{4.9}$$

$$\mathbb{E}\big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}} \|f(u(s)) - \hat{f}(u(t), u(t'))\|_{L^2}^2\big] \le C\kappa^{q-1}[(t-s)^2 + (t'-s)^2], \tag{4.10}$$

where $\hat{f}$ is defined in (3.5), and

$$\begin{aligned}C =\, &C\mathbb{E}\big[\|h_2\|_{H^1}^2\big] + C\mathbb{E}\Big[\int_0^T \|\Delta u(\zeta)\|_{H^1}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_0^T \|u(\zeta)\|_{L^{2q}}^{2q} d\zeta\Big] \\ &+ C\mathbb{E}\Big[\int_0^T \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)} d\zeta\Big] + C\mathbb{E}\Big[\int_0^T \|\nabla u(\zeta)\|_{L^4}^4 d\zeta\Big] + C.\end{aligned}$$

Proof By Lemma 1, we obtain

$$\begin{aligned}&\mathbb{E}\big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}} \|f(u(s)) - f(u(t))\|_{L^2}^2\big] \\ &\le C\mathbb{E}\Big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}} \sum_{j=1}^{q} \Big(\|u(s)\|_{H^1}^{2(j-1)} + \|u(t)\|_{H^1}^{2(j-1)} + 1\Big) \|u(s) - u(t)\|_{H^1}^2\Big]\end{aligned}$$

$$\leq C\kappa^{q-1}\mathbb{E}\Big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}}\|u(s)-u(t)\|_{H^1}^2\Big]. \tag{4.11}$$

In addition, for any integer q that satisfies the condition of Lemma 1,

$$\begin{aligned}&\left\|u^q(s)-\frac{1}{q+1}\cdot\frac{u^{q+1}(t)-u^{q+1}(t')}{u(t)-u(t')}\right\|_{L^2}^2=\left\|u^q(s)-\frac{1}{q+1}\sum_{j=0}^{q}u^j(t)u^{q-j}(t')\right\|_{L^2}^2\\&\leq C\left(\|u(s)\|_{H^1}^{2(q-1)}+\|u(t)\|_{H^1}^{2(q-1)}+\|u(t')\|_{H^1}^{2(q-1)}\right)\\&\quad\times\left(\|u(s)-u(t)\|_{H^1}^2+\|u(s)-u(t')\|_{H^1}^2\right),\end{aligned}$$

which implies that

$$\begin{aligned}&\mathbb{E}\big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}}\|f(u(s))-\hat{f}(u(t),u(t'))\|_{L^2}^2\big]\\&\quad\leq C\kappa^{q-1}\mathbb{E}\Big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}}\left(\|u(s)-u(t)\|_{H^1}^2+\|u(s)-u(t')\|_{H^1}^2\right)\Big].\end{aligned}$$

Utilizing the conclusions in Lemma 3 (Hölder continuity for u in L^2 norm) and Lemma 5 (Hölder continuity for u in H^1-seminorm) yields (4.9) and (4.10), with the dependence of the constant as stated. □

Applying Lemma 1 directly, we can obtain the following lemma.

Lemma 9 (error representation for nonlinear term on subsets) *Let u be the strong solution to problem* (1.1)–(1.3). *Under the assumptions* (1.6)–(1.8)*, we have*

$$\begin{aligned}&\mathbb{E}\big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}}\|f(u(t_{m+1}))-f(u_h^{m+1})\|_{L^2}^2\big]\\&\quad\leq C\kappa^{q-1}\mathbb{E}\big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}}\|u(t_{m+1})-u_h^{m+1}\|_{H^1}^2\big], \qquad (4.12)\\&\mathbb{E}\big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}}\|\hat{f}(u(t_{m+1}),u(t_m))-\hat{f}(u_h^{m+1},u_h^m)\|_{L^2}^2\big]\\&\quad\leq C\kappa^{q-1}\mathbb{E}\big[\mathbb{1}_{\tilde{\Omega}_{\kappa,m+1}}\|u(t_{m+1})-u_h^{m+1}\|_{H^1}^2+\mathbb{1}_{\tilde{\Omega}_{\kappa,m}}\|u(t_m)-u_h^m\|_{H^1}^2\big], \qquad (4.13)\end{aligned}$$

where the constant C is independent of u and u_h.

Now we are ready to state the following main theorem on the error estimates.

Theorem 3 (error estimates) *Let $\{u_h^\ell\}_{\ell=0}^N$ denote the numerical solutions of the finite element methods* (3.2). *Under the Assumption 1 and the Hölder continuity assumption* (2.9)*, i.e., for $\sigma\geq 2$,*

$$\begin{aligned}&\mathbb{E}[\|u(t)-u(s)\|_{H^\sigma}^2]\leq C(\sigma,u)(s-t)^2,\\&\mathbb{E}[\|v(t)-v(s)\|_{H^{\sigma-1}}^2]\leq C(\sigma,u)(s-t),\end{aligned}\quad \forall s,t\in[0,T]\ \text{with}\ s<t,$$

the following error estimate holds for any integer $\ell\in[1,N]$:

$$\begin{aligned}&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,\ell}}\|e_u^\ell\|_{L^2}^2\right]+\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,\ell}}\|\nabla e_u^\ell\|_{L^2}^2ds\right]+\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,\ell}}\|e_v^\ell\|_{L^2}^2\right]\\&\quad\leq C(\sigma,u)(\tau+h^{2\min\{r,\sigma-1\}}+\tau^2|\ln h|)h^{-\beta},\end{aligned} \tag{4.14}$$

where r is the polynomial order defined in (3.1)*, $\beta>0$ can be chosen to be small enough, κ satisfies $\kappa^{q-1}=C\ln(h^{-\beta})$ so that $\mathbb{P}\left[\tilde{\Omega}_{\kappa,\ell}\right]\to 1$ as $h\to 0$. Moreover, the explicit* *dependence of* $C(\sigma, u)$ *when* $\sigma = 2$ *is given in* (2.8)*, i.e.,*

$$
\begin{aligned}
C(2,u) = C\mathbb{E}\Big[\|h_2\|_{H^2}^2 + \int_0^T \|\Delta u(\zeta)\|_{H^2}^2 + \|\nabla^2 u(\zeta)\|_{L^4}^4 + \|\nabla u(\zeta)\|_{L^8}^8 \\
+ \|u(\zeta)\|_{L^{2q}}^{2q} + \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)}\, d\zeta\Big] \\
+ C \sup_{0\le\zeta\le T} \mathbb{E}\big[\|\nabla u(\zeta)\|_{L^4}^4 + \|u(\zeta)\|_{L^2}^2\big] + C.
\end{aligned}
$$

Proof For the sake of simplicity of the exposition, we suppress the dependence of σ and u for the generic constant C. By taking $w_h = \xi_u^{n+1}$ in (4.6) and $z_h = \xi_v^{n+1}$ in (4.7), multiplying (4.6) and (4.7) by $\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}$, and then taking the expectation, we obtain

$$
\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}(\xi_u^{n+1} - \xi_u^n, \xi_u^{n+1})\right] = \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}} \int_{t_n}^{t_{n+1}} (v(s) - v_h^{n+1}, \xi_u^{n+1}) ds\right], \tag{4.15}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}(\xi_v^{n+1} - \xi_v^n, \xi_v^{n+1})\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}} \int_{t_n}^{t_{n+1}} (\nabla u(s) - \nabla u_h^{n+1}, \nabla\xi_v^{n+1}) ds\right] \\
&\quad = \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}} \int_{t_n}^{t_{n+1}} (f(u(s)) - f_h^{n+1}, \xi_v^{n+1}) ds\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}} \int_{t_n}^{t_{n+1}} (g(u(s)) - g(u_h^n), \xi_v^{n+1}) dW(s)\right].
\end{aligned} \tag{4.16}
$$

The left-hand side of (4.15) can be bounded by

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}(\xi_u^{n+1} - \xi_u^n, \xi_u^{n+1})\right] \\
&= \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1}\|_{L^2}^2\right] - \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n}\|_{L^2}^2\right] + \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1} - \xi_u^n\|_{L^2}^2\right] \\
&= \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1}\|_{L^2}^2\right] - \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\xi_u^{n}\|_{L^2}^2\right] + \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1} - \xi_u^n\|_{L^2}^2\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[(\mathbb{1}_{\tilde{\Omega}_{\kappa,n}} - \mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}})\|\xi_u^{n}\|_{L^2}^2\right] \\
&\geq \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1}\|_{L^2}^2\right] - \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\xi_u^{n}\|_{L^2}^2\right] + \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1} - \xi_u^n\|_{L^2}^2\right].
\end{aligned} \tag{4.17}
$$

By Lemma 4 (Hölder continuity in time for v in L^2 norm) and the definition of the L^2-projection, the right-hand side of (4.15) is bounded as

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}} \int_{t_n}^{t_{n+1}} (v(s) - v_h^{n+1}, \xi_u^{n+1}) ds\right] \\
&\quad = \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}} \int_{t_n}^{t_{n+1}} (v(s) - v(t^{n+1}), \xi_u^{n+1}) ds\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}} \int_{t_n}^{t_{n+1}} (\eta_v^{n+1} + \xi_v^{n+1}, \xi_u^{n+1}) ds\right] \\
&\quad \leq \tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1}\|_{L^2}^2\right] + \tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}\|_{L^2}^2\right] + C\tau^2.
\end{aligned} \tag{4.18}
$$

Following the derivation of the inequality (4.17), the first term on the left-hand side of (4.16) can be bounded by

$$\begin{aligned}&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}(\xi_v^{n+1}-\xi_v^n,\xi_v^{n+1})\right]\\&\geq\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}\|_{L^2}^2\right]-\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\xi_v^{n}\|_{L^2}^2\right]+\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}-\xi_v^n\|_{L^2}^2\right].\end{aligned}\tag{4.19}$$

The second term on the left-hand side of (4.16) is

$$\begin{aligned}&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(\nabla u(s)-\nabla u_h^{n+1},\nabla\xi_v^{n+1})ds\right]\\&=\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(\nabla u(s)-\nabla u(t_{n+1}),\nabla\xi_v^{n+1})ds\right]\\&\quad+\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(\nabla e_u^{n+1},\nabla\xi_v^{n+1})ds\right].\end{aligned}\tag{4.20}$$

Notice that

$$\begin{aligned}\nabla\xi_v^{n+1}&=\nabla(P_hv(t_{n+1}))-\nabla v_h^{n+1}\\&=\nabla(P_hv(t_{n+1}))-\nabla(d_tu(t_{n+1}))+(\nabla(d_tu(t_{n+1}))-\nabla(d_tu_h^{n+1})).\end{aligned}\tag{4.21}$$

For the first term on the right-hand side of (4.20), we can move it to the right-hand side of (4.16) by adding a negative sign, and then bound it by

$$\begin{aligned}&-\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(\nabla u(s)-\nabla u(t_{n+1}),\nabla\xi_v^{n+1})ds\right]\\&\leq C\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\|\nabla u(s)-\nabla u(t_{n+1})\|_{L^2}^2ds\right]\\&\quad+\tau^2\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\left(\|\nabla(P_hv(t_{n+1}))\|_{L^2}^2+\|\nabla(d_tu(t_{n+1}))\|_{L^2}^2\right)\right]\\&\quad+\frac{1}{4}\tau^2\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla(d_tu(t_{n+1}))-\nabla(d_tu_h^{n+1})\|_{L^2}^2\right]\\&\leq C\tau^2+\frac{1}{4}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}-\nabla e_u^n\|_{L^2}^2\right],\end{aligned}\tag{4.22}$$

where the triangle inequality, the H^1 stability of the L^2-projection [5], Lemma 5 (Hölder continuity in time for u in H^1-seminorm), and the stability of $\mathbb{E}[\|\nabla v\|_{L^2}^2]$ (see the proof of Lemma 5) are used in the derivation of the last inequality. For the second term on the right-hand side of (4.20), we move the first term in (4.21) to the right-hand side of (4.16) by adding a negative sign, and obtain ($\sigma \geq 2$)

$$\begin{aligned}&-\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(\nabla e_u^{n+1},\nabla(P_hv(t_{n+1}))-\nabla(d_tu(t_{n+1}))ds\right]\\&\leq C\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}\|_{L^2}^2\right]\\&\quad+\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla(P_hv(t_{n+1}))-\nabla(P_hd_tu(t_{n+1}))\|_{L^2}^2\right]\\&\quad+\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla(P_hd_tu(t_{n+1}))-\nabla(d_tu(t_{n+1})\|_{L^2}^2\right]\end{aligned}$$

$$
\begin{aligned}
&\leq C\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}\|_{L^2}^2\right] + C\tau^2 \\
&\quad + C\tau h^{2\min\{r,\sigma-1\}}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|d_t u(t_{n+1})\|_{H^\sigma}^2\right] \\
&\leq C\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}\|_{L^2}^2\right] + C\tau^2 + C\tau h^{2\min\{r,\sigma-1\}},
\end{aligned} \tag{4.23}
$$

where the mean value theorem, the H^1 stability of the L^2-projection [5], and Lemma 6 (Hölder continuity in time for v in H^1-seminorm) are used in the second last inequality, and the Hölder continuity in time for u in H^σ norm are used in the last inequality. The second term in (4.21) is bounded by

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(\nabla e_u^{n+1}, \nabla d_t e_u^{n+1})ds\right] \\
&= \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}\|_{L^2}^2\right] - \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n}\|_{L^2}^2\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1} - \nabla e_u^n\|_{L^2}^2\right] \\
&\geq \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}\|_{L^2}^2\right] - \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\nabla e_u^{n}\|_{L^2}^2\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1} - \nabla e_u^n\|_{L^2}^2\right].
\end{aligned} \tag{4.24}
$$

Next, we estimate the nonlinear terms on the right-hand side of (4.16). We split the first term on the right-hand side of (4.16) as follows:

1. For fully implicit discretization (3.4),

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(f(u(s)) - f_h^{n+1}, \xi_v^{n+1})ds\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(f(u(s)) - f(u(t_{n+1})), \xi_v^{n+1})ds\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(f(u(t_{n+1}) - f(u_h^{n+1}), \xi_v^{n+1})ds\right].
\end{aligned} \tag{4.25}
$$

2. For modified Crank–Nicolson discretization (3.5),

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(f(u(s)) - f_h^{n+1}, \xi_v^{n+1})ds\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(f(u(s)) - \hat{f}(u(t_{n+1}), u(t_n)), \xi_v^{n+1})ds\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(\hat{f}(u(t_{n+1}), u(t_n)) - \hat{f}(u_h^{n+1}, u_h^n), \xi_v^{n+1})ds\right].
\end{aligned} \tag{4.26}
$$

By Lemma 8 (Hölder continuity in time for nonlinear term on subsets), the first term on the right-hand side of (4.25) or (4.26) is bounded (taking fully implicit discretization as an example, the modified Crank-Nicolson discretizatin has the same estimate) as

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(f(u(s)) - f(u(t_{n+1})), \xi_v^{n+1})ds\right] \\
&\leq C\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}\|_{L^2}^2\right] + C\kappa^{q-1}\tau^3.
\end{aligned} \tag{4.27}
$$

By Lemma 9 (error representation for nonlinear term on subsets), the second term on the right-hand side of (4.25) or (4.26) is bounded as

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(f(u(t_{n+1})-f_h^{n+1},\xi_v^{n+1})ds\right]\\
&\leq C\kappa^{q-1}\tau\mathbb{E}\Big[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|u(t_{n+1})-u_h^{n+1}\|_{H^1}^2+\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|u(t_n)-u_h^n\|_{H^1}^2\Big]\\
&\quad+\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}\|_{L^2}^2\right]\\
&\leq C\kappa^{q-1}\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1}\|_{L^2}^2+\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\xi_u^n\|_{L^2}^2\right]\\
&\quad+C\kappa^{q-1}\tau h^{2\min\{r+1,\sigma\}}\sup_{s\leq T}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|u(s)\|_{H^\sigma}^2\right]\\
&\quad+C\kappa^{q-1}\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}\|_{L^2}^2+\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\nabla e_u^n\|_{L^2}^2\right]+\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}\|_{L^2}^2\right],
\end{aligned}\tag{4.28}$$

where the standard approximation theory of the L^2 projection is applied.

By Itô isometry, the second term on the right-hand side of (4.16) is bounded as

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(g(u(s))-g(u_h^n),\xi_v^{n+1})dW(s)\right]\\
&=\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(g(u(s))-g(u(t_n)),\xi_v^{n+1}-\xi_v^n)dW(s)\right]\\
&\quad+\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\int_{t_n}^{t_{n+1}}(g(u(t_n))-g(u_h^n),\xi_v^{n+1}-\xi_v^n)dW(s)\right]\\
&\leq\frac{1}{4}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}-\xi_v^n\|_{L^2}^2\right]+C\tau^3+C\tau h^{2\min\{r+1,\sigma\}}\sup_{s\leq T}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|u(s)\|_{H^\sigma}^2\right]\\
&\quad+C\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\xi_u^n\|_{L^2}^2\right].
\end{aligned}\tag{4.29}$$

Combining (4.15)–(4.29), we obtain the estimate

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1}\|_{L^2}^2\right]-\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\xi_u^n\|_{L^2}^2\right]+\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1}-\xi_u^n\|_{L^2}^2\right]\\
&\quad+\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}\|_{L^2}^2\right]-\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\xi_v^n\|_{L^2}^2\right]+\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}-\xi_v^n\|_{L^2}^2\right]\\
&\quad+\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}\|_{L^2}^2ds\right]-\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\nabla e_u^n\|_{L^2}^2ds\right]\\
&\quad+\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}-\nabla e_u^n\|_{L^2}^2ds\right]\\
&\leq C\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1}\|_{L^2}^2+\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\xi_u^n\|_{L^2}^2\right]\\
&\quad+C\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}\|_{L^2}^2\right]+C\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}\|_{L^2}^2\right]\\
&\quad+\frac{1}{4}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1}-\xi_v^n\|_{L^2}^2\right]+\frac{1}{4}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}-\nabla e_u^n\|_{L^2}^2\right]\\
&\quad+C\kappa^{q-1}\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1}\|_{L^2}^2+\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\xi_u^n\|_{L^2}^2\right]\\
&\quad+C\kappa^{q-1}\tau\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1}\|_{L^2}^2+\mathbb{1}_{\tilde{\Omega}_{\kappa,n}}\|\nabla e_u^n\|_{L^2}^2\right]
\end{aligned}$$

$$+ C\tau^2 + C\tau h^{2\min\{r,\sigma-1\}} + C\kappa^{q-1}\tau^3 + C\kappa^{q-1}\tau h^{2\min\{r+1,\sigma\}}.$$

By choosing $\kappa^{q-1} = C\ln(h^{-\beta})$, where $\beta > 0$ is small enough, taking the summation over n from 0 to $\ell - 1$, and applying Gronwall's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,\ell}}\|\xi_u^\ell\|_{L^2}^2\right] + \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,\ell}}\|\nabla e_u^\ell\|_{L^2}^2 ds\right] + \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,\ell}}\|\xi_v^\ell\|_{L^2}^2\right] \\
&\quad + \sum_{n=0}^{\ell-1}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_v^{n+1} - \xi_v^n\|_{L^2}^2\right] + \sum_{n=0}^{\ell-1}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\xi_u^{n+1} - \xi_u^n\|_{L^2}^2\right] \\
&\quad + \sum_{n=0}^{\ell-1}\mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n+1}}\|\nabla e_u^{n+1} - \nabla e_u^n\|_{L^2}^2 ds\right] \\
&\le C(\tau + h^{2\min\{r,\sigma-1\}} + \kappa^{q-1}\tau^2 + \kappa^{q-1}h^{2\min\{r+1,\sigma\}})h^{-\beta} \\
&\le C(\tau + h^{2\min\{r,\sigma-1\}} + \tau^2|\ln h|)h^{-\beta},
\end{aligned} \tag{4.30}$$

By combining (4.30) with the properties of the L^2-projection, we get (4.14).

Using the Markov's inequality, discrete Burkholder–Davis–Gundy inequalities [6–8, 23], Eq. (1.10), and Theorem 1 (stability of discrete Hamiltonian), we have the following property

$$\begin{aligned}
\mathbb{P}\left[\tilde{\Omega}_{\kappa,\ell}\right] &\ge 1 - \frac{\mathbb{E}\left[\max\limits_{1\le n\le \ell}\|u_h^n\|_{H^1}^2 + \max\limits_{s\le t_\ell}\|u(s)\|_{H^1}^2\right]}{(C\ln(h^{-\beta}))^{\frac{1}{q-1}}} \\
&\ge 1 - \frac{C}{\beta^{\frac{1}{q-1}}(-\ln h)^{\frac{1}{q-1}}} \to 1 \text{ as } h \to 0.
\end{aligned} \tag{4.31}$$

This finishes the proof. □

Remark 3 We make the following remarks:

1. We focus on the nonlinear case when $q > 1$ since it is much easier to analyze the linear case when $q = 1$.
2. Under the assumption on the Hölder continuity in time (2.9), the H^1 error order is nearly optimal in space. Due to the limited Hölder continuity of v, only half error order in time can be proved, which seems to be sharp according to the numerical experiments (see the last column in Table 6).
3. The original form (3.2) and the mixed form (4.4)–(4.5) are mathematically equivalent, but there exist some difficulties in analyzing the noise term that might be hardly circumvented if the original form is used.

5 Numerical Tests

In this section, we provide various numerical tests to validate our theoretical results. We consider the stability and error estimates of our proposed numerical schemes based on different nonlinear drift terms $f(u)$ and different diffusion terms $g(u)$ in both one-dimensional and two-dimensional cases.

The piecewise linear ($r = 1$) Lagrangian finite element space is used in all the numerical experiments. The regular Monte-Carlo method is used to compute the stochastic term, and 5, 000 samples are used for all the tests below.

Following the algorithmic introduction to the numerical simulation of SDE [36], we briefly discuss the implementation issues. For any sample, we first generate a discretized Brownian path with a sufficiently small time step size, which is used to compute the reference solution for one sample. The same discretized Brownian path, but with (a large) time step size τ, is used to compute the numerical solution. The expectation is then computed by averaging all the errors for the 5, 000 samples.

When computing the partial expectation, strictly speaking, we need to remove some samples whose continuous or discrete H^1 norm exceeds $\sqrt{\kappa}$, due to the definition of $\tilde{\Omega}_{\kappa,m}$ in (4.8). An interesting observation is that, for a sufficiently large but fixed $\kappa > 0$, we could not find even one sample path such that the error blows up, i.e., all the sample paths in our numerical experiments satisfy

$$\max_{0\le n\le N} \|u_h^n\|_{H^1}^2 + \max_{s\le T} \|u(s)\|_{H^1}^2 \le \kappa.$$

However, such a strong (discrete) H^1 bound is difficult to prove due to the nonlinearity of f and g. Nevertheless, we take a large but fixed $\kappa > 0$ so that the partial expectation $\mathbb{E}[\mathbb{1}_{\tilde{\Omega}_{\kappa,\ell}} \cdot]$ and the standard expectation $\mathbb{E}[\cdot]$ coincide, at the numerical level (see Test 3 below).

Test 1 (Convergence Order Test) Consider the one-dimensional stochastic wave equations (1.1)–(1.4) with the initial conditions

$$h_1(x) = \cos(\pi x), \quad h_2(x) = 0,$$

and different nonlinear drift and diffusion terms outlined below.

First, we consider the nonlinear drift and diffusion terms chosen to be $f(u) = -u - u^3$ and $g(u) = u$. We evaluate the following errors, $\left\{\sup_{0\le n\le N} \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}} \|e_u^n\|_{L^2}^2\right]\right\}^{\frac{1}{2}}$ (L^2 error), $\left\{\sup_{0\le n\le N} \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}} \|\nabla e_u^n\|_{L^2}^2\right]\right\}^{\frac{1}{2}}$ (H^1 error), and $\left\{\sup_{0\le n\le N} \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}} \|d_t e_u^n\|_{L^2}^2\right]\right\}^{\frac{1}{2}}$ ($d_t L^2$ error). Thanks to Lemma 4 (Hölder continuity in time for v in L^2 norm), the $d_t L^2$ error can be estimated as

$$\begin{aligned}\sup_{0\le n\le N} \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}} \|d_t e_u^n\|_{L^2}^2\right] &\le 2 \sup_{0\le n\le N} \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}} \left(\|e_v^n\|_{L^2}^2 + \|d_t u(t_n) - v(t_n)\|_{L^2}^2\right)\right]\\ &\le 2 \sup_{0\le n\le N} \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}} \left(\|e_v^n\|_{L^2}^2 + \sup_{\xi\in(t_{n-1},t_n)} \|v(\xi) - v(t_n)\|_{L^2}^2\right)\right]\\ &\le 2 \sup_{0\le n\le N} \mathbb{E}\left[\mathbb{1}_{\tilde{\Omega}_{\kappa,n}} \|e_v^n\|_{L^2}^2\right] + C\tau.\end{aligned}$$

In view of Theorem 3 (error estimate), the theoretical bound for these errors should be nearly $\mathcal{O}(\tau^{\frac{1}{2}} + h)$.

Table 1 show these errors and their convergence rates with respect to space, when the time step is fixed to be $\tau = 1 \times 10^{-3}$, from which the spatial order of 2 (L^2 error), 1 (H^1 error), 2 ($d_t L^2$ error) can be observed. Similarly, we fix the spatial step size and report the errors and their convergence rates with respect to time in Table 2, where the temporal order of 1 are observed for all three errors.

Next, the nonlinear drift and diffusion terms are chosen to be $f(u) = -u - u^{11}$ and $g(u) = u$, and the corresponding results are demonstrated in Tables 3 and 4. The convergence rates are consistent with those of the previous test where $f(u) = -u - u^3$. Tables 1, 2, 3, and 4 indicate that the H^1 error estimate in Theorem 3 is nearly sharp.

Table 1 Test 1 (a): spatial errors and convergence rates when $\tau = 1 \times 10^{-3}$, $T = 0.01$

	L^2 error	Order	H^1 error	Order	$d_t L^2$ error	Order
$h = 1/4$	7.902×10^{-2}	–	8.798×10^{-1}	–	1.488×10^{-1}	–
$h = 1/8$	1.602×10^{-2}	2.302	4.178×10^{-1}	1.074	3.369×10^{-2}	2.143
$h = 1/16$	3.792×10^{-3}	2.079	2.063×10^{-1}	1.018	8.079×10^{-3}	2.060
$h = 1/32$	9.349×10^{-4}	2.020	1.028×10^{-1}	1.004	1.999×10^{-3}	2.015
$h = 1/64$	2.329×10^{-4}	2.005	5.138×10^{-2}	1.001	4.984×10^{-4}	2.004

Table 2 Test 1 (a): temporal errors and convergence rates when $h = 1/128$, $T = 0.4$

	L^2 error	Order	H^1 error	Order	$d_t L^2$ error	Order
$\tau = 0.1$	3.506×10^{-2}	–	1.102×10^{-1}	–	2.630×10^{-1}	–
$\tau = 0.1/2$	1.495×10^{-2}	1.230	4.708×10^{-2}	1.227	1.577×10^{-1}	0.738
$\tau = 0.1/4$	6.379×10^{-3}	1.229	2.017×10^{-2}	1.223	8.577×10^{-2}	0.879
$\tau = 0.1/8$	2.860×10^{-3}	1.157	9.063×10^{-3}	1.154	4.451×10^{-2}	0.946
$\tau = 0.1/16$	1.341×10^{-3}	1.093	4.253×10^{-3}	1.092	2.278×10^{-2}	0.966
$\tau = 0.1/32$	6.572×10^{-4}	1.029	2.084×10^{-3}	1.029	1.164×10^{-2}	0.969

Table 3 Test 1 (b): spatial errors and convergence rates when $\tau = 1 \times 10^{-3}$, $T = 0.01$

	L^2 error	Order	H^1 error	Order	$d_t L^2$ error	Order
$h = 1/4$	7.982×10^{-2}	–	8.828×10^{-1}	–	1.938×10^{-1}	–
$h = 1/8$	1.614×10^{-2}	2.306	4.207×10^{-1}	1.069	3.445×10^{-2}	2.492
$h = 1/16$	3.801×10^{-3}	2.086	2.068×10^{-1}	1.024	8.168×10^{-3}	2.076
$h = 1/32$	9.369×10^{-4}	2.020	1.031×10^{-1}	1.004	2.017×10^{-3}	2.018
$h = 1/64$	2.334×10^{-4}	2.005	5.149×10^{-2}	1.002	5.022×10^{-4}	2.006

Table 4 Test 1 (b): temporal errors and convergence rates when $h = 1/128$, $T = 0.4$

	L^2 error	Order	H^1 error	Order	$d_t L^2$ error	Order
$\tau = 0.1$	3.206×10^{-2}	–	1.008×10^{-1}	–	2.526×10^{-1}	–
$\tau = 0.1/2$	1.342×10^{-2}	1.256	4.242×10^{-2}	1.249	1.504×10^{-1}	0.748
$\tau = 0.1/4$	5.692×10^{-3}	1.237	1.830×10^{-2}	1.213	8.125×10^{-2}	0.888
$\tau = 0.1/8$	2.536×10^{-3}	1.166	8.338×10^{-3}	1.134	4.236×10^{-2}	0.940
$\tau = 0.1/16$	1.196×10^{-3}	1.084	3.996×10^{-3}	1.061	2.164×10^{-2}	0.969
$\tau = 0.1/32$	5.938×10^{-4}	1.010	1.995×10^{-3}	1.002	1.106×10^{-2}	0.968

Table 5 Test 1 (c): spatial errors and convergence rates when $\tau = 1 \times 10^{-3}$, $T = 0.01$

	L^2 error	Order	H^1 error	Order	$d_t L^2$ error	Order
$h = 1/4$	7.901×10^{-2}	–	8.798×10^{-1}	–	1.469×10^{-1}	–
$h = 1/8$	1.602×10^{-2}	2.302	4.178×10^{-1}	1.074	3.364×10^{-2}	2.127
$h = 1/16$	3.792×10^{-3}	2.079	2.063×10^{-1}	1.018	8.053×10^{-3}	2.063
$h = 1/32$	9.349×10^{-4}	2.020	1.028×10^{-1}	1.005	1.991×10^{-3}	2.016
$h = 1/64$	2.329×10^{-4}	2.005	5.138×10^{-2}	1.001	4.964×10^{-4}	2.004

Table 6 Test 1 (c): temporal errors and convergence rates when $h = 1/128$, $T = 0.4$

	L^2 error	Order	H^1 error	Order	$d_t L^2$ error	Order
$\tau = 0.1$	1.132×10^{-1}	–	3.502×10^{-1}	–	1.052×10^{-1}	–
$\tau = 0.1/2$	7.337×10^{-2}	0.626	2.292×10^{-1}	0.612	7.203×10^{-2}	0.546
$\tau = 0.1/4$	4.198×10^{-2}	0.805	1.318×10^{-1}	0.798	4.773×10^{-2}	0.594
$\tau = 0.1/8$	2.242×10^{-2}	0.905	7.068×10^{-2}	0.899	3.184×10^{-2}	0.584
$\tau = 0.1/16$	1.158×10^{-2}	0.953	3.666×10^{-2}	0.947	2.169×10^{-2}	0.554
$\tau = 0.1/32$	5.890×10^{-3}	0.975	1.871×10^{-2}	0.970	1.495×10^{-2}	0.537

Table 7 Test 2: spatial errors and convergence rates when $\tau = 5 \times 10^{-3}$, $T = 5 \times 10^{-2}$

	L^2 error	Order	H^1 error	Order	$d_t L^2$ error	Order
$h = 1/16$	2.565×10^{-4}	–	2.335×10^{-2}	–	4.586×10^{-3}	–
$h = 1/32$	6.355×10^{-5}	2.013	1.097×10^{-2}	1.089	1.896×10^{-3}	1.275
$h = 1/64$	1.642×10^{-5}	1.952	5.500×10^{-3}	0.996	5.391×10^{-4}	1.814
$h = 1/128$	4.142×10^{-6}	1.987	2.746×10^{-3}	1.002	1.410×10^{-4}	1.935
$h = 1/256$	1.038×10^{-6}	1.997	1.373×10^{-3}	1.001	3.565×10^{-5}	1.984

Lastly, the nonlinear drift and diffusion terms are chosen to be $f(u) = -u - u^3$ and $g(u) = \sqrt{u^2 + 0.01}$, and the corresponding results are included in Tables 5 and 6. We observe that the convergence rates of the L^2 and the H^1 errors are consistent with the above cases. Significantly, Table 6 indicates the $\mathcal{O}(\tau^{\frac{1}{2}})$ convergence for the $d_t L^2$ error, which is in agreement with the theoretical result.

Test 2 (Influence of Initial Conditions) Consider the one-dimensional stochastic wave equations (1.1)–(1.4) with the following initial conditions (with less regularity)

$$h_1(x) = 0, \quad h_2(x) = \max\{0, 1 - 4|x - 0.5|\}.$$

The nonlinear drift and diffusion terms are chosen to be $f(u) = -u - u^3$ and $g(u) = u$. Tables 7 and 8 show the L^2 error, the H^1 error, the $d_t L^2$ error, and their convergence rates in both space and time. The spatial convergence rates are the same as in the previous cases, and the temporal convergence rates of the L^2, H^1, and $d_t L^2$ errors are approximately 1.0, 1.0, and 0.75. This indicates that the regularity of data does affect the convergence behaviors, which is also known for the discretizations of deterministic wave equations.

Table 8 Test 2: temporal errors and convergence rates when $h = 1/512$, $T = 1$

	L^2 error	Order	H^1 error	Order	$d_t L^2$ error	Order
$\tau = 0.1/8$	1.324×10^{-3}	–	7.489×10^{-3}	–	4.653×10^{-2}	–
$\tau = 0.1/16$	6.437×10^{-4}	1.040	3.963×10^{-3}	0.918	3.130×10^{-2}	0.572
$\tau = 0.1/32$	3.163×10^{-4}	1.025	1.935×10^{-3}	1.035	1.920×10^{-2}	0.705
$\tau = 0.1/64$	1.549×10^{-4}	1.030	9.583×10^{-4}	1.014	1.152×10^{-2}	0.738
$\tau = 0.1/128$	7.659×10^{-5}	1.016	4.974×10^{-4}	0.946	6.908×10^{-3}	0.737

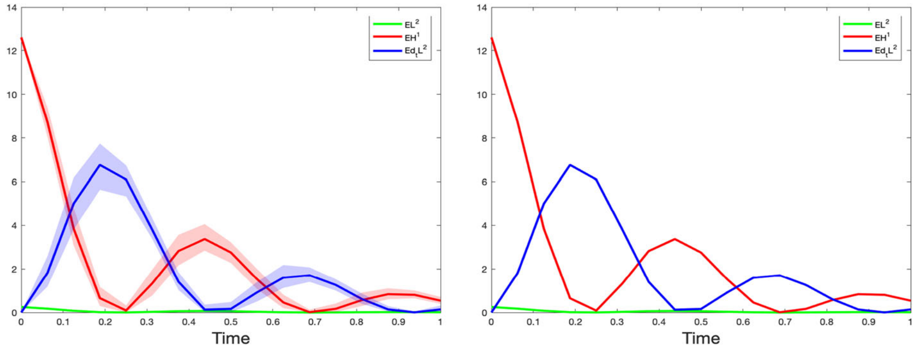

Fig. 1 Test 3: the stability in the stochastic case (left), and the stability in the deterministic case (right). Here $f(u) = -u - u^3$ and $g(u) = u$

Test 3 (Numerical Stability) Consider the two-dimensional stochastic wave equations (1.1)–(1.4) with the following initial conditions

$$h_1(x, y) = \cos(\pi x)\cos(2\pi y), \quad h_2(x, y) = 0.$$

We investigate the stability (in different norms) of the proposed methods with various nonlinear drift and diffusion terms. For comparison, we also include the results of the deterministic equations. Figures 1, 2, and 3 provide the time history of the *stability* in L^2, H^1, and $d_t L^2$ norms of the stochastic (left) and deterministic (right) solutions. The transparent shaded regions in the left figures are possible trajectories of all sample points, and the solid lines represent the average of all trajectories.

In Fig. 1, the nonlinear drift and diffusion terms are chosen to be $f(u) = -u - u^3$ and $g(u) = u$. In Fig. 2, the nonlinear drift and diffusion terms are chosen to be $f(u) = -u - u^7$ and $g(u) = u$. In Fig. 3, the nonlinear drift and diffusion terms are chosen to be $f(u) = -u - u^3$ and $g(u) = \sqrt{u^2 + 1}$. From these figures, we see that all the possible trajectories of sample points are bounded in H^1 norm. We also observe that the expectation of stochastic numerical solutions in various norms do not blow up, which is consistent with the theoretical results provided in Sect. 3.

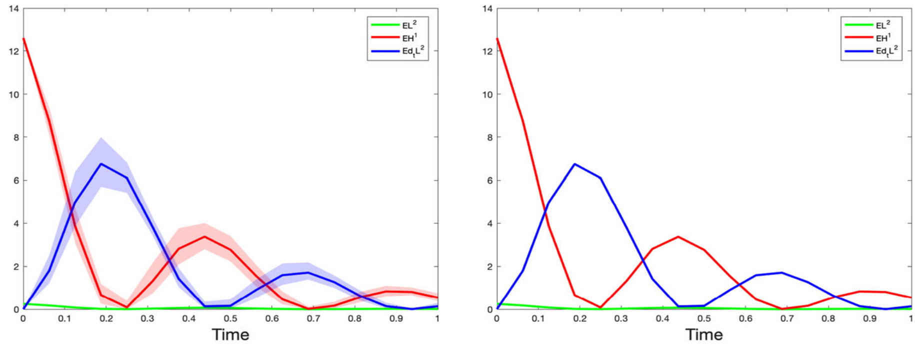

Fig. 2 Test 3: the stability in the stochastic case (left), and the stability in the deterministic case (right). Here $f(u) = -u - u^7$ and $g(u) = u$

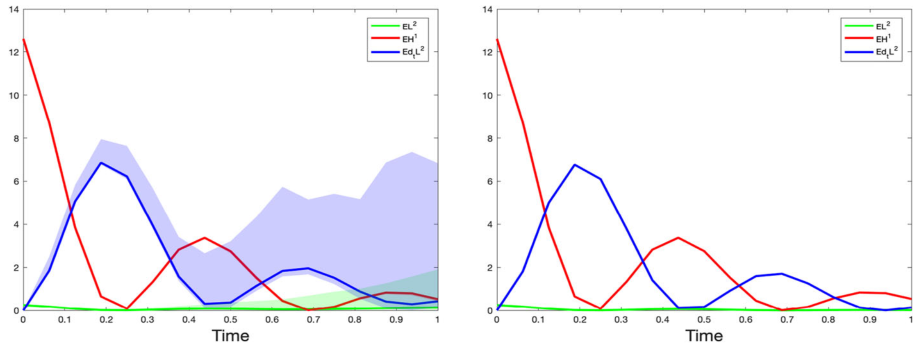

Fig. 3 Test 3: the stability in the stochastic case (left), and the stability in the deterministic case (right). Here $f(u) = -u - u^3$ and $g(u) = \sqrt{u^2+1}$

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Author Contributions The authors contributed equally to this work.

Availability of Data and Material All datasets generated during the current study are available from the corresponding author upon reasonable request.

Code Availability Available upon reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A Proofs of Hölder Continuity

In this Appendix, we prove the Hölder continuity in time for the strong solution u in various norms in Sect. 2.

Proof of Lemma 3 The SPDE (1.1) leads to

$$u_t(t) - h_2 = \int_0^t \Delta u d\zeta + \int_0^t f(u) d\zeta + \int_0^t g(u) dW(\zeta). \tag{A.1}$$

Taking the square, the spatial integral, and the expectation on both sides of (A.1), and then using the triangle inequality, the Schwarz inequality, and Itô isometry, we obtain

$$\begin{aligned}
&\mathbb{E}\big[\|u_t(t)\|_{L^2}^2\big] \\
&\quad \le C\mathbb{E}\big[\|h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_{\mathcal{D}}\Big(\int_0^t \Delta u(\zeta) d\zeta\Big)^2 dx\Big] \\
&\qquad + C\mathbb{E}\Big[\int_{\mathcal{D}}\Big(\int_0^t f(u(\zeta)) d\zeta\Big)^2 dx\Big] + C\mathbb{E}\Big[\int_{\mathcal{D}}\Big(\int_0^t g(u(\zeta)) dW(\zeta)\Big)^2 dx\Big] \\
&\quad \le C\mathbb{E}\big[\|h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_0^t \|\Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] \\
&\qquad + C\mathbb{E}\Big[\int_0^t \|f(u(\zeta))\|_{L^2}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|g(u(\zeta))\|_{L^2}^2 d\zeta\Big] \\
&\quad \le C\mathbb{E}\big[\|h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_0^t \|\Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|u(\zeta)\|_{L^{2q}}^{2q} d\zeta\Big] + C,
\end{aligned} \tag{A.2}$$

where (1.8) is used in the derivation of the last inequality. For any $s, t \in [0, T]$ with $s < t$, we have

$$\mathbb{E}\big[\|u(t) - u(s)\|_{L^2}^2\big] = \mathbb{E}\big[\|u_t(\xi)\|_{L^2}^2\big](t-s)^2 \le C(t-s)^2, \tag{A.3}$$

where $\xi \in (s, t)$ and

$$C = C\mathbb{E}\big[\|h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_0^t \|\Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|u(\zeta)\|_{L^{2q}}^{2q} d\zeta\Big] + C.$$

This finishes the proof of the lemma. □

Proof of Lemma 4 Note that $v = u_t$. By (1.1), for any $s, t \in [0, T]$ with $s < t$, we have

$$v(t) - v(s) = \int_s^t \Delta u d\zeta + \int_s^t f(u) d\zeta + \int_s^t g(u) dW(\zeta). \tag{A.4}$$

Taking the square, the spatial integral, and the expectation on both sides of (A.4), and then using the triangle inequality, the Schwarz inequality, and Itô isometry, we obtain

$$\begin{aligned}
&\mathbb{E}\big[\|v(t) - v(s)\|_{L^2}^2\big] \\
&\quad \le C\mathbb{E}\Big[\int_s^t \|\Delta u(\zeta)\|_{L^2}^2 d\zeta\Big](t-s) \\
&\qquad + C\mathbb{E}\Big[\int_s^t \|f(u(\zeta))\|_{L^2}^2 d\zeta\Big](t-s) + C\mathbb{E}\Big[\int_s^t \|g(u(\zeta))\|_{L^2}^2 d\zeta\Big], \\
&\quad \le C\mathbb{E}\Big[\int_s^t \|\Delta u(\zeta)\|_{L^2}^2 d\zeta\Big](t-s) \\
&\qquad + C\mathbb{E}\Big[\int_s^t \|u(\zeta)\|_{L^{2q}}^{2q} d\zeta\Big](t-s) + C\mathbb{E}\Big[\int_s^t \|u(\zeta)\|_{L^2}^2 d\zeta\Big] + C(t-s), \\
&\quad \le C(t-s),
\end{aligned} \tag{A.5}$$

where

$$C = C\mathbb{E}\Big[\int_s^t \|\Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_s^t \|u(\zeta)\|_{L^{2q}}^{2q} d\zeta\Big] + C \sup_{s\le\zeta\le t} \mathbb{E}\big[\|u(\zeta)\|_{L^2}^2\big] + C,$$

and this finishes the proof of the lemma. □

Proof of Lemma 5 From the Eq. (1.1), we get

$$u_t(t) - h_2 = \int_0^t \Delta u d\zeta + \int_0^t f(u) d\zeta + \int_0^t g(u) dW(\zeta). \tag{A.6}$$

Taking the gradient, the square, the spatial integral, and the expectation on both sides of (A.6), and then using the triangle inequality, the Schwarz inequality, and Itô isometry, we obtain

$$\begin{aligned}
&\mathbb{E}\big[\|\nabla u_t(t)\|_{L^2}^2\big] \\
&\le C\mathbb{E}\big[\|\nabla h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_{\mathcal{D}} \Big(\int_0^t \nabla\Delta u(\zeta) d\zeta\Big)^2 dx\Big] \\
&\quad + C\mathbb{E}\Big[\int_{\mathcal{D}} \Big(\int_0^t \nabla f(u(\zeta)) d\zeta\Big)^2 dx\Big] + C\mathbb{E}\Big[\int_{\mathcal{D}} \Big(\int_0^t \nabla g(u(\zeta)) dW(\zeta)\Big)^2 dx\Big] \\
&\le C\mathbb{E}\big[\|\nabla h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_0^t \|\nabla\Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] \\
&\quad + C\mathbb{E}\Big[\int_0^t \|\nabla f(u(\zeta))\|_{L^2}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|\nabla g(u(\zeta))\|_{L^2}^2 d\zeta\Big] \\
&\le C\mathbb{E}\big[\|\nabla h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_0^t \|\nabla\Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] \\
&\quad + C\mathbb{E}\Big[\int_0^t \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)} d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|\nabla u(\zeta)\|_{L^4}^4 d\zeta\Big] + C.
\end{aligned} \tag{A.7}$$

Therefore, for any $s, t \in [0, T]$ with $s < t$, we have

$$\mathbb{E}\big[\|\nabla(u(t) - u(s))\|_{L^2}^2\big] = \mathbb{E}\big[\|\nabla u_t(\xi)\|_{L^2}^2\big](t-s)^2 \le C(t-s)^2, \tag{A.8}$$

where $\xi \in (s, t)$ and

$$\begin{aligned}
C &= C\mathbb{E}\big[\|h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_0^t \|\nabla\Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)} d\zeta\Big] \\
&\quad + C\mathbb{E}\Big[\int_0^t \|\nabla u(\zeta)\|_{L^4}^4 d\zeta\Big] + C.
\end{aligned}$$

This finishes the proof of the lemma. □

Proof of Lemma 6 From the SPDE (1.1), for any $s, t \in [0, T]$ with $s < t$, we have

$$v(t) - v(s) = \int_s^t \Delta u d\zeta + \int_s^t f(u) d\zeta + \int_s^t g(u) dW(\zeta). \tag{A.9}$$

Taking the gradient, the square, the spatial integral, and the expectation on both sides of (A.6), and then using the triangle inequality, the Schwarz inequality, and Itô isometry, we obtain

$$
\begin{aligned}
&\mathbb{E}\big[\|\nabla(v(t)-v(s))\|_{L^2}^2\big] \\
&\quad \le C\mathbb{E}\Big[\int_s^t \|\nabla \Delta u(\zeta)\|_{L^2}^2 d\zeta\Big](t-s) \\
&\qquad + C\mathbb{E}\Big[\int_s^t \|\nabla f(u(\zeta))\|_{L^2}^2 d\zeta\Big](t-s) + C\mathbb{E}\Big[\int_s^t \|\nabla g(u(\zeta))\|_{L^2}^2 d\zeta\Big], \\
&\quad \le C\mathbb{E}\Big[\int_s^t \|\nabla \Delta u(\zeta)\|_{L^2}^2 d\zeta\Big](t-s) \\
&\qquad + C\mathbb{E}\Big[\int_s^t \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)} d\zeta\Big](t-s) + C\mathbb{E}\Big[\int_s^t \|\nabla u(\zeta)\|_{L^4}^4 d\zeta\Big] + C(t-s), \\
&\quad \le C(t-s),
\end{aligned}
\tag{A.10}
$$

where

$$
\begin{aligned}
C = C\mathbb{E}\Big[\int_s^t \|\nabla \Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_s^t \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)} d\zeta\Big] \\
+ C \sup_{s\le\zeta\le t} \mathbb{E}\big[\|\nabla u(\zeta)\|_{L^4}^4\big] + C.
\end{aligned}
$$

This finishes the proof of the lemma. □

Proof of Lemma 7 Again, from the Eq. (1.1), we get

$$
u_t(t) - h_2 = \int_0^t \Delta u d\zeta + \int_0^t f(u)d\zeta + \int_0^t g(u)dW(\zeta). \tag{A.11}
$$

Taking the Hessian, the square, the spatial integral, and the expectation on both sides of (A.6), and then using the triangle inequality, the Schwarz inequality, and Itô isometry, we obtain

$$
\begin{aligned}
&\mathbb{E}\big[\|\nabla^2 u_t(t)\|_{L^2}^2\big] \\
&\quad \le C\mathbb{E}\big[\|\nabla^2 h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_{\mathcal{D}}\Big(\int_0^t \nabla^2 \Delta u(\zeta) d\zeta\Big)^2 dx\Big] \\
&\qquad + C\mathbb{E}\Big[\int_{\mathcal{D}}\Big(\int_0^t \nabla^2 f(u(\zeta)) d\zeta\Big)^2 dx\Big] + C\mathbb{E}\Big[\int_{\mathcal{D}}\Big(\int_0^t \nabla^2 g(u(\zeta)) dW(\zeta)\Big)^2 dx\Big] \\
&\quad \le C\mathbb{E}\big[\|\nabla^2 h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_0^t \|\nabla^2 \Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] \\
&\qquad + C\mathbb{E}\Big[\int_0^t \|\nabla^2 f(u(\zeta))\|_{L^2}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|\nabla^2 g(u(\zeta))\|_{L^2}^2 d\zeta\Big] \\
&\quad \le C\mathbb{E}\big[\|\nabla^2 h_2\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_0^t \|\nabla^2 \Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] \\
&\qquad + C\mathbb{E}\Big[\int_0^t \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)} d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|\nabla^2 u(\zeta)\|_{L^4}^4 d\zeta\Big] \\
&\qquad + C\mathbb{E}\Big[\int_0^t \|\nabla u(\zeta)\|_{L^8}^8 d\zeta\Big] + C.
\end{aligned}
\tag{A.12}
$$

Therefore, for any $s, t \in [0, T]$ with $s < t$, we have

$$
\begin{aligned}
\mathbb{E}\big[\|\nabla^2(u(t)-u(s))\|_{L^2}^2\big] &= \mathbb{E}\big[\|\nabla^2 u_t(\xi)\|_{L^2}^2\big](t-s)^2 \\
&\le C(t-s)^2,
\end{aligned}
\tag{A.13}
$$

where $\xi \in (s, t)$ and

$$C = C\mathbb{E}\big[\|\nabla^2 u_t(0)\|_{L^2}^2\big] + C\mathbb{E}\Big[\int_0^t \|\nabla^2 \Delta u(\zeta)\|_{L^2}^2 d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|u(\zeta)\|_{L^{4(q-1)}}^{4(q-1)} d\zeta\Big]$$
$$+ C\mathbb{E}\Big[\int_0^t \|\nabla^2 u(\zeta)\|_{L^4}^4 d\zeta\Big] + C\mathbb{E}\Big[\int_0^t \|\nabla u(\zeta)\|_{L^8}^8 d\zeta\Big] + C.$$

This finishes the proof of the lemma. □

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