Chapter 13

Numerical Methods for the Nonlinear Shallow Water Equations

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ABSTRACT
Free surface flows often appear in ocean, engineering and atmospheric modelling. In many applications involving unsteady water flows where the horizontal length scale is much greater than the vertical length scale, the shallow water equations are commonly used to model these flows. Research on effective and accurate numerical methods for their solutions has attracted great attention in the past two decades. In this chapter, we review some work on designing positivity-preserving and well-balanced methods for solving the shallow water equations with a nonflat bottom topography. Some shallow water-related models, including the shallow water flows through channels with irregular geometry, the shallow water equations on the sphere and the two-layer shallow water equations, and their numerical approximation will also be presented.

Keywords: Shallow water equation, Source term, Well balanced, Positivity preserving, Finite volume scheme, Finite difference scheme, Discontinuous Galerkin method

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1 OVERVIEW

Free surface flows have wide applications in ocean, environmental, hydraulic engineering and atmospheric modelling, with examples including dam break and flooding problems, tidal flows in coastal water regions, nearshore wave propagation with complex bathymetry structure, tsunami wave propagation and ocean model. In many applications involving unsteady water flows where the horizontal length scale is much greater than the vertical length scale, the shallow water equations (SWEs), often known as Saint-Venant equations, are commonly used to model these flows. The two-dimensional SWEs are a nonhomogeneous system of nonlinear hyperbolic equations, which takes the form of

$$\begin{aligned} &h_t + (hu)_x + (hv)_y = 0, \\ &(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghb_x + fhv + c_f u|\mathbf{u}|, \\ &(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghb_y - fhu + c_f v|\mathbf{u}|, \end{aligned} \tag{1}$$

where h denotes the water height, $\mathbf{u} = (u, v)^T$ is the velocity vector of the fluid and g is the gravitational constant. The source terms on the right-hand side represent the effect of nonflat bathymetry, Coriolis force and friction on the bottom, with b being the bottom topography, f the Coriolis parameter and c_f the friction parameter (from the classical Manning or Darcy formulation).

SWEs play a critical role in the modelling and simulation of free surface flows in rivers and coastal areas and can predict tides, storm surge levels and coastline changes from hurricanes and ocean currents. SWEs also arise in atmospheric flows, debris flows, internal flows and certain hydraulic structures like open channels and sedimentation tanks. Due to the large scientific and engineering applications of the SWEs, research on effective and accurate numerical methods for their solutions has attracted great attention in the past two decades. The homogeneous SWEs are simply a system of hyperbolic conservation laws. Tremendous numerical methods designed for conservation laws can be applied to simulate the SWEs directly. But the SWEs also admit other numerical challenges. Two types of difficulties are often encountered at the simulation of the SWEs, coming from the preservation of steady-state solutions and the preservation of water height positivity. The first difficulty is related to the treatment of the source terms due to nonflat bottom topography. An essential part for the SWEs and other conservation laws with source terms is that they often admit steady-state solutions in which the flux gradients are exactly balanced by the source terms. SWEs admit the general moving-water equilibrium, as well as a simpler still water steady-state solution, which represents a still flat water surface, and referred as the "lake at rest" solution. Traditional numerical schemes with a straightforward handling of the bottom source term cannot balance the effect of the source term and the flux in the discrete level, and usually fail to capture the steady-state well. This

will introduce spurious oscillations near the steady state. The well-balanced schemes are specially designed to preserve exactly these steady-state solutions up to machine error with relatively coarse meshes, and therefore, it is desirable to design numerical methods which have the well-balanced property. The other major difficulty often encountered in the simulations of the SWEs is the appearance of dry regions in many engineering applications. Typical applications include the dam break problem, flood waves and run-up phenomena at a coast with tsunamis being the most impressive example. Special attention needs to be paid near the dry/wet front to preserve the water height positivity, otherwise they may produce nonphysical negative water height, which becomes problematic when calculating the eigenvalues $u \pm \sqrt{gh}$ to determine the time step size Δt, and renders the system not hyperbolic and not well posed.

The rest of this chapter is organized as follows: In Section 2, we describe the mathematical model, as well as some properties of this model in the one-dimensional setting. Review and discussion of some numerical methods for the SWEs are presented in Sections 3. In Section 4, we discuss some models which are similar to SWEs, and some references on their numerical approximation. Concluding remarks are given in Section 5.

2 MATHEMATICAL MODEL

Free surface flows appear in many engineering and atmospheric applications. Three-dimensional Navier–Stokes equations can be used to simulate such flows directly. However, in the case where the horizontal length scale is much greater than the vertical length scale, one can average over the depth to eliminate the vertical direction and reduce the model into the simpler two-dimensional nonlinear SWEs.

In one space dimension, the SWEs are defined as follows

$$\begin{aligned} &h_t + (hu)_x = 0, \\ &(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghb_x, \end{aligned} \tag{2}$$

where we ignore the Coriolis force and friction source terms and keep the only source term due to bottom topography. The two equations represent the conservation of mass and momentum. This model also admits other conserved quantities, including the energy ($E = gh + u^2/2$), vorticity (for two-dimensional model, $\omega = v_x - u_y$), potential vorticity and potential enstrophy, etc.

For the ease of presentation, we denote the SWEs (2) by

$$U_t + f(U)_x = s(U,b)$$

where $U = (h,\ hu)^T$ with the superscript T denoting the transpose, $f(U) = (hu, hu^2 + \frac{1}{2}gh^2)^T$ is the flux and $s(U, b)$ is the source term. The Jacobi matrix of the flux is given by:

$$A(U) = \begin{bmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{bmatrix},$$

with the two eigenvalues $\lambda_{1,2}(U) = u \pm \sqrt{gh}$.

When the solutions are smooth, the SWEs (2) can also be rewritten in the equivalent form of

$$\begin{aligned} &h_t + (hu)_x = 0, \\ &u_t + (u^2/2 + g(h+b))_x = 0. \end{aligned} \tag{3}$$

We can easily observe that the SWEs admit the general moving-water equilibrium state, given by

$$m := hu = \text{const} \quad \text{and} \quad E := \frac{1}{2}u^2 + g(h+b) = \text{const}, \tag{4}$$

where m and E are the moving-water equilibrium variables. A special case is the still water at rest steady state

$$u = 0 \quad \text{and} \quad h + b = \text{const}, \tag{5}$$

which represents a flat water surface.

For the homogeneous SWEs without any source term, the Riemann problem can be easily solved (LeVeque, 2002). When the source term due to the variable bottom topography is added to the system, solving the Riemann problem for the SWEs with discontinuous bathymetry becomes a less trivial issue. One approach is to augment a separate equation $b_t = 0$ for the bottom, and rewrite the SWEs in a nonconservative form with the variables $(h, hu, b)^T$. One can then investigate its Riemann solution based on this formulation. Some results on this topic can be found in Bernetti et al. (2008) and LeFloch and Thanh (2007).

3 NUMERICAL METHODS

Extensive research has been done to numerically simulate the SWEs in the past two decades. The commonly used numerical methods range from finite difference, finite volume to finite element methods. In this section, we start by reviewing numerical methods for the homogeneous SWEs, and then present the well-balanced and positivity-preserving numerical methods to overcome some numerical challenges encountered at the simulation of the SWEs.

3.1 Numerical Methods for the Homogeneous Equations

The homogeneous system of the SWEs is a hyperbolic conservation law. A tremendous amount of numerical methods for conservation laws has been designed, and most of them can be applied to the SWEs directly. In this

section, we briefly mention a few numerical methods to solve SWEs. For the ease of presentation, we use one-dimensional model (2) (without the source term) as an example to demonstrate these methods. In this chapter, we mainly focus on the spatial discretization. Total variation diminishing (TVD) Runge–Kutta time discretization (Shu and Osher, 1988) is usually used in practice for stability and to increase temporal accuracy. For example, the third-order TVD Runge–Kutta method can be coupled with all the spatial discretization introduced in this chapter:

$$\begin{aligned} U^{(1)} &= U^n + \Delta t \mathcal{F}(U^n), \\ U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}\left(U^{(1)} + \Delta t \mathcal{F}(U^{(1)})\right), \\ U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}\left(U^{(2)} + \Delta t \mathcal{F}(U^{(2)})\right), \end{aligned} \tag{6}$$

where $\mathcal{F}(U)$ is the spatial operator.

3.1.1 Finite Volume Methods

We discretize the computational domain into cells $I_j = [x_{j-1/2}, x_{j+1/2}]$ and denote the size of the j-th cell by Δx_j and the maximum mesh size by $\Delta x = \max_j \Delta x_j$. In a finite volume scheme, our computational variables are $\overline{U}_j(t)$, which approximate the cell averages $\overline{U}(x_j,t) = \frac{1}{\Delta x_j}\int_{I_j} U(x,t)\,dx$. The conservative finite volume numerical scheme is given by

$$\frac{d}{dt}\overline{U}_j(t) + \frac{1}{\Delta x_j}\left(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}\right) = 0, \tag{7}$$

with $\hat{f}_{j+1/2} = F(U^-_{j+1/2}, U^+_{j+1/2})$ being the numerical flux. Here $U^-_{j+1/2}$ and $U^+_{j+1/2}$ are the pointwise approximations to $U(x_{j+1/2}, t)$ from left and right, respectively.

In a first-order method, one can approximate U by piecewise constant in each cell, therefore, $U^-_{j+1/2} = \overline{U}_j$ and $U^+_{j+1/2} = \overline{U}_{j+1}$. The solution can also be approximated by a piecewise linear function in each cell, with the slope of the function constructed by a slope limiter based on $\overline{U}_j$ and $\overline{U}_{j\pm1}$, for example, the minmod limiter (Shu, 1987). This will lead to a second-order method. To construct a high order weighted essentially nonoscillatory (WENO) method, we can use a WENO reconstruction procedure to evaluate $U^-_{j+1/2}$ and $U^+_{j+1/2}$ through the neighbouring cell average values $\overline{U}_j$. Basically, for a $(2k-1)$-th order WENO scheme, we first compute k reconstructed boundary values $U^{(k),\pm}_{j+1/2}$ corresponding to different candidate stencils. Then by providing each value a weight which indicates the smoothness of the corresponding stencil, we define the $(2k-1)$-th order WENO reconstruction

$U^{\pm}_{j+1/2}$ as a convex combination of all these k reconstructed values. Eventually, the WENO reconstruction can be written out as:

$$U^{+}_{j+\frac{1}{2}} = \sum_{r=-k+1}^{k} w_r \overline{U}_{j+r}, \quad U^{-}_{j+\frac{1}{2}} = \sum_{r=-k}^{k-1} \widetilde{w}_r \overline{U}_{j+r}. \tag{8}$$

where $k = 3$ for the fifth-order WENO approximation and the coefficients w_r and $\widetilde{w}_r$ depend nonlinearly on the smoothness indicators involving the cell average $\overline{U}$. We refer to Crnjaric-Zic et al. (2004) and Xing and Shu (2006a,b) for more details in constructing finite volume WENO methods for the SWEs.

In order to obtain a stable scheme, the numerical flux $\hat{f}_{j+1/2} = F(U^{-}_{j+1/2}, U^{+}_{j+1/2})$ needs to be a monotone flux, namely F is a nondecreasing function of its first argument and a nonincreasing function of its second argument. One simple and inexpensive choice is the well-known Lax–Friedrichs flux

$$F(a,b) = \tfrac{1}{2}(f(a) + f(b) - \alpha(b-a)), \tag{9}$$

where $\alpha = \max(u + \sqrt{gh})$ and the maximum is taken over the whole domain. Other choices include the Roe's flux, Godunov flux, HLLC flux, etc. Second-order central-upwind methods have been studied for the SWEs in a series of papers (see Bryson et al., 2011; Kurganov and Levy, 2002; Kurganov and Petrova, 2007 and the references therein), which are based on the choice of numerical flux:

$$F(U^{-}_{j+1/2}, U^{+}_{j+1/2}) = \frac{c^{+}_{j+1/2} f(U^{-}_{j+1/2}) - c^{-}_{j+1/2} f(U^{+}_{j+1/2})}{c^{+}_{j+1/2} - c^{-}_{j+1/2}} + \frac{c^{+}_{j+1/2} c^{-}_{j+1/2}}{c^{+}_{j+1/2} - c^{-}_{j+1/2}} \left[U^{+}_{j+1/2} - U^{-}_{j+1/2} \right],$$

where the one-sided local speeds of propagation $c^{\pm}_{j+1/2}$ are determined by

$$c^{+} = \max(u^{+} + \sqrt{gh^{+}},\ u^{-} + \sqrt{gh^{-}}, 0), \quad c^{-} = \min(u^{+} - \sqrt{gh^{+}},\ u^{-} - \sqrt{gh^{-}}, 0),$$

at each cell boundary $x_{j+1/2}$.

3.1.2 Finite Difference Methods

For a finite difference scheme, we approximate the point values of the solution $U(x)$ at mesh points x_j by U_j. The spatial derivative $f(u)_x$ is then approximated by a conservative flux difference approximation

$$f(U)_x\big|_{x=x_j} \approx \frac{1}{\Delta x}\left(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}\right), \tag{10}$$

and the finite difference numerical scheme takes the form of

$$\frac{d}{dt}U_j(t)+\frac{1}{\Delta x_j}\left(\hat{f}_{j+\frac{1}{2}}-\hat{f}_{j-\frac{1}{2}}\right)=0. \tag{11}$$

The numerical flux $\hat{f}_{j+\frac{1}{2}}$ is computed through the neighbouring point values $f_j=f(U_j)$ by a flux limiter, or a finite difference WENO procedure.

For a $(2k-1)$-th order WENO scheme, we first compute k numerical fluxes

$$\hat{f}^{(r)}_{i+\frac{1}{2}}=\sum_{j=0}^{k-1}c_{rj}f_{i-r+j},\quad r=0,\ldots,k-1,$$

corresponding to k different candidate stencils $S_r(i)=\{x_{i-r},\ \ldots,\ x_{i-r+k-1}\}$, $r=0,\ \ldots,\ k-1$, and each of these k numerical fluxes is k-th order accurate. The $(2k-1)$-th order WENO flux is a convex combination of all these k numerical fluxes

$$\hat{f}_{i+\frac{1}{2}}=\sum_{r=0}^{k-1}w_r\hat{f}^{(r)}_{i+\frac{1}{2}},$$

where the nonlinear weights w_r satisfy $w_r\geq 0$, $\sum_{r=0}^{k-1}w_r=1$, and depend on the linear weights which yield $(2k-1)$-th order accuracy, and the smoothness indicators' of the stencil $S_r(i)$. An upwinding mechanism, essential for the stability of the scheme, can be realized by a global 'flux splitting'. The simplest one is the Lax–Friedrichs splitting, $f^{\pm}(u)=(f(u)\pm\alpha u)/2$, where α is taken as $\alpha=\max_u|f'(u)|$. The WENO procedure is applied to $f^{\pm}$ individually with upwind biased stencils. For hyperbolic systems such as the SWEs, we use the local characteristic decomposition, which is more robust than a component by component version. We first project the values f_j into the local characteristic direction, apply the WENO procedure on them to compute the flux, and then project them back to obtain the numerical fluxes in the physical space. We refer to Shu (1998) and Xing and Shu (2005) for further details.

3.1.3 Discontinuous Galerkin Methods

Finite element discontinuous Galerkin (DG) methods have been actively applied to hyperbolic conservation laws, especially the SWEs recently (Dawson and Proft, 2002; Eskilsson and Sherwin, 2004; Giraldo et al., 2002; Nair et al., 2005; Schwanenberg and Köngeter, 2000). We start by presenting the standard notations. In a high order DG method, we seek an approximation, still denoted by U for the ease of presentation with an abuse of notation, which belongs to the finite dimensional space

$$V_{\Delta x}=V^k_{\Delta x}\equiv\{w:w|_{I_j}\in P^k(I_j),j=1,\ldots,J\}, \tag{12}$$

where $P^k(I)$ denotes the space of polynomials of degree at most k and J is the total number of computational cells. Notice that the DG solution U can be discontinuous at the cell boundary $x_{j+\frac{1}{2}}$. The standard semidiscrete DG method is given by

$$\int_{I_j} \partial_t U v dx - \int_{I_j} f(U)\partial_x v dx + \widehat{f}_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = 0, \tag{13}$$

where $v(x)$ is a test function from the test space $V_{\Delta x}$, $\widehat{f}_{j+1/2} = F(U(x^-_{j+1/2},t), U(x^+_{j+1/2},t))$ is a numerical flux.

Another important ingredient of the DG method is that a slope limiter procedure should be performed after each inner stage in the Runge–Kutta time stepping. This is necessary for computing solutions with strong discontinuities. There are many choices for the slope limiters (see, e.g. Qiu and Shu, 2005). The total variation bounded (TVB) limiter in Shu (1987), Cockburn and Shu (1989) and Cockburn et al. (1989) is commonly used in many applications, and we refer to these references for the details of this limiter. Multidimensional problems can be handled in the same fashion. The main difference is that the fluxes are now integrals along the cell boundary, which can be calculated by Gauss-quadrature rules.

3.1.4 Residual Distribution Methods

Residual distribution (RD) schemes, also referred as Fluctuation Splitting schemes, are introduced in Roe (1981), and the idea behind the method is to decompose the local numerical error (fluctuation) to nodes to evolve the approximation of the solution. They are first designed for the steady problem with an upwinding mechanics, and later generalized to approximate unsteady problem. High order version has also been studied. We refer to the recent review paper (Abgrall, 2010) for the history and development of RD methods. Application of RD methods for the SWEs has been studied in recent years. An earlier investigation can be found in Garcia-Navarro et al. (1995), where a first-order scheme for the unsteady system is presented. We refer to Ricchiuto et al. (2007) and Ricchiuto and Bollermann (2009) for some recent advances in this topic.

3.2 Well-Balanced Methods

The nonhomogeneous SWEs with a source term due to the nonflat bottom topography, admit a nontrivial steady-state solution (4)–(5), which describes physical equilibrium such as a flowing river with a rugged bottom but flat surface (on a windless day). It is crucial to accurately simulate these physical equilibria in order to make dynamic, real-time predictions, as geophysical flows are typically perturbations of these underlying equilibrium states. In this section, we review some well-balanced methods which can preserve exactly

these steady-state solutions in the discrete level. For the ease of presentation, we will confine our attention in the framework of DG methods, but the well-balanced techniques can all be extended to finite volume or difference methods.

3.2.1 Well-Balanced Methods for the Still Water

In the past two decades, many well-balanced numerical methods have been developed for the SWEs. The well-balanced property is often referred as "exact C-property", which means that the scheme is "exact" when applied to the stationary case (5). The well-balanced concept was first proposed by Bermudez and Vazquez (1994), where they extended upwind methods to the SWEs with source terms. Following this pioneering work, many other schemes for the SWEs with such well-balanced property have been developed in the finite volume community. A quasi-steady wave propagation algorithm based upon modified Riemann problems is presented in LeVeque (1998), LeVeque and Bale (1998) and Bale et al. (2002). Another popular approach is to rewrite the equation in terms of the water surface instead of water height (also referred as the prebalanced formulation), and well-balanced methods (Kurganov and Levy, 2002; Rogers et al., 2003; Zhou et al., 2001) can be designed based on such formulation. Well-balanced methods can also be derived utilizing the idea of hydrostatic reconstruction initially proposed in Audusse et al. (2004). A kinetic approach to achieve well-balanced property has been shown in Perthame and Simeoni (2001). In the framework of RD, simulation for the SWEs with well-balanced properties is shown in Ricchiuto et al. (2007) and Ricchiuto and Bollermann (2009). For more related work, see also Gallouët et al. (2003), Delis and Katsaounis (2003), Gosse (2000), Greenberg and LeRoux (1996), Jin (2001), Liang and Marche (2009), Lukácová-Medviová et al. (2007), Russo (2005), Fjordholm et al. (2011), Xu (2002) and Xing and Shu (2014).

Most of the works mentioned above are for numerical schemes of first or second-order accuracy. In recent years, high order accurate numerical schemes (with higher than second-order accuracy) have attracted increasing attention in many computational fields. Some finite difference/volume WENO schemes with well-balanced property have been designed for the SWEs recently. In Xing and Shu (2005) and Xing and Shu (2006a), a special decomposition of the source term was introduced which leads to high order finite difference and finite volume well-balanced WENO methods. The hydrostatic reconstruction idea is extended to high order methods in Xing and Shu (2006b) and Noelle et al. (2006) with a careful high order approximation of the source term. Path-conservative methods for the nonconservative product are introduced in Castro et al. (2006) and Parés (2006) and extended to the SWEs. Other high order finite volume methods include Caleffi et al. (2006), Caleffi and Valiani (2009) and Canestrelli et al.

(2009). High order DG methods for the SWEs have also attracted increasing attention recently. Several well-balanced DG methods have been proposed in the last few years, by the idea of special decomposition of the source term (Xing and Shu, 2006a), hydrostatic reconstruction (Ern et al., 2008; Kesserwani and Liang, 2010; Xing and Shu, 2006b) and path-conservative (Rhebergen et al., 2008).

Here, we briefly review the well-balanced idea presented in Xing and Shu (2006a,b). In order to achieve the well-balanced property, we are interested in numerical methods which balance the numerical approximation of the flux and source term at the still water stationary solution (5). The key idea is to introduce high order accurate numerical discretization of the source term, which mimics the approximation of the flux term, so that the exact balance between the source term and the flux can be achieved at the steady state numerically. Here we present two different approaches to achieve such goal. The first approach focuses on a nonstandard discretization of the source term, by following the idea of decomposing the source terms (Xing and Shu, 2005, 2006a). The second approach employs the idea of hydrostatic reconstruction (Audusse et al., 2004) to modify the approximation of the numerical flux and keep a simple source term approximation. We notice that the traditional DG methods are capable of maintaining certain steady states exactly, if a small modification on the numerical flux is provided. The computational cost to obtain such a well-balanced DG method is basically the same as the traditional DG method.

The first well-balanced approach is to decompose the integral of the source term on the DG method as:

$$\begin{aligned}\int_{I_j} -ghb_x v dx &= \int_{I_j}\left(\frac{1}{2}gb^2\right)_x v dx - g\overline{(h+b)}_j\int_{I_j} b_x v dx - \int_{I_j} g\left(h+b-\overline{(h+b)}_j\right)b_x v dx\\ &= \left(\frac{1}{2}gb^2\right)v\left(x^-_{j+\frac{1}{2}}\right) - \left(\frac{1}{2}gb^2\right)v\left(x^+_{j-\frac{1}{2}}\right) - \int_{I_j}\frac{1}{2}gb^2 v_x dx\\ &\quad - g\overline{(h+b)}_j\left(bv\left(x^-_{j+\frac{1}{2}}\right) - bv\left(x^+_{j-\frac{1}{2}}\right) - \int_{I_j} bv_x dx\right)\\ &\quad - \int_{I_j} g(h+b-\overline{(h+b)}_j)b_x v dx.\end{aligned}$$

We then replace this source term with a high order approximation of it given by

$$\begin{aligned}s_j &= \left\{\frac{1}{2}gb^2_{j+\frac{1}{2}}\right\}v\left(x^-_{j+\frac{1}{2}}\right) - \left\{\frac{1}{2}gb^2_{j-\frac{1}{2}}\right\}v\left(x^+_{j-\frac{1}{2}}\right) - \int_{I_j}\frac{1}{2}gb^2 v_x dx\\ &\quad - g\overline{(h+b)}_j\left(\left\{b_{j+\frac{1}{2}}\right\}v\left(x^-_{j+\frac{1}{2}}\right) - \left\{b_{j-\frac{1}{2}}\right\}v\left(x^+_{j-\frac{1}{2}}\right) - \int_{I_j} bv_x dx\right)\\ &\quad - \int_{I_j} g(h+b-\overline{(h+b)}_j)b_x v dx,\end{aligned} \tag{14}$$

where the notation $\{\phi\}$ is defined as the average of $\phi^{\pm}$. Combined with the semidiscrete form (13), this gives our well-balanced high order DG schemes. Usually, we perform the limiter on the function U after each Runge–Kutta stage. Note that the slope limiter procedure could destroy the preservation of still water steady state, since if the limiter is enacted, the resulting modified solution h may no longer satisfy $h + b =$ constant. We therefore propose to first check whether any limiting is needed based on the function $h + b$ in each Runge–Kutta stage. If a certain cell is flagged by this procedure needing limiting, then the actual limiter is implemented on h, not on $h + b$, so that the slope limiter will not conflict with the well-balanced property. The well-balanced property for the still water (5) can be easily proved, and we refer to Xing and Shu (2006a) for the details.

A different approach to achieve well-balanced property is to utilize the hydrostatic reconstruction idea in the numerical flux. As mentioned in Xing and Shu (2006b), our well-balanced numerical scheme has the form:

$$\int_{I_j} \partial_t U v dx - \int_{I_j} f(U)\partial_x v dx + \widehat{f}^{\,l}_{j+\frac{1}{2}} v\left(x^-_{j+\frac{1}{2}}\right) - \widehat{f}^{\,r}_{j-\frac{1}{2}} v\left(x^+_{j-\frac{1}{2}}\right) = \int_{I_j} s(h,b) v dx. \quad (15)$$

The left and right fluxes $\widehat{f}^{\,l}_{j+1/2}$ and $\widehat{f}^{\,r}_{j-1/2}$ are given by:

$$\begin{aligned} \widehat{f}^{\,l}_{j+\frac{1}{2}} &= F\left(U^{*,-}_{j+\frac{1}{2}}, U^{*,+}_{j+\frac{1}{2}}\right) + \begin{pmatrix} 0 \\ \frac{g}{2}\left(h^-_{j+\frac{1}{2}}\right)^2 - \frac{g}{2}\left(h^{*,-}_{j+\frac{1}{2}}\right)^2 \end{pmatrix}, \\ \widehat{f}^{\,r}_{j-\frac{1}{2}} &= F\left(U^{*,-}_{j-\frac{1}{2}}, U^{*,+}_{j-\frac{1}{2}}\right) + \begin{pmatrix} 0 \\ \frac{g}{2}\left(h^+_{j-\frac{1}{2}}\right)^2 - \frac{g}{2}\left(h^{*,+}_{j-\frac{1}{2}}\right)^2 \end{pmatrix}, \end{aligned} \quad (16)$$

with the left and right values of U^* defined as:

$$U^{*,\pm}_{j+\frac{1}{2}} = \begin{pmatrix} h^{*,\pm}_{j+\frac{1}{2}} \\ h^{*,\pm}_{j+\frac{1}{2}} u^{\pm}_{j+\frac{1}{2}} \end{pmatrix}, \quad h^{*,\pm}_{j+\frac{1}{2}} = \max\left(0, h^{\pm}_{j+\frac{1}{2}} + b^{\pm}_{j+\frac{1}{2}} - \max\left(b^+_{j+\frac{1}{2}}, b^-_{j+\frac{1}{2}}\right)\right). \quad (17)$$

We also require that all the integrals in formula (15) should be calculated exactly at the still water state. This can be easily achieved by using suitable Gauss-quadrature rules since the numerical solutions h, b and v are polynomials at the still water state in each cell I_j, hence $f(U)$ and $s(h, b)$ are both polynomials. We have proven in Xing and Shu (2006b) that the above methods (15), combined with the choice of fluxes (16), are actually well-balanced for the still water steady state of the SWEs.

3.2.2 Well-Balanced Methods for the Moving Water

The well-balanced methods mentioned above target to preserve the still water steady state (5). They cannot preserve the moving water steady state (4), and it is significantly more difficult to obtain well-balanced schemes for such equilibrium. In a recent paper (Xing et al., 2011), several numerical examples are shown to demonstrate the advantage of moving-water well-balanced schemes over still-water well-balanced schemes for the SWEs. Those numerical examples clearly demonstrate the importance of utilizing moving-water well-balanced methods for solutions near a moving-water equilibrium. There have been a few attempts in developing well-balanced methods for the moving water equilibrium. A class of first-order accurate flux-vector-splitting schemes based on the theory of nonconservative products was proposed in Gosse (2000). Well-balanced second-order central schemes on staggered grids can be found in Russo (2005). Numerical methods based on local subsonic steady-state reconstruction, which are exactly well-balanced for subsonic moving equilibria, were shown in Bouchut and Morales (2010). A few high order accurate well-balanced methods for the moving water equilibrium have been introduced recently. In Noelle et al. (2007), well-balanced finite volume WENO methods are designed for arbitrary equilibria of the SWEs. The key component there is a special way to recover the moving water equilibrium and a well-balanced quadrature rule of the source term. Other high order well-balanced methods for the moving water equilibrium include the central WENO methods (Russo and Khe, 2009), path-conservative WENO methods (Castro et al., 2013) and DG methods (Xing, 2014).

In this section, we present high order finite element DG methods for the SWEs (1), with the objective to maintain the general moving steady state (4). The main structure of well-balanced methods for moving water equilibrium (4) follows the one (15) for still water. As explained in Xing (2014), we could define the transformation between the conservative variables $U := (h, m)^T$ and the equilibrium variables $V := (m, E)^T$, as well as the recovery of well-balanced states $\overline{V}_j = (\bar{m}_j, \bar{E}_j)$. The reference equilibrium values $\overline{V}_j$ lead to the reference equilibrium functions $U(\overline{V}_j, b(x))$. Since they may not be polynomials, we consider their projection into the finite element space $V_{\Delta x}$ and denote it by

$$U_j^e(x) = (h_j^e(x), m_j^e(x)) = P\,U(\overline{V}_j, b(x)), \tag{18}$$

in each cell I_j, where P denotes the L^2 projection operator. Therefore, the numerical solutions U, which are piecewise polynomials, can be decomposed as

$$U = U^e + U^r, \tag{19}$$

where $U^r = U - U^e \in V_{\Delta x}$. The source term approximation now becomes

$$\int_{I_j} s(h,b)v dx = \int_{I_j} s(h^e,b)v dx + \int_{I_j} s(h^r,b)v dx, \tag{20}$$

since $s(h,\ b) = -ghb_x$ is linear with respect to h. Given the fact that $U(\overline{V}_j,b) = (h(\overline{V}_j,b), \bar{m}_j)^T$ is the equilibrium state, we have the relationship

$$\int_{I_j} s(h(\overline{V}_j,b),b)v dx = -\int_{I_j} f(U(\overline{V}_j,b))v_x dx + f\left(U\left(\overline{V}_j, b^-_{j+\frac{1}{2}}\right)\right)v^-_{j+\frac{1}{2}} - f\left(U\left(\overline{V}_j, b^+_{j-\frac{1}{2}}\right)\right)v^+_{j-\frac{1}{2}}.$$

Since U^e is the L^2 projection of $U(\overline{V}_j,b)$, we conclude that

$$\int_{I_j} s(h^e,b)v dx + O(\Delta x^{k+1}) = -\int_{I_j} f(U^e)v_x dx + f\left(U^{e,-}_{j+\frac{1}{2}}\right)v^-_{j+\frac{1}{2}} - f\left(U^{e,+}_{j-\frac{1}{2}}\right)v^+_{j-\frac{1}{2}}, \tag{21}$$

and can approximate the source term integral (20) by

$$\int_{I_j} s(h,b)v dx \approx -\int_{I_j} f(U^e)v_x dx + f\left(U^{e,-}_{j+\frac{1}{2}}\right)v^-_{j+\frac{1}{2}} - f\left(U^{e,+}_{j-\frac{1}{2}}\right)v^+_{j-\frac{1}{2}} + \int s(h^r,b)v dx. \tag{22}$$

Since U^e is always smooth inside a cell, the relation (21) is always true regardless of the smoothness of the solution U. Therefore, numerical methods with this source term approximation (22) will satisfy the Lax–Wendroff theorem and converge to the weak solution.

The well-balanced numerical fluxes are computed by a generalized hydrostatic reconstruction. At each time step t^n, one can compute the cell boundary values $U^{\pm}_{j+1/2}$ from the solution $U(x)$. But in the case of moving water equilibrium, suppose $U(x)$ are computed from the exact solution, these cell boundary values $U^{\pm}_{j+1/2}$ are not equal to the exact solution value at the same point, as $U(x)$ is the projection of the exact solution into the polynomial space and this projection does not preserve the equilibrium state. To overcome this problem, we redefine an updated boundary value as:

$$\tilde{U}^{\pm}_{j+\frac{1}{2}} = U\left(\overline{V}_j, b^{\pm}_{j+\frac{1}{2}}\right) + U^{r,\pm}_{j+\frac{1}{2}}, \tag{23}$$

where U^r is defined in (19). One can easily verify that $\tilde{U}^{\pm}_{j+1/2} = U(\overline{V}_j, b^{\pm}_{j+1/2})$ in the case of moving water equilibrium. We follow the idea of hydrostatic reconstruction to compute the numerical fluxes and define

$$\tilde{V}^{\pm}_{j+\frac{1}{2}} = V\left(\tilde{U}^{\pm}_{j+\frac{1}{2}}, b^{\pm}_{j+\frac{1}{2}}\right). \tag{24}$$

The cell boundary values (used to evaluate the numerical fluxes) are then defined by:

$$\begin{aligned} U^{*,\pm}_{j+\frac{1}{2}} &= \left(\max\left(0, h\left(\widetilde{V}^{\pm}_{j+\frac{1}{2}}, b^{*}_{j+\frac{1}{2}}\right)\right), \widetilde{m}^{\pm}_{j+\frac{1}{2}} \right)^T \\ &= \left(\max\left(0, h\left(\widetilde{V}^{\pm}_{j+\frac{1}{2}}, b^{*}_{j+\frac{1}{2}}\right)\right), m^{\pm}_{j+\frac{1}{2}} \right)^T, \end{aligned} \tag{25}$$

as one can easily observe that $\widetilde{m}^{\pm}_{j+1/2} = m^{\pm}_{j+1/2}$. At the end, the left and right fluxes $\widehat{f}^{\,l}_{j+1/2}, \widehat{f}^{\,r}_{j-1/2}$ are given by:

$$\begin{aligned} \widehat{f}^{\,l}_{j+\frac{1}{2}} &= F\left(U^{*,-}_{j+\frac{1}{2}}, U^{*,+}_{j+\frac{1}{2}}\right) + f\left(U^{-}_{j+\frac{1}{2}}\right) - f\left(U^{*,-}_{j+\frac{1}{2}}\right), \\ \widehat{f}^{\,r}_{j-\frac{1}{2}} &= F\left(U^{*,-}_{j-\frac{1}{2}}, U^{*,+}_{j-\frac{1}{2}}\right) + f\left(U^{+}_{j-\frac{1}{2}}\right) - f\left(U^{*,+}_{j-\frac{1}{2}}\right). \end{aligned} \tag{26}$$

This completes the well-balanced DG methods for the moving water (4), and we have proven their well-balanced property in Xing (2014).

3.3 Positivity-Preserving Methods

The other difficulty in simulating the SWEs is associated with the robustness of the numerical methods near the wet/dry front. This problem relates to the fact that there is no water in these areas, while the SWEs (1) are only defined in wet regions. Therefore, we may need to deal with moving boundary problems. One could use the mesh adaption technique (Bokhove, 2005) which tracks the dry front by changing the meshes. It has the advantage in accuracy but is computationally expensive. A more popular approach is the thin-layer technique, which maintains a very thin layer in dry elements and includes these dry elements in the computation. The difficulty then reduces to the issue of preserving the nonnegativity of water height for the SWEs during the computation. Another related problem is the computation of velocity given height and discharge in the nearly dry region. One usually introduces a threshold on the velocity (or on the water height) to avoid extremely large velocity when $h \ll 1$. There have been a number of positivity-preserving schemes (Audusse et al., 2004; Berthon and Marche, 2008; Bryson et al., 2011; Castro et al., 2007; Gallardo et al., 2007; Kurganov and Petrova, 2007; Liang and Marche, 2009) in the finite volume framework. They usually rely on the positivity-preserving Riemann solver, for example the HLL solver (Harten et al., 1983). Some positivity-preserving DG methods with P^1 polynomial spaces (Bunya et al., 2009; Ern et al., 2008; Kesserwani and Liang, 2012) have been developed in the past few years, mainly relying on modifying the slope to avoid negative values of water height. For both finite volume and DG methods, the issue of positivity-preserving property for high order methods is nontrivial. Most existing high order wetting and drying treatments are focused on postprocessing

reconstruction of the data obtained from the numerical solution at each time level. One example is to project the solution to a nonnegative linear element in the cell near the wet/dry front. Even though the postprocessing can bring the reconstruction to satisfy nonnegative water height, this alone usually does not guarantee that the solution (e.g. cell average from a finite volume or DG scheme) at the next time step still maintains the nonnegative water height property. If negative cell averages for the water height are obtained at the next time level, the positivity reconstruction postprocessing will destroy the conservation. Recently, following the general approach introduced in Zhang and Shu (2010), a sufficient condition on the time step size to ensure the positivity of cell averages of water height, plus a simple high order positivity-preserving limiter, has been studied in Xing and Shu (2011) for the finite volume methods, and in Xing et al. (2010), Xing and Zhang (2013) and Wen et al. (2016) for the DG methods in one dimension and two dimensions on unstructured meshes.

In this section, we review the positivity-preserving limiter studied in Xing et al. (2010), Xing and Shu (2011) and Xing and Zhang (2013) for the SWEs (1) with dry areas. In the previous section, we discussed several different approaches to design well-balanced methods, which focus on the approximation of the momentum equation that has nonzero source term. The positivity-preserving property mainly relies on the numerical approximation to the mass equation, therefore, the positivity-preserving limiter discussed in this section can be applied to all these well-balanced methods.

We only consider the Euler forward in time in this subsection. The same results can be generalized to TVD high order Runge–Kutta and multistep time discretizations, since TVD time discretizations are convex combinations of the Euler forward operators. By plugging (17) and (16) into (13), the scheme satisfied by the cell averages of the water height in the well-balanced finite volume WENO or DG methods can be written as

$$\overline{h}_j^{n+1} = \overline{h}_j^n - \lambda\left[\widehat{F}\left(h_{j+\frac{1}{2}}^{*,-}, u_{j+\frac{1}{2}}^{-}; h_{j+\frac{1}{2}}^{*,+}, u_{j+\frac{1}{2}}^{+}\right) - \widehat{F}\left(h_{j-\frac{1}{2}}^{*,-}, u_{j-\frac{1}{2}}^{-}; h_{j-\frac{1}{2}}^{*,+}, u_{j-\frac{1}{2}}^{+}\right)\right], \quad (27)$$

where $\lambda = \Delta x/\Delta t$, $h_{j+1/2}^{*,\pm}$ are defined in (17) and

$$\widehat{F}\left(h_{j+\frac{1}{2}}^{*,-}, u_{j+\frac{1}{2}}^{-}; h_{j+\frac{1}{2}}^{*,+}, u_{j+\frac{1}{2}}^{+}\right) = \frac{1}{2}\left(h_{j+\frac{1}{2}}^{*,-}u_{j+\frac{1}{2}}^{-} + h_{j+\frac{1}{2}}^{*,+}u_{j+\frac{1}{2}}^{+} - \alpha\left(h_{j+\frac{1}{2}}^{*,+} - h_{j+\frac{1}{2}}^{*,-}\right)\right). \quad (28)$$

We now consider the $(2k-1)$-th order WENO scheme (27). For the ease of presentation, we consider a reconstructed polynomial $p_j(x)$ of degree $2k-2$, which satisfies

$$p_j(x_{j-\frac{1}{2}}) = h_{j-\frac{1}{2}}^{+}, \quad p_j(x_{j+\frac{1}{2}}) = h_{j+\frac{1}{2}}^{-}, \quad \frac{1}{\Delta x}\int_{I_j} p_j(x)dx = \overline{h}_j^n. \quad (29)$$

Moreover, $p_j(x)$ should be a $(2k-1)$-th order accurate approximation to the exact solution on I_j. For the WENO method, this polynomial only serves the theoretical purpose to understand the derivation of the limiter and will not need to be explicitly constructed in the implementation. For the DG method, this polynomial $p_j(x)$ is simply the DG solution $h_j(x)$.

Let us introduce the N-point Legendre Gauss–Lobatto quadrature rule on the interval $I_j = [x_{j-1/2}, x_{j+1/2}]$, which is exact for the integral of polynomials of degree up to $2N-3$. We choose N such that $2N-3 \geq 2k-2$, therefore this N-point Gauss–Lobatto quadrature is exact for polynomial of degree $2k-2$. We denote these quadrature points on I_j as

$$S_j = \left\{ x_{j-\frac{1}{2}} = \widehat{x}_j^1, \widehat{x}_j^2, \ldots, \widehat{x}_j^{N-1}, \widehat{x}_j^N = x_{j+\frac{1}{2}} \right\}.$$

Let $\widehat{w}_r$ be the quadrature weights for the interval $[-1/2, 1/2]$ such that $\sum_{r=1}^{N} \widehat{w}_r = 1$. Since $p_j(x)$ is polynomial of degree $2k-2$ and this quadrature is exact, we have

$$\overline{h}_j^n = \frac{1}{\Delta x} \int_{I_j} p_j(x)dx = \sum_{r=1}^{N} \widehat{w}_r p_j(\widehat{x}_j^r) = \sum_{r=2}^{N-1} \widehat{w}_r p_j(\widehat{x}_j^r) + \widehat{w}_1 h_{j-\frac{1}{2}}^+ + \widehat{w}_N h_{j+\frac{1}{2}}^-. \tag{30}$$

If we introduce the variable

$$\xi_j = \frac{1}{\sum_{r=2}^{N-1} \widehat{w}_r} \sum_{t=2}^{N-1} \widehat{w}_r p_j(\hat{x}_j^r) = \frac{\overline{h}_j^n - \widehat{w}_1 h_{j-\frac{1}{2}}^+ - \widehat{w}_N h_{j+\frac{1}{2}}^-}{1 - \widehat{w}_1 - \widehat{w}_N}, \tag{31}$$

we have

$$\overline{h}_j^n = (1 - \widehat{w}_1 - \widehat{w}_N)\xi_j + \widehat{w}_1 h_{j-\frac{1}{2}}^+ + \widehat{w}_N h_{j+\frac{1}{2}}^-. \tag{32}$$

Following the approaches in Perthame and Shu (1996), Zhang and Shu (2010), Xing et al. (2010) and Xing and Shu (2011), we can conclude that: If $h_{j-1/2}^{\pm}$, $h_{j+1/2}^{\pm}$ and ξ_j are all nonnegative, then $\overline{h}_j^{n+1}$ is also nonnegative under the CFL condition

$$\lambda\alpha \leq \widehat{w}_1. \tag{33}$$

This result tells us that for the scheme (27), we need to modify $p_j(x)$ (satisfying (29)) such that $p_j(x_{j\pm1/2})$ and ξ_j are all nonnegative. At time level n, given $\overline{h}_j^n \geq 0$, we consider the following limiter on the piecewise polynomial $p_j(x)$ introduced in Zhang and Shu (2010). It is a linear scaling around the cell average:

$$\widetilde{p}_j(x) = \theta\left(p_j(x) - \overline{h}_j^n\right) + \overline{h}_j^n, \quad \theta = \min\left\{1, \frac{\overline{h}_j^n}{\overline{h}_j^n - m_j}\right\}, \tag{34}$$

with

$$m_j = \min\left(h^+_{j-\frac{1}{2}}, h^-_{j+\frac{1}{2}}, \xi_j\right). \tag{35}$$

It is easy to observe that these conditions are satisfied with this limiter. Moreover, it can also be shown that this limiter does not destroy the high order accuracy, and we refer to Zhang and Shu (2010) and Xing and Shu (2011) for the detailed proof. Let $\widetilde{h}^+_{j-1/2} = \widetilde{p}_j(x_{j-1/2})$, $\widetilde{h}^-_{j+1/2} = \widetilde{p}_j(x_{j+1/2})$, and define $\widetilde{h}^{*,+}_{j-1/2}$, $\widetilde{h}^{*,-}_{j+1/2}$ following (17). Then, the revised positivity-preserving version of the scheme (27) takes the form

$$\overline{h}^{n+1}_j = \overline{h}^n_j - \lambda\left[\widehat{F}\left(\widetilde{h}^{*,-}_{j+\frac{1}{2}}, u^-_{j+\frac{1}{2}}; \widetilde{h}^{*,+}_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}\right) - \widehat{F}\left(\widetilde{h}^{*,-}_{j-\frac{1}{2}}, u^-_{j-\frac{1}{2}}; \widetilde{h}^{*,+}_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}\right)\right]. \tag{36}$$

The limiter (34) and (35) is a high order accurate positivity-preserving limiter, and preserves the conservation of $p_j(x)$.

We would like to mention that in wet regions, where m_j is $O(1)$ above zero, the limiter does not take effect, i.e. $\widetilde{p}_j(x) = p_j(x)$. Therefore, this positivity-preserving limiter is active only in the dry or nearly dry region. For high order time discretizations, we need to apply the limiter in each stage for a Runge–Kutta method or in each step for a multistep method. To be efficient, we could implement the time step restriction (33) only when a preliminary calculation to the next time step produces negative water height. This limiter has also been extended to two-dimensional SWEs on unstructured triangular meshes (Xing and Zhang, 2013).

4 SHALLOW WATER-RELATED MODELS

In the engineering and environmental applications, there are many physical models that are related to the SWEs, but obtained in slightly different context or assumptions. In this section, we will briefly discuss some numerical methods to solve these shallow water-related models.

4.1 Shallow Water Flows Through Channels With Irregular Geometry

As simplified models of some free surface flows, the shallow water equations for flows in an open channel with variable cross-section take the form of

$$\begin{aligned} &H_t + Q_x = 0 \\ &Q_t + \left(\frac{Q^2}{H} + I_1\right)_x = I_2 - g\sigma_b h b_x, \end{aligned} \tag{37}$$

where $\sigma^0(x, z)$ is the breadth of the channel, $\sigma_b(x) = \sigma^0(x, b(x))$ is the bottom channel width, $H = \int_b^{h+b} \sigma^0(x,z)dz$ is the cross-sectional wet area, and $Q = Hu$ is the mass flow rate. I_1 is given by $I_1 = g\int_h^{h+b}(h+b-z)\sigma^0(x,z)dz$ which equals to the cross-sectional average of the hydrostatic pressure multiplied by H, and $I_2 = g\int_h^{h+b}(h+b-z)\sigma_x^0(x,z)dz$.

In the simplified case when the channel has rectangular cross-section, i.e. $\sigma^0(x, z) \equiv \sigma(x)$, the hydrostatic pressure can be written as a function of variables and the model becomes a system of standard conservation laws. We have $\sigma_b(x) = \sigma(x)$, $H = \sigma h$, $I_1 = g\sigma h^2/2$, $I_2 = g\sigma_x h^2/2$, and the model (37) can be reduced to

$$\begin{aligned} &H_t + Q_x = 0 \\ &Q_t + \left(\frac{Q^2}{H} + \frac{1}{2}g\sigma h^2\right)_x = \frac{1}{2}gh^2\sigma_x - g\sigma h b_x. \end{aligned} \tag{38}$$

The system is hyperbolic and has two eigenvalues given by $u \pm \sqrt{gh}$. Note that when the channel width σ is a constant and independent of x, the model (38) becomes the SWEs with a nonflat bottom topography. Like the SWEs, the shallow water model (38) in rectangular channel admits the general moving water and the still water steady-state solutions.

Well-balanced methods for the shallow water model in rectangular channel have been designed in Vazquez-Cendon (1999), Garcia-Navarro and Vazquez-Cendon (2000), Balbas and Karni (2009), Hernández-Duenas and Karni (2011), Murillo and García-Navarro (2014) and Xing (2016), and an extension of the positivity-preserving limiter in Section 3.3 to this model has been studied in Xing (2016).

4.2 Shallow Water Equations on the Sphere

The SWEs on a rotating sphere have been known as a common test bed for numerical methods used in modelling global atmospheric flows. They describe the behaviour of a shallow homogeneous incompressible and inviscid fluid layer, and present the major difficulties found in the horizontal aspects of three-dimensional global atmospheric modelling. The SWEs on the sphere can be written in the flux form of

$$\begin{aligned} &h_t + \nabla\cdot(h\mathbf{u}) = 0, \\ &(h\mathbf{u})_t + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + gh\nabla h = -f\hat{\mathbf{k}}\times h\mathbf{u} - gh\nabla b, \end{aligned} \tag{39}$$

in the latitude–longitude coordinate, where the horizontal (on the sphere) vector velocity $\mathbf{u}$ has the longitudinal (λ) component u and the latitudinal (θ) component v. The operators ∇ and $\nabla\cdot$ are the spherical horizontal gradient and divergence operators given by

$$\nabla(\ ) = \frac{\hat{\mathbf{i}}}{a\cos\theta}\frac{\partial}{\partial\lambda}(\ ) + \frac{\hat{\mathbf{j}}}{a}\frac{\partial}{\partial\theta}(\ ), \quad \nabla\cdot\mathbf{u} = \frac{1}{a\cos\theta}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\theta)}{\partial\theta}\right].$$

The longitudinal, latitudinal and outwards radial unit vectors are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively, and a is the radius of the earth.

The intrinsic curvature properties of the spherical computational domain lead to new numerical difficulties for the choice of the computational mesh. Many successes have been observed using the traditional latitude–longitude mesh (Williamson et al., 1992). To remove its disadvantage of pole-singularity, other choices of meshes have been proposed and studied, including the Yin–Yang mesh, cubed-sphere mesh, among others. Well-balanced and positivity-preserving methods under these meshes, including spectral element, finite volume, continuous low order finite elements and DG methods, have also been investigated, and we refer to Taylor et al. (1997), Giraldo et al. (2002), Nair et al. (2005), Verkley (2009), Ullrich et al. (2010) and Duben et al. (2012) for recent development on numerical methods for the SWEs on the sphere.

4.3 Two-Layer Shallow Water Equations

One primary source of error in storm surge modelling is due to the fact that the vertical structure is not taken into account by the single-layer depth averaged SWEs. The multilayer SWEs have obtained increasing attention as a model for primarily long wave phenomena where vertical structure plays an important role in the flow. The two-layer SWEs are used to describe incompressible flows in the shallow water regime, in the situation where two layers with different densities can be identified. They appear in oceanographic models when a warm, light upper layer flows over a lower layer of cooler, heavier water with larger salinity. In one dimension, they take the form of

$$\begin{aligned}
&(h_1)_t + (h_1u_1)_x = 0,\\
&(h_1u_1)_t + \left(h_1u_1^2 + \frac{1}{2}gh_1^2\right)_x = -gh_1b_x - gh_1(h_2)_x,\\
&(h_2)_t + (h_2u_2)_x = 0,\\
&(h_2u_2)_t + \left(h_2u_2^2 + \frac{1}{2}gh_2^2\right)_x = -gh_2b_x - rgh_2(h_1)_x,
\end{aligned} \tag{40}$$

where h_1, u_1 denote the water height and velocity in upper layer, and h_2, u_2 denote the water height and velocity in lower layer. $r = \rho_1/\rho_2$ is the ratio of the layer densities.

Solving the two-layer SWEs is a challenging problem due to several reasons: they are only conditionally hyperbolic; they contain nonconservative product terms; they admit steady state to be exactly preserved; their water heights should stay nonnegative, and their eigenstructure cannot be obtained

in explicit form. Despite these difficulties, some numerical methods were developed recently from different approaches to overcome some of these difficulties. We refer to Kurganov and Petrova (2009), Abgrall and Karni (2009), Bouchut and de Luna (2008), Bouchut and Zeitlin (2010), Mandli (2013) and references therein for the details.

5 CONCLUSION REMARKS

In this chapter, we provide an overview of some numerical schemes, including finite difference, finite volume schemes and finite element DG methods for the SWEs. We have discussed two commonly encountered difficulties in simulating SWEs numerically and presented well-balanced methods and positivity-preserving methods to overcome these challenges. A list of shallow water-related models, as well as some numerical methods for them, has also been shown at the end. The numerical methods reviewed in this chapter can be very useful to model, simulate and help us understand the scientific and engineering applications related to the shallow water flows.

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