Journal of Scientific Computing, Vol. 27, Nos. 1–3, June 2006 (© 2005)
DOI: 10.1007/s10915-005-9027-y

High-Order Well-Balanced Finite Difference WENO Schemes for a Class of Hyperbolic Systems with Source Terms

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Received October 23, 2004; accepted (in revised form) September 12, 2005; Published online December 22, 2005

In this paper, we generalize the high order well-balanced finite difference weighted essentially non-oscillatory (WENO) scheme, designed earlier by us in Xing and Shu (2005, *J. Comput. phys.* **208**, 206–227) for the shallow water equations, to solve a wider class of hyperbolic systems with separable source terms including the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow and a two phase flow model. Properties of the scheme for the shallow water equations (Xing and Shu 2005, *J. Comput. phys.* **208**, 206–227), such as the exact preservation of the balance laws for certain steady state solutions, the non-oscillatory property for general solutions with discontinuities, and the genuine high order accuracy in smooth regions, are maintained for the scheme when applied to this general class of hyperbolic systems.

KEY WORDS: Hyperbolic balance laws; WENO scheme; high-order accuracy; source term; conservation laws; shallow water equation; elastic wave equation; chemosensitive movement; nozzle flow; two phase flow.

1. INTRODUCTION

In this paper, we generalize the high order well-balanced finite difference weighted essentially non-oscillatory (WENO) scheme, designed earlier by us in [19] for the shallow water equations, to solve a wider class of hyperbolic systems with separable source terms including the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow

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and a two phase flow model. Properties of the scheme for the shallow water equations [19], such as the exact preservation of the balance laws for certain steady state solutions, the non-oscillatory property for general solutions with discontinuities, and the genuine high-order accuracy in smooth regions, are maintained for the scheme when applied to this general class of hyperbolic systems.

We describe the procedure only for the one dimensional case in this paper. Generalizations to multi-dimensions can be done for some cases, following the same procedure as in [19]. The one-dimensional hyperbolic system of conservation laws with source terms (also called balance laws) under consideration is:

$$u_t + f(u, x)_x = g(u, x), \tag{1.1}$$

where u is the solution vector, $f(u, x)$ is the flux and $g(u, x)$ is the source term. This balance law admits steady state solutions in which the source term is exactly balanced by the flux gradient. The objective of well balanced schemes is to preserve exactly some of these steady state solutions.

A typical example considered extensively in the literature for balance laws is the shallow water equation with a non-flat bottom topology (see Sec. 3.1). Bermudez and Vazquez [4] first introduced the concept of the "exact C-property", which means that the scheme is exact when applied to the still water stationary solution. Many well balanced schemes satisfying the exact C-property have been developed in the literature, mostly for first and second order accuracy (e.g. [4, 10, 11–14, 17, 20]).

In a recent paper [19], we have developed a well balanced high order finite difference WENO scheme for solving the shallow water equation, which is non-oscillatory, well balanced (satisfying the exact C-property) for still water, and genuinely high order in smooth regions. Extensive one and two-dimensional numerical experiments are provided in [19] to demonstrate the good behavior of this scheme. The main idea in [19] is a special decomposition of the source term, allowing a discretization to the source term to be both high order accurate for general solutions and exactly well balanced with the flux gradient for still water. In this paper, we extend this idea of decomposition of source terms to a general class of balance laws with separable source terms, allowing the design of well balanced high order finite difference WENO scheme for all balance laws falling into this category. This class is quite general, including the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow and a two phase flow model. In Sec. 2, we describe the class of balance laws under consideration and develop well balanced finite difference WENO schemes for such balance laws. In Sec. 3, we give several exam-

ples in applications which fall into the category of balance laws discussed in Sec. 2, and show selective numerical results to demonstrate the behavior of our schemes. Concluding remarks are given in Sec. 4.

2. A GENERAL CLASS OF BALANCE LAWS

The main idea in [19] to design a high order finite difference WENO scheme for the shallow water equation is to decompose its source term into a sum of two terms, each of which is discretized independently using a finite difference formula consistent with that of approximating the flux derivative terms in the conservation law. In this section, we generalize this idea to a class of general balance laws (1.1) We first consider the case that (1.1) is a scalar balance law. The case of systems will be explored later. We are interested in preserving exactly certain steady state solutions u of (1.1):

$$f(u,x)_x = g(u,x). \tag{2.1}$$

We make two assumptions on Eq. (1.1) and the steady state solution u of (2.1) that we are interested to preserve exactly:

Assumption 2.1. The steady state solution u of (2.1) that we are interested to preserve satisfies

$$a(u,x) = constant \tag{2.2}$$

for a known function $a(u,x)$.

Assumption 2.2. The source term $g(u,x)$ in (1.1) can be decomposed as

$$g(u,x) = \sum_i s_i(a(u,x))\, t_i'(x) \tag{2.3}$$

for some functions s_i and t_i.

Before proceeding further, let us comment on Assumption 2.1. We consider a special case of (1.1):

$$u_t + f(u)_x = g(u)\, z'(x), \tag{2.4}$$

i.e., when the flux f does not depend explicitly on x and the source term $g(u,x)$ in (1.1) is separable as a product of a function in u and a function in x. Notice that the case $g(u,x) = g(u)$ falls into this category with $z(x) = x$. The steady state solution of Eq. (2.4) is given by:

$$f(u)_x = g(u)\, z'(x).$$

Clearly

$$\int \frac{f'(u)}{g(u)}\,du = \int \frac{f'(u)}{g(u)} u_x\,dx = \int \frac{f(u)_x}{g(u)} dx = \int z'(x)dx = z(x) + constant.$$

Hence we have

$$a(u,x) \equiv b(u) + z(x) = constant, \tag{2.5}$$

if we denote $b(u) = -\int \frac{f'(u)}{g(u)} du$. This is an example of (2.2).

We would like to preserve exactly the steady state solutions u which satisfy Assumption 2.1, for a balance law (1.1) with a source term satisfying Assumption 2.2. Following the ideas in [19], we will first consider a linear scheme with an identical finite difference operator for the flux derivative and the derivatives in the decomposed source terms. As usual, we define a linear finite difference operator D to be one satisfying $D(af_1 + bf_2) = aD(f_1) + bD(f_2)$ for constants a, b and arbitrary grid functions f_1 and f_2. A scheme for (1.1) with a source term given by (2.3) is said to be a linear scheme if all the spatial derivatives are approximated by linear finite difference operators. Such a linear scheme would have a truncation error

$$D_0(f(u,x)) - \sum_i s_i(a(u,x))\, D_i(t_i(x)),$$

where D_i are linear finite difference operators used to approximate the spatial derivatives. We further restrict our attention to linear schemes which satisfy

$$D_0 = D_1 = \cdots = D \tag{2.6}$$

for the steady state solution. For such linear schemes we have

Proposition 2.3. For the balance law (1.1) with its source term given by (2.3), linear schemes with (2.6) for the steady state solutions satisfying (2.2) can preserve these steady state solutions exactly.

Proof. For the steady state solution u satisfying (2.2), the truncation error for such linear schemes with (2.6) reduces to

$$\begin{aligned} &D\,(f(u,x)) - \sum_i s_i(a(u,x))\, D\,(t_i(x)) \\ &\quad = D\left(f(u,x) - \sum_i s_i(a(u,x))\, t_i(x)\right), \end{aligned} \tag{2.7}$$

where the linearity of D and the fact that $a(u,x)=constant$ for the steady state solution u are used. Clearly, for such steady state solution u,

$$\frac{d}{dx}\left(f(u,x)-\sum_i s_i(a(u,x))\,t_i(x)\right) = f(u,x)_x-\sum_i s_i(a(u,x))\,t_i'(x)$$
$$= f(u,x)_x - g(u,x)=0,$$

that is, $f(u,x)-\sum_i s_i(a(u,x))\,t_i(x)$ is a constant. Hence the truncation error (2.7) is 0 for any consistent finite difference operator D. This finishes the proof. □

We now consider high-order nonlinear finite difference WENO schemes [3, 8], in which the non-linearity comes from the non-linear weights and the smooth indicators. We follow the procedures described in [19] for the shallow water equations, to treat the general balance laws (1.1) to obtain well balanced high-order finite difference WENO schemes.

To present the basic ideas, we first consider the situation when the WENO scheme is used without a flux splitting (e.g., the WENO-Roe scheme as described in [8]). We notice that the WENO approximation to d_x where $d=f(u,x)$ can be eventually written out as

$$d_x|_{x=x_j} \approx \sum_{k=-r}^{r} a_k d_{k+j} \equiv D_d(d)_j, \tag{2.8}$$

where $r=3$ for the fifth-order WENO approximation and the coefficients a_k depend non-linearly on the smoothness indicators involving the grid function d. As explained in [19], the key idea now is to use the finite difference operator D_d with $d=f(u,x)$ *fixed*, and apply it to approximate $t_i'(x)$ in the source terms. Thus

$$t_i'(x_j) \approx \sum_{k=-r}^{r} a_k\, t_i(x_{k+j}) = D_d\,(t_i(x))_j\,.$$

Clearly, the finite difference operator D_d, obtained from the high-order WENO procedure and when $d=f(u,x)$ is fixed, is a high-order accurate *linear* approximation to the first derivative for any grid function. Therefore the proof for Proposition 2.3 will go through and we conclude that the high-order finite difference WENO scheme as stated above, without the flux splitting, and with the special handling of the source terms described above, maintains exactly the steady state.

Now, we consider WENO schemes with a Lax-Friedrichs flux splitting, such as the WENO-LF and WENO-LLF schemes described in [8].

Here the flux $f(u,x)$ is written as a sum of $f^+(u,x)$ and $f^-(u,x)$, defined by

$$f^{\pm}(u,x)=\frac{1}{2}[f(u,x)\pm\alpha u], \tag{2.9}$$

where $\alpha = max_u \left|\frac{\partial f(u,x)}{\partial u}\right|$ with the maximum being taken over either a local region (WENO-LLF) or a global region (WENO-LF) (see [8, 16] for more details). We now make a modification to this flux splitting, by replacing $\pm\alpha u$ in (2.9) with $\pm\alpha\,\text{sign}\left(\frac{\partial a(u,x)}{\partial u}\right) a(u,x)$. We would need to assume here that $\frac{\partial a(u,x)}{\partial u}$ does not change sign. The constant α should be suitably adjusted by the size of $\frac{\partial a(u,x)}{\partial u}$ in order to maintain enough artificial viscosity. The term $a(u,x)$ can also be replaced by $p(a(u,x))$ for any function p, whose choice should be such that $p(a(u,x))$ is as close to u as possible in order to emulate the original LF flux splitting with $\pm\alpha u$. This modification does not affect accuracy, which relies only on the fact $f(u,x)=f^+(u,x)+f^-(u,x)$. For the steady state solution satisfying (2.2), the artificial viscosity term $\pm\alpha\,\text{sign}\left(\frac{\partial a(u,x)}{\partial u}\right) a(u,x)$ (or $\pm\alpha\,\text{sign}\left(\frac{\partial p(a(u,x))}{\partial u}\right) p(a(u,x))$) in the Lax-Friedrichs flux splitting becomes a constant, and by the consistency of the WENO approximation, the effect of these viscosity terms towards the approximation of $f(u,x)_x$ is zero. The flux splitting WENO approximation in this situation becomes simply $f^{\pm}(u,x)=\frac{1}{2}f(u,x)$, hence the steady state solution is preserved as before, if we simply split the derivatives in the source term as:

$$t_i'(x)=\frac{1}{2}t_i'(x)+\frac{1}{2}t_i'(x) \tag{2.10}$$

and apply the same flux splitting WENO procedure to approximate them with the nonlinear coefficients a_k coming from the WENO approximations to $f^{\pm}(u,x)$, respectively. This will guarantee (2.6). We have thus proved that

Proposition 2.4. The WENO-Roe, WENO-LF, and WENO-LLF schemes as implemented above are exact for steady state solutions satisfying (2.2) and can maintain the original high-order accuracy.

We now discuss the system case. The framework described for the scalar case can be applied to systems provided that we have certain knowledge about the steady state solutions to be preserved in the form of (2.2). Typically, for a system with m equations, we would have m relationships in the form of (2.2):

$$a_1(u,x)=constant, \quad \cdots \quad a_m(u,x)=constant \tag{2.11}$$

for the steady state solutions that we would like to preserve exactly. We would then still aim for decomposing each component of the source term in the form of (2.3), where s_i could be arbitrary functions of $a_1(u,x),\ldots,a_m(u,x)$, and the functions s_i and t_i could be different for different components of the source vector. The remaining procedure is then the same as that for the scalar case and we again obtain well balanced high order WENO schemes. Examples of such systems will be given in next section. We should also mention that local characteristic decomposition is typically used in high order WENO schemes in order to obtain better non-oscillatory property for strong discontinuities. When computing the numerical flux at $x_{i+\frac{1}{2}}$, the local characteristic matrix R, consisting of the right eigenvectors of the Jacobian at $u_{i+\frac{1}{2}}$, is a constant matrix for fixed i. Hence this characteristic decomposition procedure does not alter the argument presented above for the scalar case (we refer to [19] for more details).

3. APPLICATIONS

In this section we give several examples from applications which fall into the category of balance laws considered in the previous section, and present well balanced high order finite difference WENO schemes for them. Due to page limitation, only selected numerical results are shown to give a glimpse of how these methods work. In the numerical tests, time discretization is by the classical fourth order Runge–Kutta method, and the CFL number is taken as 0.6.

3.1. Shallow Water Equations

The shallow water equations have wide applications in ocean and hydraulic engineering and river, reservoir, and open channel flows, among others. We consider the system with a geometrical source term due to the bottom topology. In one space dimension, the equations take the form

$$\begin{aligned} &h_t+(hu)_x=0,\\ &(hu)_t+\left(hu^2+\frac{1}{2}gh^2\right)_x=-ghb_x, \end{aligned} \tag{3.1}$$

where h denotes the water height, u is the velocity of the fluid, $b(x)$ represents the given bottom topography and g is the gravitational constant.

We are interested in preserving the still water solution, which satisfies (2.11) in the form

$$a_1 \equiv h + b = constant, \quad a_2 \equiv u = 0.$$

The first component of the source term is 0. A decomposition of the second component of the source term in the form of (2.3) is

$$-ghb_x = -g(h+b)b_x + \frac{1}{2}g\left(b^2\right)_x,$$

i.e., $s_1 = s_1(a_1) = -g(h+b)$, $s_2 = \frac{1}{2}g$, $t_1(x) = b(x)$, and $t_2(x) = b^2(x)$.

More details of the high-order finite difference WENO scheme applied to this system, and extensive numerical results, can be found in [19].

3.2. Elastic Wave Equation

We consider the propagation of compressional waves [2, 18] in an one-dimensional elastic rod with a given media density $\rho(x)$. The equations of motion in a Lagrangian frame are given by the balance laws:

$$\begin{aligned} &(\rho\varepsilon)_t + (-\rho u)_x = -u\frac{d\rho}{dx}, \\ &(\rho u)_t + (-\sigma)_x = 0, \end{aligned} \tag{3.2}$$

where $\varepsilon = \varepsilon(x,t)$ is the strain, $u = u(x,t)$ is the velocity and σ is a given stress–strain relationship $\sigma(\varepsilon, x)$. The equation of linear acoustics can be obtained from the elasticity problem if the stress–strain relationship is linear,

$$\sigma(\varepsilon, x) = K(x)\varepsilon,$$

where $K(x)$ is the given bulk modulus of compressibility.

The steady state we are interested to preserve for this problem is characterized by

$$a_1 \equiv \sigma(\varepsilon, x) = constant, \qquad a_2 \equiv u = constant,$$

which is of the form (2.11). The second component of the source term is 0. The first component of the source term is already in the form of (2.3) with $s_1 = s_1(a_2) = -u$ and $t_1 = \rho(x)$.

We now show two numerical examples to demonstrate the fifth-order well balanced finite difference WENO scheme for (3.2). The first example, from [18], is to test the fifth order accuracy for smooth solutions, for which we take the initial conditions as

Table I. L^1 Errors and Numerical Orders of Accuracy for the Example in Sec. 3.2

	Balanced WENO			
	$\rho\varepsilon$		ρu	
Number of points	L^1 error	Order	L^1 error	Order
20	2.33E-002		2.80E-002	
40	3.21E-003	2.86	3.50E-003	3.00
80	3.75E-004	3.10	2.30E-004	3.93
160	1.59E-005	4.56	1.10E-005	4.38
320	5.20E-007	4.93	3.92E-007	4.81
640	1.65E-008	4.97	1.25E-008	4.97

$$\rho\varepsilon(x,0)=\frac{-1-1.5e^{-(8x)^2}}{(1-0.5\sin(\pi x))^2},\qquad u(x,0)=0$$

with the density $\rho(x)$ and bulk modulus of compressibility $K(x)$ given by:

$$\rho(x)=\frac{1}{1-0.5\sin(\pi x)},\qquad K(x)=1-0.5\sin(\pi x).$$

The computational domain is $[-1,1]$ and periodic boundary condition is used. The exact solution is unknown in this case, hence we use the same fifth-order well balanced WENO scheme with $N=5120$ grid points to compute a reference solution and use this reference solution as the exact solution in computing the numerical errors at $t=0.1$ s. Table I contains the L^1 errors and numerical orders of accuracy. We can clearly see that fifth-order accuracy is achieved for this example.

Next, we present the numerical result for a linear acoustic test [2]. The properties of the media are given by

$$c(x)=\sqrt{\frac{K(x)}{\rho(x)}}=1+0.5\sin(10\pi x),\qquad Z(x)=\rho(x)c(x)=1+0.25\cos(10\pi x)$$

and are shown in Fig. 1. The initial conditions are given by

$$\rho\varepsilon(x,0)=\begin{cases}\dfrac{-1.75+0.75\cos(10\pi x)}{c^2(x)}, & \text{if } 0.4<x<0.6,\\[2ex] \dfrac{-1}{c^2(x)}, & \text{otherwise,}\end{cases}\qquad u(x,0)=0.$$

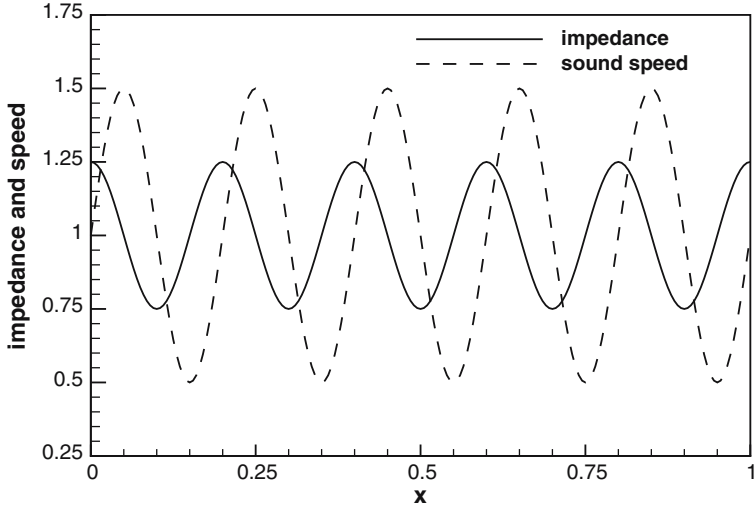

Fig. 1. The impedance $Z(x)$ and the sound speed $c(x)$ for the smooth media.

It is a test case where the impedance $Z(x)$ and hence the eigenvectors are both spatially varying. We perform the computation with 200 uniform cells, with the ending time $t = 0.4$ s. An "exact" reference solution is computed with the same scheme over a 2000 grid point uniform cells. The simulation results are shown in Fig. 2. The numerical resolution shows very good agreement with the "exact" reference solution.

3.3. Chemosensitive Movement

Originated from biology, chemosensitive movement [5, 7] is a process by which cells change their direction reacting to the presence of a chemical substance, approaching chemically favorable environments and avoiding unfavorable ones. Hyperbolic models for chemotaxis are recently introduced [7] and take the form

$$\begin{aligned} &n_t + (nu)_x = 0, \\ &(nu)_t + (nu^2 + n)_x = n\chi'(c)\frac{\partial c}{\partial x} - \sigma nu, \end{aligned} \tag{3.3}$$

where the chemical concentration $c = c(x, t)$ is given by the parabolic equation

$$\frac{\partial c}{\partial t} - D_c \triangle c = n - c.$$

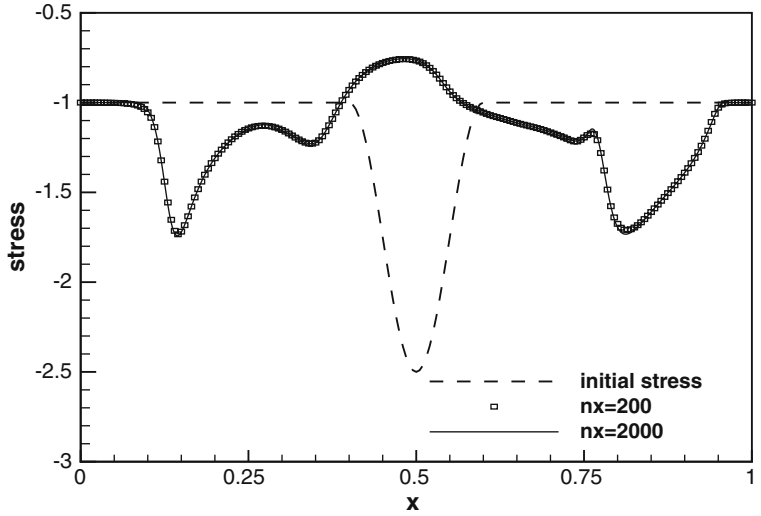

Fig. 2. The numerical (symbols) and the "exact" reference (solid line) stress $\sigma(x)$ at time $t = 0.4$ s.

Here, $n(x,t)$ is the cell density, $nu(x,t)$ is the population flux and σ is the friction coefficient. In [5], a well balanced WENO scheme is constructed based on a different approach, which can maintain the steady state solutions with zero population flux to the size of a small parameter ε in the non-linear WENO weights. Here we construct well balanced WENO schemes based on the framework in Sec. 2, which does not have this restriction.

We would like to preserve the steady state solution to (3.3) with a zero population flux, which satisfies

$$n\chi'(c)c_x - n_x = 0, \qquad nu = 0, \tag{3.4}$$

where $c = c(x)$ does not depend on t in steady state. The first equality above does not seem to be of the form (2.11). However, (3.4) is equivalent to

$$a_1 \equiv \log(n) - \chi(c) = constant, \qquad a_2 \equiv nu = 0,$$

which is clearly in the form of (2.11). The first component of the source term is 0. A decomposition of the second component of the source term in the form of (2.3) is

$$n\chi'(c)\frac{\partial c}{\partial x} - \sigma nu = e^{\log(n)-\chi(c)}\frac{d}{dx}e^{\chi(c)} - \sigma nu$$

Table II. L^1 Errors for Different Precisions for the Stationary Solution in Sec. 3.3

	L^1 error	
Precision	n	nu
Single	7.34E-07	3.16E-07
Double	1.02E-15	3.96E-16
Quadruple	9.13E-34	2.32E-34

i.e. $s_1 = s_1(a_1) = e^{\log(n) - \chi(c)}$, $s_2 = s_2(a_2) = \sigma nu$, $t_1(x) = e^{\chi(c(x))}$, and $t_2(x) = x$.

We now show two numerical examples to demonstrate the fifth order well balanced finite difference WENO scheme for (3.3). The first example is to test the well balancedness property of the scheme. We take the initial conditions as

$$n(x, 0) = \frac{1}{10}(1 + c(x)), \qquad nu(x, 0) = 0$$

with

$$c(x) = \begin{cases} 1, & \text{if } |x| \leqslant 0.5, \\ 0.125, & \text{otherwise,} \end{cases} \qquad \chi(c) = \log(1 + c), \qquad \sigma = 1.$$

The initial condition is a steady state solution which should be exactly preserved. We compute the solution until $t = 2.0\,\mathrm{s}$ using $N = 500$ uniform mesh points. In order to demonstrate that the steady state is indeed maintained up to round-off error, we use single precision, double precision and quadruple precision to perform the computation, and show the L^1 errors for the cell density n (note: n in this case is a discontinuous function!) and the population flux nu in Table II for these different precisions. We can clearly see that the L^1 errors are at the level of round-off errors for different precisions, verifying the steady state conservation.

The second example is to test the fifth order accuracy for smooth solutions, for which we take the initial conditions as

$$n(x, 0) = 1 + 0.2\cos(\pi x), \qquad u(x, 0) = 0, \qquad x \in [-1, 1]$$

with

$$c(x) = e^{-16x^2}, \qquad \chi(c) = \log(1 + c), \qquad \sigma = 0$$

Table III. L^1 Errors and Numerical Orders of Accuracy for the Example in Sec. 3.3.

	Balanced WENO			
	n		nu	
Number of points	L^1 error	Order	L^1 error	Order
20	1.02E-002		5.99E-003	
40	1.05E-003	3.29	7.70E-004	2.96
80	1.31E-004	3.00	1.07E-004	2.85
160	6.57E-006	4.32	5.49E-006	4.29
320	2.44E-007	4.75	2.06E-007	4.73
640	7.58E-009	5.01	6.43E-009	5.00

with a periodic boundary condition. Since the exact solution is not known explicitly for this problem, we use the same fifth-order WENO scheme with $N = 5120$ points to compute a reference solution and treat it as the exact solution when computing the numerical errors. Final time $t = 0.5$ s is used to avoid the development of shocks. Table III contains the L^1 errors and numerical orders of accuracy. We can clearly see that fifth-order accuracy is achieved for this example.

3.4. Nozzle Flow

In this subsection we consider the balance laws for a quasi one-dimensional nozzle flow [6]. The governing equations for the quasi-one-dimensional unsteady flow through a duct of varying cross-section can be written in conservation form as:

$$\begin{aligned} &(\rho A)_t + (\rho u A)_x = 0, \\ &(\rho u A)_t + \Big((\rho u^2 + p)A\Big)_x = pA'(x), \\ &(EA)_t + ((E+p)uA)_x = 0, \end{aligned} \tag{3.5}$$

where the quantities ρ, u, p and $E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}$ represent the density, velocity, pressure, and total energy, respectively. $A = A(x)$ denotes the area of the cross section. γ is the ratio of specific heats.

As in [6], we are interested in preserving the steady state solution

$$\rho(x,t) = \bar{\rho}(x), \quad p(x,t) = \bar{p}, \quad \text{and} \quad u(x,t) = 0, \tag{3.6}$$

Table IV. L^1Errors and Numerical Orders of Accuracy for the Example in Sec. 3.4

	Balanced WENO					
	ρA		$\rho u A$		EA	
Number of points	L^1 error	Order	L^1 error	Order	L^1 error	Order
20	6.13E-003		3.90E-003		4.58E-003	
40	2.14E-003	1.52	9.46E-004	2.04	8.90E-004	2.38
80	2.10E-004	3.35	9.72E-005	3.28	8.49E-005	3.37
160	1.01E-005	4.38	4.79E-006	4.34	4.11E-006	4.37
320	3.44E-007	4.88	1.60E-007	4.91	1.40E-007	4.87
640	1.04E-008	5.04	5.08E-009	4.98	4.29E-009	5.03

where $\bar{\rho}(x)$ is an arbitrary function in x and $\bar{p}$ is a constant. The second condition in (3.6)

$$a_1 \equiv p = \bar{p}$$

is of the form (2.11). The first and third components of the source term are 0. The second component of the source term is already in the form of (2.3) with $s_1 = s_1(a_1) = p$ and $t_1 = A(x)$.

We now show two numerical examples to demonstrate the fifth order well balanced finite difference WENO scheme for (3.5). The first example is to test the fifth-order accuracy for smooth solutions, for which we take the cross section area and the initial conditions as

$$\begin{aligned} A(x) &= 1 + \sin^2(\pi x), \qquad \rho(x) = \cos(\sin(2\pi x)), \\ u(x) &= 0, \qquad E(x) = e^{\sin(2\pi x)} \end{aligned}$$

with periodic boundary conditions. As before, we compute a reference solution using the same fifth-order WENO scheme with $N = 10240$ points. Final time is chosen as $t = 0.25$ s when the solution is still smooth. Table IV contains the L^1 errors and numerical orders of accuracy. We can clearly see that fifth order accuracy is achieved for this example.

The purpose of the second test case is to study the convergence in time towards steady flow. Proposed by Anderson in [1], it is concerned with a convergent-divergent nozzle flow with a parabolic area distribution, which is given by

$$A(x) = 1 + 2.2(x - 1.5)^2, \qquad 0 \leqslant x \leqslant 3. \tag{3.7}$$

The shape of this section is illustrated in Fig. 3.

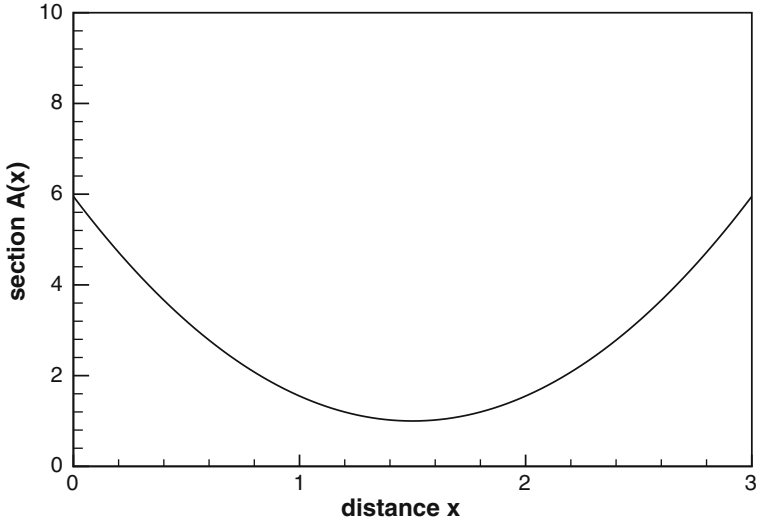

Fig. 3. The shape of a convergent-divergent nozzle.

The initial conditions are taken as

$$\rho(x,0)=1, \quad u(x,0)=0 \quad \text{and} \quad p(x,0)=1.$$

The boundary conditions are taken as one bar of pressure at the left, 0.6784 bar of pressure at the right, and 300°K of temperature at both ends. A shock is established inside the pipe, and the exact solution for this, a steady state, can be easily calculated. In this case, the Froude number $Fr=u/c$ increases to a value larger than one, and then decreases to less than one.

The computation is performed using $N=100$ points. The pressure $p(x)$ is plotted in Fig. 4, which shows very good agreement with the analytical solution. The numerical resolution is very good without oscillations, verifying the essentially non-oscillatory property of the modified WENO-LF scheme.

3.5. Two Phase Flow

The dynamics of fluids consisting of several fluid components is of great interest in a wide range of physical flows. In this subsection we are interested in a flow model for the compressible 2-velocity 2-pressure system [15, 9], which is suitable to describe liquid suspensions and bubbly flows. The balance equations are written for the individual phases:

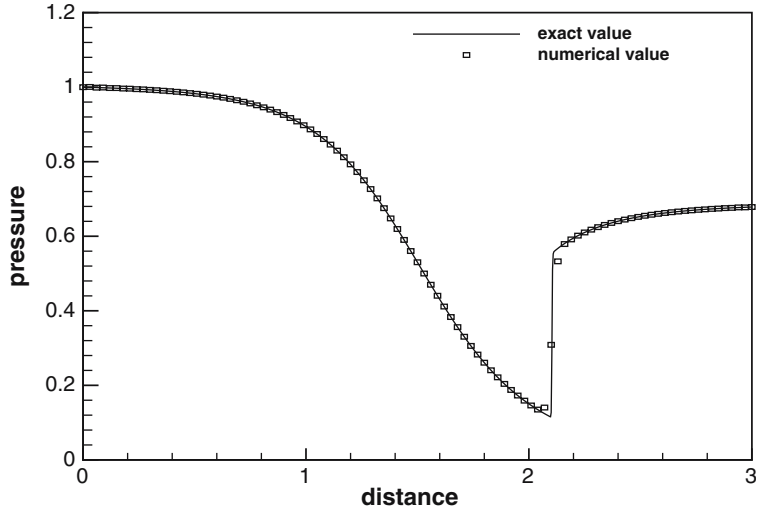

Fig. 4. Steady state pressure for the nozzle flow.

$$\begin{pmatrix} a_g\rho_g \\ a_g\rho_g u_g \\ a_g E_g \\ a_l\rho_l \\ a_l\rho_l u_l \\ a_l E_l \end{pmatrix}_t + \begin{pmatrix} a_g\rho_g u_g \\ a_g(\rho_g u_g^2 + p_g) \\ u_g a_g(E_g + p_g) \\ a_l\rho_l u_l \\ a_l(\rho_l u_l^2 + p_l) \\ u_l a_l(E_l + p_l) \end{pmatrix}_x$$

$$= \begin{pmatrix} 0 \\ p_i(a_g)_x + \quad \lambda(u_l - u_g) + \quad a_g\rho_g g \\ u_i p_i(a_g)_x + \lambda u_i(u_l - u_g) + \mu p_i(p_l - p_g) + a_g\rho_g u_g g \\ 0 \\ p_i(a_l)_x - \quad \lambda(u_l - u_g) + \quad a_l\rho_l g \\ u_i p_i(a_l)_x - \lambda u_i(u_l - u_g) - \mu p_i(p_l - p_g) + a_l\rho_l u_l g \end{pmatrix}, \quad (3.8)$$

coupled with an additional equation for the volume fraction

$$(a_g)_t + u_i(a_g)_x = -\mu(p_l - p_g) \quad (3.9)$$

and the algebraic relation for the volume fractions

$$a_g + a_l = 1.$$

Here, a_k, $k \in \{l, g\}$ is the volume fraction of the kth phase, and ρ_k, u_k, $E_k = \frac{1}{2}\rho_k u_k^2 + \frac{p_k}{\gamma_k - 1}$ denote its density, velocity, and energy, respectively. λ

is a velocity relaxation parameter and μ is a pressure relaxation parameter. The closure equations for the interface pressure p_i and the interface velocity u_i are

$$p_i = a_g p_g + a_l p_l, \quad u_i = \frac{a_g \rho_g u_g + a_l \rho_l u_l}{a_g \rho_g + a_l \rho_l}.$$

If the gravitation effect is ignored, one stationary solution for (3.8) is given by:

$$\rho_g = \bar{\rho}_1(x), \quad \rho_l = \bar{\rho}_2(x), \quad u_g = u_l = 0, \quad p_g = p_l = \bar{p}, \tag{3.10}$$

where $\bar{\rho}_1(x)$ and $\bar{\rho}_2(x)$ are arbitrary functions of x and $\bar{p}$ is a constant. We would like to preserve this steady state solution exactly. (3.10) can clearly be written in the form of (2.11):

$$a_1 \equiv p_i = \bar{p}, \quad a_2 \equiv u_l - u_g = 0, \quad a_3 \equiv p_l - p_g = 0, \quad a_4 \equiv u_i = 0$$

and hence the source terms are already in the form of (2.3). For example, the second component of the source term is of the form (2.3) with $s_1 = s_1(a_1) = p_i$, $t_1 = a_g(x)$, $s_2 = s_2(a_2) = \lambda(u_l - u_g)$, $t_2 = x$; the third component of the source term is of the form (2.3) with $s_1 = s_1(a_4, a_1) = u_i p_i$, $t_1 = a_g(x)$, $s_2 = s_2(a_4, a_2) = \lambda u_i (u_l - u_g)$, $t_2 = x$, $s_3 = s_3(a_1, a_3) = \mu p_i (p_l - p_g)$, $t_3 = x$; etc.

4. CONCLUDING REMARKS

Well balanced, high order finite difference WENO schemes are designed for a class of hyperbolic systems with separable source terms including the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow and a two phase flow model, following the ideas of suitable decomposition of source terms in [19] for the shallow water equations. The schemes maintain properties of the exact preservation of the balance laws for certain steady state solutions, the non-oscillatory property for general solutions with discontinuities, and the genuine high order accuracy in smooth regions. The approach is quite general and can be adapted to high order finite volume and discontinuous Galerkin finite element methods on arbitrary triangulations, which constitutes an ongoing work.

ACKNOWLEDGMENTS

Research supported by ARO grant W911NF-04-1-0291, NSF grants DMS-0207451 and DMS-0510345, and AFOSR grants F49620-02-1-0113 and FA9550-05-1-0123.

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