

Application of high order well-balanced schemes to a class of hyperbolic systems with source terms *

YULONG XING AND CHI-WANG SHU

Department of Mathematics, Brown University, Providence, USA
Division of Applied Mathematics, Brown University, Providence, USA

xing@dam.brown.edu, shu@dam.brown.edu

Abstract

In this short note, we survey and apply our recently designed high order accurate well-balanced finite volume WENO (weighted essentially non-oscillatory) schemes and discontinuous Galerkin finite element schemes for solving a class of hyperbolic systems. Two different approaches to obtain high order well balanced schemes, which are also essentially non-oscillatory for general solutions with discontinuities, have been designed. Such schemes are particularly useful in computing small perturbations to such steady state solutions. Some applications of these well balanced schemes on hyperbolic balance laws, including the shallow water equation, the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow, a model in fluid mechanics, the Goldstein-Taylor model of the Boltzmann equation and a two phase flow model, are introduced at the end.

Key words: *hyperbolic balance laws; WENO scheme; discontinuous Galerkin method; high order accuracy; source term; conservation laws; shallow water equation; elastic wave equation; chemosensitive movement; nozzle flow; two phase flow; Goldstein-Taylor model.*

AMS subject classifications: *65M60, 65N30, 35L65*

1 Introduction

We are interested in numerically solving the following hyperbolic conservation laws with source terms, also referred to as hyperbolic balance laws:

$$u_t + f_1(u, x, y)_x + f_2(u, x, y)_y = g(u, x, y) \quad (1)$$

*Research supported by ARO grant W911NF-04-1-0291, NSF grant DMS-0510345 and AFOSR grant FA9550-05-1-0123.

or in the one dimensional case

$$u_t + f(u, x)_x = g(u, x) \quad (2)$$

where u is the solution vector, $f_1(u, x, y)$ and $f_2(u, x, y)$ (or $f(u, x)$) are the fluxes and $g(u, x, y)$ (or $g(u, x)$) is the source term. Hyperbolicity refers to the fact that the Jacobians $\frac{\partial f_1(u, x, y)}{\partial u}$ and $\frac{\partial f_2(u, x, y)}{\partial u}$ (or $\frac{\partial f(u, x)}{\partial u}$) always have real eigenvalues and complete sets of eigenvectors. Often, this balance law would admit steady state solutions in which the source term is exactly balanced by the flux gradient. Notice that in such situations the solution u is typically a non-trivial function, hence a straightforward numerical scheme may fail to preserve exactly this balance. Many physical phenomena come from small perturbations of these steady state solutions, which are very difficult to capture numerically, unless the numerical schemes can preserve the unperturbed steady state at the discrete level. Schemes which can preserve the unperturbed steady state at the discrete level are the so called well balanced schemes. Our purpose is to design well balanced schemes without sacrificing the high order accuracy and non-oscillatory properties of the scheme when applied to general, non-steady state solutions.

A prototype example for the balance laws (2), which has been investigated extensively in the literature, is the shallow water equation with a non-flat bottom topology. Many geophysical flows are modeled by the variants of the shallow water equations. In one space dimension, they take the form

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghb_x, \end{cases} \quad (3)$$

where h denotes the water height, u is the velocity of the fluid, b represents the bottom topography and g is the gravitational constant. The steady state solutions are given by

$$hu = \text{constant} \quad \text{and} \quad \frac{1}{2}u^2 + g(h + b) = \text{constant}. \quad (4)$$

Often, we are particularly interested in the still water stationary solution, denoted by

$$u = 0 \quad \text{and} \quad h + b = \text{constant}. \quad (5)$$

Bermudez and Vazquez [2] first introduced the concept of the ‘‘exact C-property’’, which refers to the ability of the scheme to exactly preserve the still water stationary solution. Many well balanced schemes satisfying the exact C-property have been developed in the literature, mostly for first and second order accuracy, e.g. [2, 8, 9, 11, 12, 13, 14, 19]. It is technically difficult to construct schemes which are genuinely high order accurate in smooth regions for general solutions, are essentially non-oscillatory for general solutions with discontinuities, and at the same time can preserve exactly certain steady state solutions.

Recently, we have developed high order well-balanced finite difference and finite volume WENO (weighted essentially non-oscillatory) schemes and discontinuous Galerkin finite element schemes for solving a class of hyperbolic systems with separable source terms. In [15], we started our investigation by designing a class of high order finite difference WENO schemes which are well balanced for the still water solution of the shallow water equations. In [16], the well balanced high order finite difference WENO schemes designed in [15] are generalized to a general class of balance laws with separable source terms, including the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow and a two phase flow model. In [17], well balanced high order finite volume WENO schemes and finite element discontinuous Galerkin schemes are designed for the same class of balance laws as those in [16], which are more suitable for computations in complex geometry and / or for using adaptive meshes. The key ingredient of the technique used in [15, 16, 17] to obtain well balanced property is a special decomposition of the source term, allowing a discretization to the source term to be both high order accurate for general solutions and exactly well balanced with the flux gradient for some steady states.

In [18] we discuss a new approach of high order well balanced finite volume WENO schemes and discontinuous Galerkin finite element schemes. We show that the traditional RKDG methods can achieve the well balanced property with a small modification of either the initial value or the flux. This is by far the simplest approach to obtain a high order well balanced scheme. Very little additional computational cost over the traditional RKDG methods is involved to obtain a well balanced property. Similar ideas are then applied to obtain well balanced finite volume WENO schemes. Comparing with our previous work [17], the new approach [18] requires less computational cost, and is easier to understand and to code, at least for the RKDG methods and for the application problems with simple functions in numerical integration. For the cases we have tested, this simpler approach seems to yield comparably good numerical results.

Here, we survey the new approach of high order well balanced finite element discontinuous Galerkin methods developed in [18], and present several applications not covered in [18]. In Section 2, we describe the algorithm to construct genuine high order well-balanced RKDG schemes for the shallow water equations. In Section 3, we give several examples in applications, and indicate how the algorithm discussed in Section 2 is implemented for each of them. Concluding remarks are given in Section 4.

2 A well balanced RKDG schemes for shallow water equations

We first present the well balanced schemes for the shallow water equation. The generalization of these schemes to other balance laws will be presented in Section 3. We will show that, for one-dimensional and two-dimensional shallow water equations, the traditional high order RKDG method is indeed a well balanced

scheme for still water, based on a suitable choice of the initial value or the flux. This choice will not affect the property of the scheme, such as high order accuracy in smooth region and non-oscillatory shock resolution, and it increases the computational cost only slightly.

For the shallow water equations (3), the purpose of well balanced schemes is to preserve the still water stationary solution (5) exactly. Recall in a traditional RKDG method, U is usually approximated by the piecewise polynomial U_h , which belongs to V_h , the space of piecewise polynomials of degree at most k . (We refer to [4] for more details of the traditional RKDG methods.) Here we also project the bottom function b into the same space V_h , to obtain an approximation b_h . This implies that $h_h + b_h = \text{constant}$ if $h + b = \text{constant}$. Because the first equation $(hu)_x = 0$ is satisfied exactly for any consistent scheme since $hu = 0$, here we only concentrate on the second equation, and denote it by

$$(hu)_t + f(U)_x = g(h, b)$$

where $U = (h, hu)^T$ with the superscript T denoting the transpose.

Following the idea first introduced by Audusse et al. [1], and later used in the recent paper by Noelle et al. [10], we write our numerical scheme in the following form:

$$\begin{aligned} \int_{I_j} \partial_t (hu)_h v_h dx - \int_{I_j} f(U_h) \partial_x v_h dx + \hat{f}_{j+\frac{1}{2}}^l v_h(x_{j+\frac{1}{2}}^-) - \hat{f}_{j-\frac{1}{2}}^r v_h(x_{j-\frac{1}{2}}^+) = \\ \int_{I_j} g(h_h, b_h) v_h dx. \end{aligned} \quad (6)$$

Comparing with the standard RKDG scheme, we can see that the single valued fluxes $\hat{f}_{j+\frac{1}{2}}$ and $\hat{f}_{j-\frac{1}{2}}$ have been replaced by the left flux $\hat{f}_{j+\frac{1}{2}}^l$ and the right flux $\hat{f}_{j-\frac{1}{2}}^r$, respectively. We can rewrite the above scheme in a new form, where the left side is the traditional RKDG scheme, and the right side is an approximation to the source term with some high order correction terms at the level of $O(\Delta x^{k+1})$. Therefore, the scheme (6) is a $(k+1)$ -th order conservative scheme and will converge to the weak solution.

In order to obtain the well balanced property, we need the residue in the scheme (6) to be zero if the still water stationary state (5) is reached. The following three conditions, *which only need to be valid for the still water stationary state*, are sufficient to guarantee this zero residue property.

- All the integrals should be calculated exactly for the still water. This can be easily achieved by using suitable Gauss-quadrature rules since h_h , b_h and v_h are polynomials in each cell I_j , hence f , g are both polynomials. Note that $(hu)_h = 0$ for the still water.
- We assume that

$$\hat{f}_{j+\frac{1}{2}}^l = f(U_h(x_{j+\frac{1}{2}}^-, t)), \quad \hat{f}_{j-\frac{1}{2}}^r = f(U_h(x_{j-\frac{1}{2}}^+, t)) \quad (7)$$

for the still water. Note that this condition is not obvious. Later we will comment on how to make it possible for the RKDG method.

- We assume that U_h , which is the numerical approximation of U , is a steady state solution of the equation $(hu)_t + f(U)_x = g(h, b_h)$, where b_h has substituted b , hence we have $\partial_x f(U_h) = g(h_h, b_h)$.

Proposition 1: RKDG schemes which satisfy the above three conditions for the shallow water equations are exact for the still water stationary state (5).

The proof of this result is rather straightforward and can be found in [18].

We also need to comment on the limiter procedure in the discontinuous Galerkin method. To prevent the limiter procedure from destroying the preservation of $h+b = \text{constant}$, we apply the limiter procedure on the function $(h_h + b_h, (hu)_h)^T$ instead. The modified RKDG solution is then defined by $h_h^{mod} \equiv (h+b)_h^{mod} - b_h$. We can easily observe that this procedure will not destroy the conservativity of h_h , which should be maintained during the limiter process. We refer to [17, 18] for more details.

Here the only thing remaining is to check that the above three conditions are satisfied for shallow water equations. It is easy to observe that the first and third conditions are true. As presented in [18], we have two choices for the second condition.

Choice A: We define the initial value and the approximation b_h by continuous piecewise polynomials by using the idea of essentially non-oscillatory (ENO) procedure [7]. Based on the values $u_{j+\frac{1}{2}}$, we can choose suitable stencils for each individual cell I_j by an ENO procedure, and then obtain a polynomial on I_j through an interpolation. If the steady state $h_h + b_h = \text{constant}$ is reached, because b_h is continuous, we will have a continuous h_h , which makes

$$\hat{f}_{j+\frac{1}{2}} = F(U_h(x_{j+\frac{1}{2}}^-, t), U_h(x_{j+\frac{1}{2}}^+, t)) = f(U_h(x_{j+\frac{1}{2}}^-, t)) = f(U_h(x_{j+\frac{1}{2}}^+, t)).$$

We can therefore simply define the left and right fluxes as the usual flux $\hat{f}_{j+\frac{1}{2}}$, and it will fulfill the second condition. This makes our scheme (6) to be identical to the traditional RKDG scheme without any modification.

Choice B: This idea follows the one of Audusse et al. [1]. After computing boundary values $U_{h,j+\frac{1}{2}}^\pm$, we set

$$h_{h,j+\frac{1}{2}}^{*,+} = \max\left(0, h_{h,j+\frac{1}{2}}^+ + b_{h,j+\frac{1}{2}}^+ - \max(b_{h,j+\frac{1}{2}}^+, b_{h,j+\frac{1}{2}}^-)\right) \quad (8)$$

$$h_{h,j+\frac{1}{2}}^{*,-} = \max\left(0, h_{h,j+\frac{1}{2}}^- + b_{h,j+\frac{1}{2}}^- - \max(b_{h,j+\frac{1}{2}}^+, b_{h,j+\frac{1}{2}}^-)\right) \quad (9)$$

and redefine the left and right values of $U_{h,j+\frac{1}{2}}^{*,\pm}$ by replacing $h_{h,j+\frac{1}{2}}^\pm$ with $h_{h,j+\frac{1}{2}}^{*,\pm}$. Then the left and right fluxes $\hat{f}_{j+\frac{1}{2}}^l$ and $\hat{f}_{j-\frac{1}{2}}^r$ are given by:

$$\hat{f}_{j+\frac{1}{2}}^l = F(U_{h,j+\frac{1}{2}}^{*,-}, U_{h,j+\frac{1}{2}}^{*,+}) + \left(\begin{array}{c} 0 \\ \frac{g}{2}(h_{h,j+\frac{1}{2}}^-)^2 - \frac{g}{2}(h_{h,j+\frac{1}{2}}^{*,-})^2 \end{array} \right) \quad (10)$$

$$\hat{f}_{j-\frac{1}{2}}^r = F(U_{h,j-\frac{1}{2}}^{*, -}, U_{h,j-\frac{1}{2}}^{*, +}) + \left(\begin{array}{c} 0 \\ \frac{g}{2}(h_{h,j-\frac{1}{2}}^+)^2 - \frac{g}{2}(h_{h,j-\frac{1}{2}}^{*,+})^2 \end{array} \right) \quad (11)$$

Here F is a monotone flux. It is easy to check that the second condition is satisfied. We again refer to [18] for details.

Remark 2: Choice A provides a simpler scheme with smaller computational cost, hence it would be preferred. Unfortunately, although it works well for small perturbation solutions from still water for a smooth bottom, the numerical resolution for a discontinuous bottom is not ideal. On the other hand, Choice B provides good numerical results for all the test cases we have experimented. \square

The extension of the well-balanced high order RKDG schemes to 2-D shallow water equations

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghb_x \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghb_y \end{cases} \quad (12)$$

with the still water stationary solution

$$h + b = \text{constant}, \quad hu = 0, \quad hv = 0, \quad (13)$$

is straightforward. This scheme can be applied on any triangulation.

The generalization of this algorithm to finite volume WENO schemes is simple. We refer to [18] for details.

3 Other applications

In this section, we generalize the high order well balanced RKDG schemes, designed in Section 2, to other balance laws. The applications are quite general and include the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow, a model of fluid mechanics, a two phase flow model and the Goldstein-Taylor model of the Boltzmann equation. Our well balanced high order finite difference and finite volume WENO schemes and discontinuous Galerkin methods presented in [15, 16, 17] can also be designed for these cases. We refer to [15, 16, 17] for details. Due to page limitation, only the nozzle flow, fluid mechanics model and Goldstein-Taylor model are investigated here, however our technique can also be applied to the other cases. We will not present many numerical experiment results here, and refer to [18] for such details.

3.1 A model in fluid mechanics with spherical symmetry

A classical singularity arising in fluid mechanics in case of spherical symmetry leads to the following model equation

$$u_t + \left(\frac{u^2}{2}\right)_x = \frac{1}{x}u^2, \quad (14)$$

which has been considered in [3]. Notice that here the source term is a nonlinear function of u . The steady state we are interested to preserve for this problem is given by

$$\frac{u}{x} = \text{constant}.$$

First, we project the initial value to obtain u_h , and also project x to obtain x_h . Since x is a polynomial, we know that x_h is the same as x . Then, we check the three conditions in Section 2 one by one.

1: The source term can be written as: $\frac{u}{x}u$. If the steady state is reached, $\frac{u_h}{x}$ is constant and u_h is a polynomial, hence the integral of the source term can be calculated exactly. Similarly, the integral of the flux can be calculated exactly.

2: We set

$$u_{h,j+\frac{1}{2}}^{*,+} = \frac{u_{h,j+\frac{1}{2}}^+}{x_{h,j+\frac{1}{2}}^+} \max(x_{h,j+\frac{1}{2}}^+, x_{h,j+\frac{1}{2}}^-) = u_{h,j+\frac{1}{2}}^+ \quad (15)$$

$$u_{h,j+\frac{1}{2}}^{*,-} = \frac{u_{h,j+\frac{1}{2}}^-}{x_{h,j+\frac{1}{2}}^-} \max(x_{h,j+\frac{1}{2}}^+, x_{h,j+\frac{1}{2}}^-) = u_{h,j+\frac{1}{2}}^-. \quad (16)$$

Then we define the left and right fluxes as:

$$\begin{aligned} \hat{f}_{j+\frac{1}{2}}^l &= F(u_{h,j+\frac{1}{2}}^{*,-}, u_{h,j+\frac{1}{2}}^{*,+}) + \frac{1}{2} \left(u_{h,j+\frac{1}{2}}^- \right)^2 - \frac{1}{2} \left(u_{h,j+\frac{1}{2}}^{*,-} \right)^2 = F(u_{h,j+\frac{1}{2}}^-, u_{h,j+\frac{1}{2}}^+) \\ \hat{f}_{j-\frac{1}{2}}^r &= F(u_{h,j-\frac{1}{2}}^{*,-}, u_{h,j-\frac{1}{2}}^{*,+}) + \frac{1}{2} \left(u_{h,j-\frac{1}{2}}^+ \right)^2 - \frac{1}{2} \left(u_{h,j-\frac{1}{2}}^{*,+} \right)^2 = F(u_{h,j-\frac{1}{2}}^-, u_{h,j-\frac{1}{2}}^+). \end{aligned}$$

3: We note that u_h , satisfying $\frac{u_h}{x}$, is the steady state solution of :

$$\left(\frac{u_h^2}{2} \right)_x = \frac{1}{x} u_h^2.$$

With these three conditions, we can repeat the proof of Proposition 1 to show that our new schemes are indeed well balanced and high order accurate. We can observe that the well balanced schemes for this problem are actually the traditional RKDG methods, without any modification.

Next, we present a numerical result to demonstrate the well balanced property. The initial and boundary conditions are given by

$$u(x, 0) = 0, \quad x \in [-5, 5] \quad (17)$$

$$u(x = -5, t) = 10, \quad u(x = 5, t) = -10. \quad (18)$$

The choice of this information allows us to compute the steady state, which is $u = -2x$. Numerical computations are performed by the well-balanced version of finite volume WENO schemes and RKDG methods. To see the benefit of well balanced schemes, we also compute with non well balanced finite volume WENO schemes and RKDG methods, and compare the results. We use 100 uniform cells here. The comparison of the convergence history, measured by the L^1 norm of the difference with the steady state, is given in Figure 1. The advantage of the well balanced schemes can be easily observed.

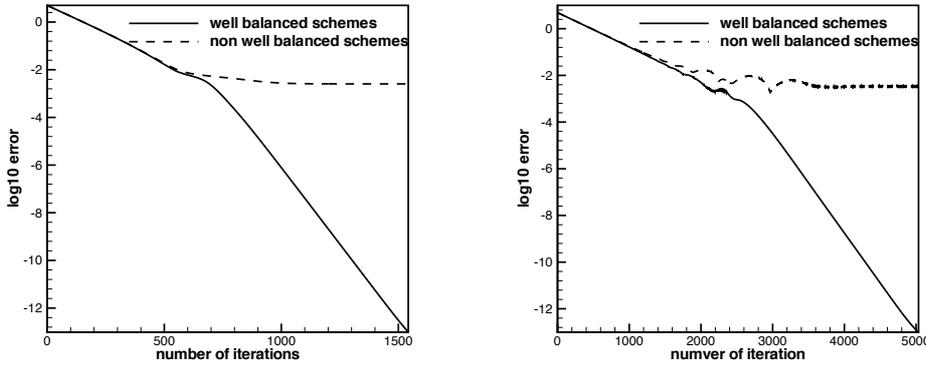


Figure 1: Comparison of the convergence history in L^1 error. Left: FV WENO schemes; right: RKDG schemes.

3.2 Nozzle flow

In this subsection we consider the balance laws for a quasi one-dimensional nozzle flow [5]. The governing equations for the quasi-one-dimensional unsteady flow through a duct of varying cross-section can be written in conservation form as:

$$\begin{cases} (\rho A)_t + (\rho u A)_x = 0 \\ (\rho u A)_t + ((\rho u^2 + p)A)_x = pA'(x) \\ (EA)_t + ((E + p)uA)_x = 0 \end{cases} \quad (19)$$

where the quantities ρ , u , p and $E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}$ represent the density, velocity, pressure and total energy, respectively. $A = A(x)$ denotes the area of the cross section. γ is the ratio of specific heats.

As in [5], we are interested in preserving the steady state solution

$$\rho(x, t) = \bar{\rho}(x), \quad p(x, t) = \bar{p}, \quad \text{and} \quad u(x, t) = 0 \quad (20)$$

where $\bar{\rho}(x)$ is an arbitrary function in x and \bar{p} is a constant.

First, we project the initial value to obtain $U_h = ((\rho A)_h, (\rho u A)_h, (EA)_h)^T$, and also project A to obtain A_h . Then, we check the three conditions in Section 2 one by one. Only the second equation in (19) is relevant for the well balanced property.

1: If the steady state is reached, p is constant and A_h is a polynomial, hence the integral of the source term can be calculated exactly. Similarly, the integral of the flux can be calculated exactly.

2: We set

$$(EA)_{h,j+\frac{1}{2}}^{*,+} = \frac{(EA)_{h,j+\frac{1}{2}}^+}{A_{h,j+\frac{1}{2}}^+} \max(A_{h,j+\frac{1}{2}}^+, A_{h,j+\frac{1}{2}}^-) \quad (21)$$

$$(EA)_{h,j+\frac{1}{2}}^{*,-} = \frac{(EA)_{h,j+\frac{1}{2}}^-}{A_{h,j+\frac{1}{2}}^-} \max(A_{h,j+\frac{1}{2}}^+, A_{h,j+\frac{1}{2}}^-), \quad (22)$$

and similarly, we can define $(\rho A)_{h,j+\frac{1}{2}}^{*,\pm}$. Then we redefine the left and right values of U as:

$$U_{h,j+\frac{1}{2}}^{*,\pm} = \begin{pmatrix} (\rho A)_{h,j+\frac{1}{2}}^{*,\pm} \\ (\rho u A)_{h,j+\frac{1}{2}}^{\pm} \\ (EA)_{h,j+\frac{1}{2}}^{*,\pm} \end{pmatrix} \quad (23)$$

At last, we define the left and right fluxes as:

$$\hat{f}_{j+\frac{1}{2}}^l = F(U_{h,j+\frac{1}{2}}^{*,-}, U_{h,j+\frac{1}{2}}^{*,+}) + \begin{pmatrix} 0 \\ (pA)_{h,j+\frac{1}{2}}^- - (pA)_{h,j+\frac{1}{2}}^{*,-} \\ 0 \end{pmatrix}. \quad (24)$$

$$\hat{f}_{j-\frac{1}{2}}^r = F(U_{h,j-\frac{1}{2}}^{*,-}, U_{h,j-\frac{1}{2}}^{*,+}) + \begin{pmatrix} 0 \\ (pA)_{h,j-\frac{1}{2}}^+ - (pA)_{h,j-\frac{1}{2}}^{*,+} \\ 0 \end{pmatrix} \quad (25)$$

where $(pA)_{h,j+\frac{1}{2}}^-$ and $(pA)_{h,j+\frac{1}{2}}^{*,\pm}$ are calculated by the relationship between pA and EA , ρA , $\rho u A$.

3: We note that U_h , satisfying $u_h \equiv 0$ and $p = \text{constant}$, is the steady state solution of :

$$((\rho u^2 + p)A_h)_x = pA'_h(x).$$

With these three conditions, we can repeat the proof of Proposition 1 to show that our new schemes are indeed well balanced and high order accurate.

The limiter procedure is performed similarly as in Section 2. We refer to [17, 18] for more details.

3.3 Goldstein-Taylor model of the Boltzmann equation

In the kinetic theory of rarefied gases classically described by the Boltzmann equation, the following two velocity model [6] is frequently used to describe the evolution of the density distribution of a fictitious gas made of two kinds of particles. Both of them move with equal speed parallel to the x-axis, either in the positive direction with a density u , or in the negative one with a density v . This model takes the form:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{\epsilon} \frac{\partial u}{\partial x} = \frac{1}{\epsilon^2} (v - u) \\ \frac{\partial v}{\partial t} - \frac{1}{\epsilon} \frac{\partial v}{\partial x} = \frac{1}{\epsilon^2} (u - v) \end{cases}. \quad (26)$$

The steady state we are interested to preserve for this problem is given by

$$u - v = \text{constant } c, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{1}{\epsilon} c.$$

First, we project the initial value to obtain u_h . Then, we check the three conditions in Section 2 one by one. Both equations in (26) need to be considered for the well balanced property.

1: If the steady state is reached, $u_h - v_h$ and ϵ are both constant, hence the integral of the source term can be calculated exactly. Similarly, the integral of the flux can be calculated exactly.

2: If the steady state is reached, both u and v are linear functions. Hence their approximations u_h and v_h in the space V_h are exactly the same as them. We know the fact $U_{h,j+\frac{1}{2}}^- = U_{h,j+\frac{1}{2}}^+$ and

$$F(U_{h,j+\frac{1}{2}}^-, U_{h,j+\frac{1}{2}}^+) = f(U_{h,j+\frac{1}{2}}^-) = f(U_{h,j+\frac{1}{2}}^+).$$

From these facts, we can define

$$\hat{f}_{j+\frac{1}{2}}^l = \hat{f}_{j+\frac{1}{2}}^r = F(U_{h,j+\frac{1}{2}}^-, U_{h,j+\frac{1}{2}}^+).$$

and prove that the second condition in Section 2 is satisfied.

3: We note that u_h and v_h are the steady state solution of :

$$\begin{aligned} \frac{1}{\epsilon} \frac{\partial u_h}{\partial x} &= \frac{1}{\epsilon^2} (v_h - u_h) \\ -\frac{1}{\epsilon} \frac{\partial v_h}{\partial x} &= \frac{1}{\epsilon^2} (u_h - v_h). \end{aligned}$$

With these three conditions, we can repeat the proof of Proposition 1 to show that our schemes are indeed well balanced and high order accurate.

4 Concluding remarks

Two different approaches of well balanced high order finite volume WENO schemes and finite element discontinuous Galerkin schemes can be designed for a class of hyperbolic systems, including the shallow water equations, the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow, a model in fluid mechanics, the Goldstein-Taylor model of the Boltzmann equation and a two phase flow model. The key idea of the first approach is a suitable decomposition of source terms [15, 16, 17], and the second approach relies on the exact calculation of the integral and special treatment of the flux term [18]. These schemes maintain properties of the exact preservation of the balance laws for certain steady state solutions, the non-oscillatory property for general solutions with discontinuities, and the genuine high order accuracy in smooth regions. The generalization of these techniques to treat more general PDEs and more general steady state solutions constitutes an ongoing work.

References

- [1] E. AUDUSSE, F. BOUCHUT, M.-O. BRISTEAU, R. KLEIN AND B. PERTHAME, *A fast and stable well-balanced scheme with hydrostatic*

reconstruction for shallow water flows, SIAM Journal on Scientific Computing, 25 (2004), pp.2050-2065.

- [2] A. BERMUDEZ AND M.E. VAZQUEZ, *Upwind methods for hyperbolic conservation laws with source terms*, Computers and Fluids, 23 (1994), pp.1049-1071.
- [3] R. BOTCHORISHVILI, B. PERTHAME AND A. VASSEUR, *equilibrium schemes for scalar conservation laws with stiff sources*, Mathematics of Computation, 72 (2003), pp.131-157. Also, an extended version containing more numerical examples is located at <http://www.inria.fr/rrrt/rr-3891.html>
- [4] B. COCKBURN, *An introduction to the discontinuous Galerkin method for convection-dominated problems*, in *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations*, B. Cockburn, C. Johnson, C.-W. Shu and E. Tadmor (Editor: A. Quarteroni), Lecture Notes in Mathematics, volume 1697, Springer, 1998, pp.325-432.
- [5] L1. GASCÓN AND J.M. CORBERÁN, *Construction of second-order TVD schemes for nonhomogeneous hyperbolic conservation laws*, Journal of Computational Physics, 172 (2001), pp.261-297.
- [6] L. GOSSE AND G. TOSCANI, *An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations*, Comptes Rendus Mathematique, 334 (2002), pp.337-342.
- [7] A. HARTEN, B. ENGQUIST, S. OSHER AND S. CHAKRAVATHY, *Uniformly high order accurate essentially non-oscillatory schemes, III*, Journal of Computational Physics, 71 (1987), pp.231-303.
- [8] A. KURGANOV AND D. LEVY, *Central-upwind schemes for the Saint-Venant system*, Mathematical Modelling and Numerical Analysis, 36 (2002), pp.397-425.
- [9] R.J. LEVEQUE, *Balancing source terms and flux gradients on high-resolution Godunov methods: the quasi-steady wave-propagation algorithm*, Journal of Computational Physics, 146 (1998), pp.346-365.
- [10] S. NOELLE, N. PANKRATZ, G. PUPPO AND J.R. NATVIG, *Well-balanced finite volume schemes of arbitrary order of accuracy for shallow water flows*, Journal of Computational Physics, to appear.
- [11] B. PERTHAME AND C. SIMEONI, *A kinetic scheme for the Saint-Venant system with a source term*, Calcolo, 38 (2001), pp.201-231.
- [12] T.C. REBOLLO, A.D. DELGADO AND E.D.F. NIETO, *A family of stable numerical solvers for the shallow water equations with source terms*, Computer Methods in Applied Mechanics and Engineering, 192 (2003), pp.203-225.

- [13] G. RUSSO, *Central schemes for balance laws*, Proceedings of the VIII International Conference on Nonlinear Hyperbolic Problems, Magdeburg, 2000.
- [14] S. VUKOVIC AND L. SOPTA, *ENO and WENO schemes with the exact conservation property for one-dimensional shallow water equations*, Journal of Computational Physics, 179 (2002), pp.593-621.
- [15] Y. XING AND C.-W. SHU, *High order finite difference WENO schemes with the exact conservation property for the shallow water equations*, Journal of Computational Physics, 208 (2005), pp.206-227.
- [16] Y. XING AND C.-W. SHU, *High order well-balanced finite difference WENO schemes for a class of hyperbolic systems with source terms*, Journal of Scientific Computing, to appear.
- [17] Y. XING AND C.-W. SHU, *High order well-balanced finite volume WENO schemes and discontinuous Galerkin methods for a class of hyperbolic systems with source terms*, Journal of Computational Physics, to appear.
- [18] Y. XING AND C.-W. SHU, *A new approach of high order well-balanced finite volume WENO schemes and discontinuous Galerkin methods for a class of hyperbolic systems with source terms*, Communications in Computational Physics, 1 (2006), pp.100-134.
- [19] J.G. ZHOU, D.M. CAUSON, C.G. MINGHAM AND D.M. INGRAM, *The surface gradient method for the treatment of source terms in the shallow-water equations*, Journal of Computational Physics, 168 (2001), pp.1-25.