RESEARCH STATEMENT

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My research interests are in the analysis of nonlinear partial differential equations (PDEs), especially conservation laws which change type from hyperbolic to elliptic. My main interest is studying the well-posedness of and dynamics of the solutions to multidimensional problems, especially those problems related to compressible gases or fluids which are modelled by the full Euler system and various simplified systems related to the full Euler system. While well-posedness for conservation laws in one space dimension is relatively well-understood, the question of well-posedness for conservation laws in more than one space dimension is still widely open. Since the full Euler system and related systems model physical reality and they have been thoroughly tested by numerical simulations and experiments, one might expect well-posedness of these systems with suitable physical data. Studying prototype cases with these systems will not only provide insights on the underlying physical and engineering problems but also on the general theory of well-posedness for conservation laws in more than one space dimension.

1. Sonic Shock Formation

In contrast to the "linear" phenomena of sound, light or electromagnetic signals which are governed by linear PDEs, the motion of compressible gas exhibits new nonlinear features and is governed by the full Euler system

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0 \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0 \\ (\rho E)_t + (\rho u H)_x + (\rho v H)_y = 0 \end{cases} \tag{1}$$

where

$$H = E + \frac{p}{\rho}; \quad E = \frac{1}{2}u^2 + e$$

For a polytropic gas

$$e = \frac{p}{(\gamma - 1)\rho}$$

where $\gamma > 1$ is a constant.

The nonlinearity of the system (1) gives rise to many novel phenomena. Among all these new phenomena, the most conspicuous one is the occurrence of shock discontinuity. Across shock the gas undergoes sudden changes in velocity, density and pressure. Even when the initial data are smooth, shock discontinuities will arise in many cases.

The study of shocks plays a central role in the theory of well-posedness for the systems of conservation laws in one space dimension. In contrast to one space dimensional cases, we encounter great difficulties in analysing the shocks interaction in more than one space dimension. One noticeable breakthrough in studying the multidimensional shocks interaction is the technique introduced by Barbara Keyfitz and her co-workers for analysis of self-similar 2-D conservation laws. Since their pioneering work, a wide range of works have been carried out to various simplified system related to the full Euler system for different prototype problems. The essence of this technique involves analysis of transonic shocks. The transonic shock is different from the traditional 1-D shock in the sense that for the transonic shock, the underlying PDE system changes from hyperbolic (corresponds to the case where the underlying flow is supersonic) to elliptic (corresponds to the case where the underlying flow is subsonic). Because of this difference between the transonic shock and the traditional 1-D shock, a shock does not usually form as transonic shock at its initiation point. The only possible way a shock can be transonic shock at the moment of its formation is forming on the sonic line (the boundary of the hyperbolic region). We call this special shock a sonic shock. However, the existence of sonic shocks is controversial, because it is very difficult for them to be detected by experiments and numerical simulations. [1]

Recent papers of Hunter and Tesdall [2] and [4] consider a problem where a weak plane shock hits a semi-finite screen (shock-screen problem). Their numerical results show with reasonable accuracy that a sonic shock forms in their problem. In their papers, Hunter and Tesdall further analyze the behavior of the sonic shock and show that the mechanism of sonic shock formation is different from the mechanism of traditional 1-D shock formation.

To confirm the existence of this sonic shock, I set to prove the existence of solution rigorously in my doctoral dissertation [5]. Because my interest is the formation of the sonic shock rather than the global structure of the shock-screen problem, instead of the full Euler system, I use the asymptotic model used in Hunter and Tesdall's papers – unsteady transonic small disturbance (UTSD) equation

$$u_t + \left(\frac{1}{2}u^2\right)_x + v_y = 0 \qquad (2)$$
$$v_x - u_y = 0.$$

and study the existence of solution locally near the shock formation point.

Because of the self-similarity of this problem, the solution to this problem depends only on $\xi = \frac{x}{t}$ and $\eta = \frac{y}{t}$. After assigning appropriate far field data in the $\xi - \eta$ plane and introducing new independent variables $\rho = \xi + \eta^2/4$ and $\tau = \eta$, the problem can be reduced to solving the following free boundary problem in the subsonic region Ω (see Figure 1) using the procedure proposed in Keyfitz et al's framework

$$((u-\rho)u_\rho + \frac{u}{2})_\rho + u_{\tau\tau} = 0 \qquad in\ \Omega, \qquad (3)$$

$$(2\rho - \frac{7}{4}u)\rho' u_\rho + (2\rho - \frac{5}{4}u)u_\tau = 0 \qquad on\ \Sigma, \qquad (4)$$

$$u = \varphi \qquad on\ \sigma, \qquad (5)$$

$$u = 0 \qquad on\ \sigma_0. \qquad (6)$$

where $\Sigma = \{(\rho,\tau)|\rho = \rho(\tau)\}$ is the shock, and

$$\rho'(\tau) = \sqrt{\rho(\tau) - \frac{1}{2}u(\rho(\tau),\tau)}, \qquad \rho(0) = 0. \qquad (7)$$

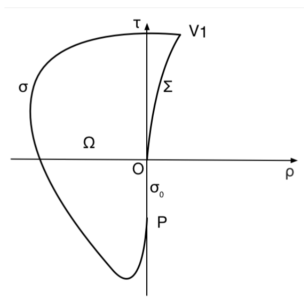

FIGURE 1. A sketch of subsonic region. Σ is the shock and σ_0 is the sonic line.

The above free boundary problem contains several noticeable features and difficulties:

1. v in the UTSD equation does not appear in the free boundary problem. The correct u will be obtained by solving this free boundary problem, and v will be recovered from the original UTSD equation afterward.

[1] Shock and sonic line are usually 1-D curves on 2-D plane and the locations of them depend on the actual solution of the PDE system. The requirement of the sonic shock forming exactly on the sonic line is difficult to be verified, because it is very sensitive to any experimental errors or numerical errors. In addition to that, the viscosity effect in experiments affects the result greatly.

2. The position of the shock Σ is unknown a priori, therefore is a free boundary. The Rankine-Hugoniot conditions for the shock are replaced by the oblique derivative condition (4) on the shock Σ and the shock evolution equation (7). We use these conditions instead of the Rankine-Hugoniot conditions so that we can set iteration scheme to determine the position of the shock Σ.
3. The quasilinear elliptic PDE for u (3) is degenerate on the sonic line σ_0.
4. The shock Σ forms at O on the sonic line σ_0. At this particular point O, both the ellipticity of the PDE (3) and the obliqueness of the oblique derivative condition (4) are lost simultaneously.

We prove the existence of solution to the free boundary problem

THEOREM 1.1. *There exist a* $u \in C^{2,\alpha}(\overline{\Omega}/\overline{\sigma_0 \cup \Sigma}) \cap C^{1,\alpha}(\overline{\Omega}/\{\overline{\sigma}_0, V_1\}) \cap C^{\gamma}(\overline{\Omega}/\overline{\sigma}_0) \cap C(\overline{\Omega})$ *for some* $0 < \alpha < 1$ *and* $0 < \gamma < 1$ *and a* $\rho(\tau) \in \mathcal{K}$ *such that* u *satisfies* (3)– (4), $\Sigma = \{(\rho, \tau) | \rho = \rho(\tau)\}$ *and* $\rho(\tau)$ *satisfies the following modified shock evolution equation*

$$\rho'(\tau) = \sqrt{\rho(\tau) - \frac{1}{2} g(u(\rho(\tau), \tau))}, \qquad \rho(0) = 0. \tag{8}$$

The modified shock evolution equation is obtained from the shock evolution equation (7) by adding the cut-off function g. The cut-off function g here is added to ensure the term under the square root is positive and it cannot be removed due to some technical difficulty.

The proof of Theorem 1.1 consists of the following four steps:

Step 1. Fix an approximate position for the free boundary Σ, given by $\rho = \rho(\tau)$. Here, ρ belongs to a closed, convex subset $\mathcal{K}$ of the Banach space $H_{1+\alpha_\Sigma}$. The PDE (3) and the oblique derivative condition (4) are regularized to ensure the ellipticity and obliqueness.

Step 2. Solve the regularized fixed boundary problem with the chosen position of Σ from step 1. This step involves solving a quasilinear PDE through linearization and an application of the Leray-Schauder fixed point theorem (Theorem 11.6 in [1]). It relies on the extensive theory for mixed boundary value problems in Lipschitz domains which was developed by Lieberman.

Step 3. Construct appropriate supersolutions and subsolutions to remove cut-off functions introduced in the regularization procedure in step 1 and ensure the ellipticity of the PDE away from the sonic line σ_0 and the obliqueness of the oblique derivative condition away from O. Obtain uniform estimate independent of the small parameters introduced in the regularization procedure and therefore obtain a subsequence converges to solution of unmodified fixed boundary problem as the small parameters go to zero.

Step 4. Map ρ to $\tilde{\rho} = J\rho$ by the modified shock evolution equation (8). We show that J maps $\mathcal{K}$ back to $\mathcal{K}$ and $J\mathcal{K}$ is precompact. Therefore, from the fixed point Theorem we conclude that there is a fixed point ρ such that $J\rho = \rho$.

In completing the proof of this result, some details of the proof lead me to think about the following further problems:

1. As indicated in Hunter and Tesdall's papers, the mechanism of the shock formation should be different from the mechanism in one space dimension. As a new type of shock in two space dimension, a further investigation on the mechanism of sonic shock formation is worthwhile.
2. In a separate attempt to establish similar result for the steady flow, certain structures are found to not exist. While my result indicates that the sonic shock can exist for unsteady flow, it still worths the effort to answer the controversial question of whether sonic shock can form for the steady flow.
3. In the process of proving the existence result, I found that some estimates needed rely on special choice of the far field data. An extension of this result to a global existence result will remove the artificial far field data posed in this problem and make this result more complete.

2. The Euler System

Although progress has been made using different simplified models of the full Euler system to analyze the behavior of gas flows in different situations, a lot of features of real gas flows are not captured because of the simplification. To better understand the dynamics for the gas flow, we have started a project to analyze 2-D shock interactions with the Euler system. Jointly with Katarina Jegdić, Barbara Keyfitz and Sunčica Čanić, we recently proved a local existence result near the reflection point for isentropic gas dynamics equations (essentially the full Euler system (1) with additional assumption p being a function of ρ) in [3]. Unlike the UTSD equation described in Section 1 or any other simplified models people have worked with, the isentropic gas dynamics equations can not be reduced to a second-order PDE for a single function. Instead, it is written as a second-order PDE for density ρ with coefficients depending not only on ρ itself but also on the pseudo-velocity (U, V),

$$\begin{aligned}
&(c^2-U^2)\rho_{\xi\xi} - 2UV\rho_{\xi\eta} + (c^2-V^2)\rho_{\eta\eta} + 2cc'(\rho_\xi^2+\rho_\eta^2) \\
&- 2\rho_\xi(U(1+U_\xi+V_\eta) - c^2\frac{U(V_\eta+1)-VV_\xi}{U^2+V^2}) \\
&- 2\rho_\eta(V(1+U_\xi+V_\eta) - c^2\frac{UU_\eta-V(U_\xi+1)}{U^2+V^2}) = 0,
\end{aligned} \tag{9}$$

where $c^2 = p'(\rho)$, and transport equations for U and V with dependence on ρ,

$$\begin{cases} (U,V)\cdot\nabla U + U + \frac{p_\xi}{\rho} = 0 \\ (U,V)\cdot\nabla V + V + \frac{p_\eta}{\rho} = 0. \end{cases} \tag{10}$$

To handle the extra difficulty due to the fact that the system can not be decoupled, we extend the method described in Section 1 to the following procedure:

Step 1. We fix approximate velocity and shock position and we solve the nonlinear problem for density using a compactness argument.
Step 2. Using the density found in the previous step, we solve the system of transport equations for velocity.
Step 3. In a small neighborhood of the reflection point, we formulate this as a mapping on the space of velocities, and show it is a contraction mapping, therefore a fixed point exists. Thus the problem is solved with the fixed shock position.
Step 4. From the solution of the fixed boundary problem, we solve free boundary problem using another compactness argument as discussed in Section 1.

Right now, with Jegdić and Keyfitz, we seek to extend our result in [3] to the full Euler system (1) and also plan later to solve the global problem. The full Euler system is similar to the isentropic gas dynamics equations in our paper, the only difference is that for the full Euler system, we do not assume the pressure is a function of the density. Instead of a PDE for the density, we obtain a second-order PDE for pressure p with coefficients depending not only on p itself but also on the density ρ and pseudo-velocity (U, V), and transport equations for ρ, U and V with dependence on p. We believe the same procedure will solve the local problem with the full Euler system and we are working on the details.

To extend the local result to a global result, we will encounter a few new difficulties. Compared to those simplified systems solved previously, one novel feature of the full Euler system is that the PDE for the pressure and the system of transport equations are degenerate at the stagnation point (a point where $(U, V) = (0, 0)$). A careful examination of the pseudo-velocity on the boundary of the subsonic region shows that though the stagnation point can be avoided in a small neighborhood of the reflection point, it is inevitable in solving the global problem. To gain some understanding of the behavior of the solution around a stagnation point, I studied the stagnation point of the full Euler system with additional axisymmetry assumption in [6]. Surprisingly, I found that any axisymmetric and self-similar smooth solution to the full Euler system with a stagnation point at the origin will either have a vacuum ($p = \rho = 0$) at the origin or have zero tangential velocity in a whole region where the solution is smooth. The asymptotic behavior of a possible solution with a vacuum at the origin was obtained in [6] as well. However, we expect the non-axisymmetric case will have structure different from the axisymmetric case and we are currently studying this problem.

3. General Self-Similar Data

Consider a general 2-D system of conservation laws

$$\boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U})_x + \boldsymbol{G}(\boldsymbol{U})_y = 0 \tag{11}$$

with initial data

$$\boldsymbol{U}(0, x, y) = \boldsymbol{U}_0(\theta) \tag{12}$$

where θ is defined by $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$. One expects (11) with the initial data (12) has a self-similar solution,

$$\boldsymbol{U}(t, x, y) = \boldsymbol{U}(\xi, \eta) = \boldsymbol{U}(\xi, \eta).$$

Because sectorially constant data for this initial value problem is a natural generalization of a 1-D Riemann problem in two space dimension, people usually study (11) with initial data (12) where $\boldsymbol{U}_0(\theta)$ is piecewise constant. However, even with only four different constants prescribed in four quadrants, it gives rise to 21 cases for the isentropic Euler system (please refer to Zheng's book [8] for more details). For most of these 21 cases, shocks appear and are not confined to a compact set in the $\xi - \eta$ plane. Notice that if we take $\boldsymbol{U}_0(\theta)$ to be smooth, the initial data $\boldsymbol{U}(0, x, y)$ is still not continuous at the origin. Motivated by this observation, I studied a general scalar conservation law in two space dimension,

$$u + f(u)_x + g(u)_y = 0 \tag{13}$$

with initial data

$$u(0, x, y) = u_0(\theta) \tag{14}$$

where u_0 is taken to be C^2. Because of the symmetry in this problem, the solution to this problem ought to be self-similar, i.e.

$$u(t, x, y) = u(\xi, \eta) = u(\frac{x}{t}, \frac{y}{t}). \tag{15}$$

In [7], we proved that in the $\xi - \eta$ plane, the solution to (13) with initial condition (14) is C^1 outside a circle of radius R with center at the origin. This result shows that in contrast to the solution of (13) with sectorially constant initial data, the solution to our problem has shocks only in a compact set.

To understand this new phenomenon related to this "new Riemann problem" in two space dimension, further research to generalize this result to systems of conservation laws is needed. However, the lack of any existence theory for weak solutions of multidimensional systems and the absence of a priori bounds for weak solutions remain to be obstacles to tackle this problem.

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