1. Introduction

1.1. Flatness. Let \( k \) be a field. We will be dealing throughout this course with properties of flatness. So we will need some facts about flatness which we list at the outset.

Definition 1.1. If \( B \) is a (commutative) \( k \)-algebra, a \( B \)-module \( M \) is flat if \( M \otimes_B \) is an exact functor in the category of \( B \)-modules.

Property 1: (Transitivity) If \( A \to B \) is a flat morphism of \( k \)-algebras and \( M \) is a flat \( B \)-module, then \( M \) becomes a flat \( A \)-module with respect to the \( A \)-module structure induced by the morphism.

Property 2: (Base extension of rings) If \( A \to B \) is a morphism of \( k \)-algebras and \( M \) is a flat \( A \)-module, then \( M \otimes_A B \) is a flat \( B \)-module.

Property 3: (Localization) \( M \) is flat over \( B \) if and only if, for every prime ideal \( p \leq B \), the localization \( M_p \) is flat over \( A_p \).

Property 4: (Ideal criterion) Suppose that, for a \( B \)-module \( M \) and every ideal \( b \leq B \), the map \( b \otimes_B M \to M \) is injective. Then \( M \) is flat over \( B \).

Property 5: (Freeness over local noetherian rings) A finitely generated module \( M \) over a local noetherian ring \( B \) is flat if and only if it is free.

Property 6: (Exact sequence criteria) For the exact sequence

\[
0 \to M' \to M \to M'' \to 0
\]

of \( B \)-modules, \( M' \) is free if the other two are, \( M \) is free if the other two are.

1.2. Snake Lemma. We will also use the Snake Lemma throughout:

Lemma 1.1. Let

\[
\begin{array}{c}
0 \to M' \to M \to M'' \to 0 \\
\downarrow \alpha' & \downarrow \alpha & \downarrow \alpha'' \\
N' \to N \to N'' \to 0
\end{array}
\]

be commutative with exact rows. Then there is a natural exact sequence

\[
0 \to \ker \alpha' \to \ker \alpha \to \ker \alpha'' \to \ker \alpha' \to \ker \alpha \to \ker \alpha'' \to 0.
\]

Date: January 4, 2007.
1.3. Schlessinger’s criteria. Let $\mathcal{A}$ denote the category of Artinian $k$-algebras. The “deformation functors”

$$ F : \mathcal{A} \to \text{Sets} $$

we define in these notes will all satisfy the four criteria for a “good” deformation functor four properties for the map

$$ F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''), $$

first codified by M. Schlessinger:

**H1:** (1) is a surjection whenever $A'' \to A$ is a small extension (see below).

**H2:** (1) is a bijection if $A'' = k[\epsilon]/(\epsilon^2)$ and $A = k$.

**H3:** Given $F(k)$, the set

$$ t_{F(k)} = \left\{ F \left( \frac{k[\epsilon]}{(\epsilon^2)} \right) \to F(k) \right\} $$

has the structure of a $k$-vector space by $H1 - H2$. This $k$-vector space is finite dimensional.

**H4:** Using $H2$ to define an action of $t_{F(k)}$ on small extensions

$$ 0 \to J \to A' \to A \to 0, $$

the set of extensions $F(A')$ of $F(A)$ is a principal homogeneous space for $t_{F(k)}$.

We usually leave as exercises the verification that $H1 - H4$ are satisfied in the cases we consider.

These notes owe a considerable debt to János Kollár and Marco Mannetti, from whose lectures on the algebraic theory and differential graded Lie algebras respectively we have borrowed extensively. Errors, however, are the responsibility of the first author of these notes.

### Part 1. Algebraic theory

2. Flat deformations of $k$-algebras over Artinian rings

2.1. Small extensions. Next let $\mathcal{A}$ denote the category of Artinian $k$-algebras. Let

$$ B \to A $$

be an epimorphism in $\mathcal{A}$ with kernel $J$ such that, for the maximal ideal $m$ of $B$ we have

$$ m \cdot J = 0. $$

We call $B$ a small extension of $A$ and write the exact sequence

$$ 0 \to J \to B \to A \to 0 $$

We will use the above exact sequence notation exclusively for small extensions in this section.
Let $S_B$ be a (commutative) $B$-algebra which is flat over $B$. Suppose $S = \frac{S_B}{m_S B}$ and suppose that

$$I \leq S$$

is an ideal in $S$ which has an extension $I_A \leq S_A$ such that the quotient

$$Q_A = \frac{S_A}{I_A}$$

is flat over $A$. We want to study the obstructions to extending $I_A$ to $I_B/B$ so that

$$Q_B = \frac{S_B}{I_B}$$

is flat over $B$.

We begin with a presentation

$$R_A \xrightarrow{(r_A^i)} G_A \xrightarrow{(g_A^i)} S_A \rightarrow Q_A \rightarrow 0$$

of the ideal $I_A$ by free $S_A$-modules. Maps are given by the matrices of elements of $S_A$ indicated. We extend these matrix entries arbitrarily to elements of $S_B$. If we denote the image of $g_B$ as $I_B$ and set $Q_B = \frac{S_B}{I_B}$, we achieve the diagram in which rows and columns are exact except the middle row is not necessarily exact at $G_B$.

$$\begin{array}{cccccc}
R \otimes J & \rightarrow & G \otimes J & \rightarrow & S \otimes J & \rightarrow & Q \otimes J & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
R_B & \xrightarrow{r_B} & G_B & \xrightarrow{g_B} & S_B & \rightarrow & Q_B & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
R_A & \xrightarrow{r_A} & G_A & \xrightarrow{g_A} & S_A & \rightarrow & Q_A & \rightarrow & 0 \\
\end{array}$$

(2)

Tensoring the bottom row by

$$k = \frac{B}{m}$$

yields the exact sequence

$$R \xrightarrow{r_0} G \xrightarrow{g_0} S \rightarrow Q \rightarrow 0$$

Now in (2) the image of the composition

$$R_B \xrightarrow{g_B g_B} I_B$$

goes to 0 in $I_A$ and so lifts to a map

$$R_B \rightarrow S \otimes J \rightarrow Q \otimes J$$

which drops to a map

$$f : R \rightarrow Q \otimes J.$$  

**Lemma 2.1.** $Q_B$ in (2) is flat over $B$ if and only if

$$(S \otimes J) \cap I_B = I \otimes J.$$  

**Proof.** If $Q_B/B$ is flat, then $Q \otimes J = \ker(Q_B \rightarrow Q_A)$. But the Snake Lemma applied to

$$\begin{array}{cccccc}
0 & \rightarrow & I_B & \rightarrow & S_B & \rightarrow & Q_B & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \sigma & \\
0 & \rightarrow & I_A & \rightarrow & S_A & \rightarrow & Q_A & \rightarrow & 0 \\
\end{array}$$

gives

$$0 \rightarrow (S \otimes J) \cap I_B \rightarrow S \otimes J \rightarrow \ker(\sigma) \rightarrow 0$$
exact. So \((S \otimes J) \cap I_B = I \otimes J\). Conversely, if this last equality holds, then \(\ker(\sigma) = Q \otimes J\). So, for any ideal \(b \leq B\), apply \(Q_B \boxtimes B\) to the diagram

\[
\begin{array}{cccccc}
0 & \to & b \cap J & \to & b & \to & a & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & J & \to & B & \to & A & \to & 0
\end{array}
\]

to get

\[
\begin{array}{cccccc}
Q \otimes (b \cap J) & \to & Q_B \boxtimes B b & \to & Q_A \boxtimes A a & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & Q \otimes J & \to & Q_B & \to & Q_A & \to & 0
\end{array}
\]

But then the composition \(Q \otimes (b \cap J) \to Q \otimes J \to Q_B\) is injective, so \(Q \otimes (b \cap J) \to Q_B \boxtimes B b\) is injective. So by the Snake Lemma \(Q_B \boxtimes B b \to Q_B\) is injective. So, by the ideal criterion for flatness, \(Q_B / B\) is flat. \(\square\)

2.2. Associated extension class. From \(f\) we construct an extension of \(S\)-modules

\[
0 \to Q \otimes J \to \frac{Q \otimes J + G}{\{ (f(r), r_0(r)) : r \in R \}} \to I \to 0
\]

and so determine an extension class

\[E \in \text{Ext}^1_S(I, Q \otimes J).\]

Lemma 2.2. \(E\) is independent of the choice of liftings \(r_B\) of \(r_A\) and \(g_B\) of \(g_A\).

Proof. If we change the lifting \(r_B\) the change is in \(R \otimes J\) and so goes to 0 by exactness of the top row of (2). If we change \(g_B\) to \(\tilde{g}_B\),

\[\tilde{g}_i - g_i = h_i \in S \otimes J\]

so that

\[
\sum_i r_{ij}^B \cdot \tilde{g}_i - \sum_i r_{ij}^B \cdot g_i = \sum_i r_{ij}^B \cdot h_i = \sum_i r_{ij}^0 \cdot h_i.
\]

So, letting \(\{e_i\}\) denote a standard basis for \(G\), we get a map

\[
\varphi : G \to Q \otimes J \\
\{e_i\} \mapsto h_i
\]

which composes with \(r_0\) to send the \(j\)-th generator of \(R\) to the class \(\sum_i r_{ij}^0 \cdot h_i\) in \(Q \otimes J\). \(\varphi\) can be used to give a commutative diagram

\[
\begin{array}{ccc}
Q \otimes J + G & \to & I \\
\downarrow & & \downarrow \{(id. + \varphi, id.)\} \\
\frac{Q \otimes J + G}{\{ (f(r), r_0(r)) : r \in R \}} & \to & \frac{Q \otimes J + G}{\{ (f(r), r_0(r)) : r \in R \}}
\end{array}
\]

showing \(E = \tilde{E}\). \(\square\)

Lemma 2.3. In (2) \(E = 0\) if and only if, for suitable choice of extension \((g_i^B)\) of \((g_i^A)\),

\[
(S \otimes J) \cap I_B = I \otimes J.
\]
Proof. In one direction, if (6) holds, then, since the image of $g_B \circ r_B$ lies in $(S \otimes J) \cap I_B$, it must lie in $I \otimes J$ and so go to 0 in $Q \otimes J$. So, referring to (2), the map $f$ in (3) is the zero map. So $E = 0$. Conversely, if $E = 0$, let $\varphi$ denote the composition

$$G_B \to G \to \frac{Q \otimes J + G}{\{(f(r), r_0(r)) : r \in R\}}$$

and let $\tilde{\varphi}$ denote a lifting of $\varphi$ to $G_B \to S \otimes J$.

Notice that, by construction, the diagram

$$\begin{array}{ccc}
R & \xrightarrow{g_B \circ r_B} & G \\
\downarrow & & \downarrow \varphi \\
Q \otimes J & & Q \otimes J
\end{array}$$

is commutative. So, referring to (2), if replace $g_B$ by $\bar{g}_B = g_B - \tilde{\varphi}$ we will have $f = 0$ so that

$$\text{image } (\bar{g}_B \circ r_B) \subseteq I \otimes J.$$ 

But now, if $(\bar{g}_i^B) \mapsto 0 \in I_A$, $(\bar{g}_i^A) \in \text{image } (r_A)$. So there is an element $x \in R_B$ such that

$$((\bar{g}_i^B) - r_B(x)) = (h_i) \in G \otimes J.$$ 

But by freeness this means that

$$h_i \in S \otimes J$$

for each $i$. Thus each element in $\sum_i \bar{g}_i^B \in ((S \otimes J) \cap I_B)$ can be written an element of $\text{image } (\bar{g}_B \circ r_B)$ and an element $\sum_i h_i \cdot g_i^0 \in (I \otimes J)$. So

$$((S \otimes J) \cap I_B) \subseteq I \otimes J.$$ 

Since the opposite inclusion is always true, the proof is complete. \qed

2.3. The space of all flat extensions. Next suppose $E = 0$ so that there exists at least one flat extension, that is, at least one extension $I_B \leq S_B$ such that $Q_B = \frac{S_B}{I_B}$ is flat over $B$. Compare with a presentation

$$G_B \xrightarrow{(g_i^0)} S_B \to Q_B \to 0$$

of $\bar{I}_B$. Now

$$\text{image } (g_B \circ r_B) \subseteq I \otimes J$$

by Lemma 2.1 since $Q_B$ is flat. Also we have that

$$((\bar{g}_i^B - g_i^B) \subseteq \ker (Q_B \to Q_A) = Q \otimes J.$$ 

Define

$$\varphi : G \to Q \otimes J$$

by sending the $i$-th generator of $G$ to $(\bar{g}_i^B - g_i^B)$. We know that $\bar{Q}_B$ is flat over $B$ if and only if

$$\text{image } (\bar{g}_B \circ r_B) \subseteq I \otimes J,$$

that is, if and only if

$$\text{image } ((\bar{g}_B \circ r_B) - (g_B \circ r_B)) \subseteq I \otimes J,$$
that is, if and only if
\[ \varphi|_R = 0. \]
Conversely, given any
\[ \varphi : G \to Q \otimes J \]
such that
\[ \varphi|_R = 0, \]
we can let
\[ \tilde{Q}_B = \frac{S_B}{\text{image } (\bar{g}_B)} \]
for
\[ \bar{g}_B = g_B^0 + \tilde{\varphi}(g_i^0) \]
where \( \tilde{\varphi} \) is any lifting of \( \varphi \) to a map to \( S \otimes J \). (Which lifting one picks will not affect \( \tilde{I}_B \).) We conclude:

**Lemma 2.4.** If \( Q_B \) is a flat extension of \( Q_A \), the set of all flat extensions, that is, other extensions \( \tilde{I}_B \) such that \( \tilde{Q}_B \) is flat, is a principal homogeneous space for the group \( \text{Hom}_S(I, Q \otimes J) \).

### 2.4. Independence of the presentation of \( I_A \)

Finally we had better check that nothing done up to this point in this chapter depends on the choice of the presentation

\[ R_A \xrightarrow{r_A} G_A \xrightarrow{g_A} S_A \to Q_A \to 0 \]
of \( I_A \). To see this, suppose we had a second presentation

\[ \tilde{R}_A \xrightarrow{\tilde{r}_A} \tilde{G}_A \xrightarrow{\tilde{g}_A} S_A \to Q_A \to 0 \]

Then since \( G_A \) and \( R_A \) are free, you can map the first resolution to the second one then extend the maps to (2) to get that \( f \) factors through \( \tilde{f} \). Alternatively form

\[ R_A \xrightarrow{r_A} G_A \xrightarrow{g_A} S_A \xrightarrow{g_A + \tilde{g}_A} S_A \to Q_A \to 0 \]

Since \( R_A \) and \( \tilde{G}_A \) are free, there exists morphisms

\[ R_A \to \tilde{R}_A \]

Extending each of these three resolutions of \( I_A \) to diagrams (2) the vertical arrows give then a commutative diagram

\[ R \xrightarrow{f} Q \otimes J. \]

In any case \( E = \tilde{E} = \tilde{E} \).
2.5. **Maximal small extension.** Suppose we have a diagram

\[
\begin{array}{cccccc}
0 & \to & J & \to & B & \to & A & \to & 0 \\
\downarrow & & \downarrow & & \uparrow & & \uparrow & & \downarrow \\
0 & \to & J' & \to & B' & \to & A & \to & 0
\end{array}
\]

So again we construct a morphism from (2) for the top row to (2) for the bottom row, and from that obtain the commutative diagram

\[
\begin{array}{cccccc}
R & \to & Q \otimes J & \to & I \\
\uparrow & & \downarrow & & \downarrow & & \uparrow \\
R & \to & Q \otimes J' & \to & I
\end{array}
\]

and so obtain \(E'\) as the image of \(E\) under the natural map

\[
(7) \quad Ext^1_S(I, Q \otimes J) \to Ext^1_S(I, Q \otimes J').
\]

Using this construction for all rank-1 quotients of \(J\) we obtain a natural bilinear map

\[
(8) \quad Ext^1_S(I, Q \otimes J) \otimes \mathcal{J} \to Ext^1_S(I, Q).
\]

**Proposition 2.5.** There exists a unique maximal quotient \(J'\) of \(J\) such that \(E \mapsto E = 0\). If we call this quotient \(J_{\text{max}}\), then \(J_{\text{max}}\) is the dual \(k\)-vector space to the kernel of the morphism

\[
J' \to Ext^1_S(I, Q)
\]

given by the element

\[
E \in Ext^1_S(I, Q \otimes J) = (Ext^1_S(I, Q)) \otimes J = \text{Hom} (J', Ext^1_S(I, Q)).
\]

In particular

\[
\dim_k J_{\text{max}} \geq \dim_k J - \dim_k Ext^1_S(I, Q).
\]

**Proof.** Let \(J'\) and \(J''\) be quotients of \(J\) for which the map (7) is the zero map, and let \(J^+\) be the span of \(J'\) and \(J''\). For the injection

\[
(J^+)' \to (J')' \oplus (J'')'
\]

select a basis \((h^j)\) for \((J^+)'\) taken and use the commutativity of the diagram

\[
\begin{array}{cccccc}
Ext^1_S(I, Q \otimes J) & \to & Ext^1_S(I, Q \otimes J') \\
\downarrow & & \downarrow & & \downarrow \\
Ext^1_S(I, Q) & \to & Ext^1_S(I, Q \otimes J') & \overset{k^j}{\to} & Ext^1_S(I, Q \otimes J^+)
\end{array}
\]

and the analogous one for \(J''\). \(\square\)

**Definition 2.1.** We denote the subspace of \(Ext^1_S(I, Q)\) generated by the images of all map

\[
J' \to Ext^1_S(I, Q)
\]

over all small extensions as

\[
\text{Obst}_S(I, Q).
\]
3. The local ring of the Hilbert scheme

3.1. Local to global. Let $Y_W/W$ be a flat family of projective schemes of finite type over a field $k$. Let

$$Z \subseteq Y$$

be a closed subscheme with sheaf of ideals

$$\mathcal{I} = \mathcal{I}_Z \subseteq \mathcal{O}_Y.$$  

**Fact:** There exists a projective scheme

$$\text{Hilb}(Y_W/W) \subseteq \mathbb{P} \times W$$

parametrizing all subschemes of fibers of $Y_W/W$ with the same Hilbert polynomial as $Z$.

Let $Y_B/B$ be a flat family of schemes over $B$, that is, over $\text{Spec}(B)$, that is $\mathcal{O}_{x,Y_B}$ for each $x \in Y_B$. For

$$0 \to J \to B \to A \to 0$$

as in the last chapter, we wish to study flat extensions of $Z_A$ to $Z_B \subseteq Y_B$. To do this we begin by choosing a finite, flat affine cover $\{U^\alpha_B\}$ of $Y_B$, that is, the $U^\alpha_B$ are affine varieties which are flat over $B$ with

$$\bigcup_{\alpha} U^\alpha_B = Y_B.$$

Let

$$S^0_B = k [U^0_B]$$

$$I^0_A = I_{Z \cap U^0_B} \leq S^0_B.$$  

So flat extensions of each $I^0_A$ are covered by the last chapter. They exist if and only if each

$$E^\alpha \in \text{Ext}^1_{S^0_A}(I^\alpha, Q_{\alpha} \otimes J)$$

is zero. Suppose that we have that. Next, we have that flat extensions on $U^\alpha$ and $U^\beta$ differ on $U^{\alpha \beta} = U^\alpha \cap U^\beta$ by an element

$$h^{\alpha \beta} \in \text{Hom}_{S^{\alpha \beta}}(I^{\alpha \beta}, Q_{\alpha \beta} \otimes J).$$

Thus local flat extension differ by a Cech 1-cochain with coefficients in the sheaf

$$\text{Hom}_S (I, Q \otimes J) = \text{Hom}_{\mathcal{O}_Y} (\mathcal{I}_Z, \mathcal{O}_Z \otimes J).$$

Clearly the cochain $h = (h^{\alpha \beta})$ is a cocycle. Since we are free to adjust flat extensions on $U^0_B$ by an arbitrary element of $\text{Hom}_{S^0} (I^0, Q_0 \otimes J)$, we can make the local extensions fit together to a global one if and only if $h$ is a coboundary.

The Grothendieck spectral sequence for the composition of the functors $\text{Hom}$ and $\Gamma$ gives the exact sequence

$$0 \to H^1(\text{Hom}_S(I, \mathcal{Q} \otimes J)) \to \text{Ext}^1_B(I, \mathcal{Q} \otimes J) \to H^0(\text{Ext}^1_S(I, \mathcal{Q} \otimes J)) \to 0.$$  

Recall that, in the construction of the Grothendieck spectral sequence for the composition of functors, we take a projective resolution

$$\ldots \to R \to G \to I \to 0.$$
and that the sheaves in
\[ \mathcal{H}om_S (\mathcal{G}, \mathcal{Q} \otimes J) \to \mathcal{H}om_S (\mathcal{R}, \mathcal{Q} \otimes J) \to \ldots \]
are \( \Gamma \)-acylic. So the data \( \left( \left( g_B^\beta - g_B^\alpha, (f^\alpha) \right) \right) \) given in the last section is a 1-cocycle and so gives an element \( E \in \text{Ext}^1_S (\mathcal{I}, \mathcal{Q} \otimes J) \). A slightly tedious check gives that this \( E \) is independent of choices. Also, just as in the previous chapter any two local two extensions differ by an element of \( \text{Hom}_{S^{\alpha \beta}} (I^\alpha, Q_\alpha \otimes J) \) which agree on intersections if and only if their coboundary is 0 in \( \text{Hom}_{S^{\alpha \beta}} (I^\alpha, Q_\alpha \otimes J, J) \). Thus we have:

**Proposition 3.1.** 1) The obstruction to extending \( Z_A \) to a subvariety \( Z_B \) of \( Y_B \) is flat over \( B \) is given by an element \( E \in \text{Ext}^1_{\mathcal{O}_Y} (I_Z, \mathcal{O}_Z) = \text{Ext}^0_{\mathcal{O}_Y} (I_Z, \mathcal{O}_Z) \).

2) Given one extension, the set of extensions form a principal homogeneous space over the group \( \text{Hom}_{\mathcal{O}_Y} (I_Z, \mathcal{O}_Z) = \text{Ext}^0_{\mathcal{O}_Y} (I_Z, \mathcal{O}_Z) \).

3) As before, there is a unique largest quotient \( J_{\text{max}} \) over which \( I_Z \) extends given by the kernel of the morphism \( J^\vee \to \text{Ext}^1_{\mathcal{O}_Y} (I_Z, \mathcal{O}_Z) \) given by \( E \).

### 3.2. The local ring of \( \text{Hilb}(Y) \) at \( \{Z\} \).

**Theorem 3.2.** Let \( Y \) be projective over \( k \) and let \( Z \subset Y \) be a closed subvariety. Let
\[
\hat{S} = \text{lim} \frac{\mathcal{O}_{\text{Hilb}(Y), \{Z\}}}{m^n_{\{Z\}}}
\]
denote the completion of the local ring of \( \text{Hilb}(Y) \) at \( \{Z\} \). Then:

1) \( T_{\text{Hilb}(Y), \{Z\}} = \text{Hom}_Y (I_Z, \mathcal{O}_Z) \).

2) Let \( \hat{T} \) be the completion of the local ring of \( T_{\text{Hilb}(Y), \{Z\}} \) with respect to the maximal ideal \( \hat{m} \) at 0. (That is, \( \hat{T} \) is formal power series in the variables \( (u_1, \ldots, u_r) \) where the \( u_i \) are uniformizing parameters for \( \text{Hilb}(Y) \) at \( \{Z\} \).) For the natural map \( \hat{T} \to \hat{S} \), let
\[
\hat{I} = \hat{I}_{\text{Hilb}} = \text{ker} \left( \hat{T} \to \hat{S} \right).
\]

There is a natural injection
\[
\hat{I} / \hat{m} \cdot \hat{I} \to \text{Ext}^1_{\mathcal{O}_Y} (I_Z, \mathcal{O}_Z).
\]

In particular, the number of equations defining \( \text{Hilb}(Y) \) at \( \{Z\} \) is no greater than \( \dim_k \text{Ext}^1_{\mathcal{O}_Y} (I_Z, \mathcal{O}_Z) \).
Proof. 1) \[ T_{\text{Hilb}(Y),\{Z\}} = \{ h \in \text{Hom}(\text{Spec}(k[e]), \text{Hilb}(Y)) : h(\text{Spec}(k)) = \{Z\} \}. \]

Let \( A = k \) and \( B = k[e] \) above, and let the “base” extension be the trivial extension \( Z \times \text{Spec}(k[e]) \). The set of all extensions is a principal homogeneous space for the group \( \text{Hom}_Z(\mathcal{I}_Z, \mathcal{O}_Z \otimes \{e\}) \), but the trivial extension gives a natural zero in this case.

2) Let \( \mathfrak{m} \) denote the maximal ideal of the power-series ring \( \hat{T} \) at 0. We have natural surjections \[
\frac{\hat{T}}{\mathfrak{m}^n \cdot \hat{T}} \to \frac{\mathcal{O}_{\text{Hilb}(Y),\{Z\}}}{\mathfrak{m}^n(\{Z\})}
\]
for each \( n \). Let \( K_n \) denote the kernel of the \( n \)-th surjection. Suppose for some diagram

\[
\begin{array}{cccc}
0 & \to & J_n & \to & \frac{\hat{T}}{\mathfrak{m} \cdot K_n + \mathfrak{m}^{n+1}} & \to & \frac{\mathcal{O}_{\text{Hilb}(Y),\{Z\}}}{\mathfrak{m}^n(\{Z\})} & \to & 0 \\
\downarrow & & \downarrow \rho & & \downarrow & & \downarrow \\
0 & \to & k & \to & \frac{\hat{T}}{\mathfrak{m} \cdot K_n + J' + \mathfrak{m}^{n+1}} & \to & \frac{\mathcal{O}_{\text{Hilb}(Y),\{Z\}}}{\mathfrak{m}^n(\{Z\})} & \to & \\
\end{array}
\]

with \( J' = \ker(\rho|_{J_n}) \neq 0 \), the extension over the bottom was unobstructed. Since this extension agrees with the given one on \( \mathcal{O}_{\text{Hilb}(Y),\{Z\}} = \frac{\hat{T}}{K_{n+1} + \mathfrak{m}^{n+1}} \), we would have a flat extension of \( Z \) over

\[
S' = \frac{\hat{T}}{\mathfrak{m} \cdot K_n + J' + \mathfrak{m}^{n+1}} \times \frac{\mathcal{O}_{\text{Hilb}(Y),\{Z\}}}{\mathfrak{m}^n(\{Z\})} \mathcal{O}_{\text{Hilb}(Y),\{Z\}}.
\]

This extended family would continue to have its tangent space equal to that of \( \text{Hilb}(Y) \) at \( \{Z\} \) and so \( \hat{T} \to S' \) would continue to be surjective by Nakayama’s lemma. So, by Section 2.5,

\[
J'_n \to \text{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}_Z, \mathcal{O}_Z)
\]
is injective for each \( n \). Since \( I_{\{Z\},\text{Hilb}} \) is finitely generated and since, by Nakayama’s lemma, a minimal set of generators is given by the vector space

\[
I_{\{Z\},\text{Hilb}} = \frac{\hat{T}}{\mathfrak{m} \cdot I_{\{Z\},\text{Hilb}}},
\]
the proof is complete.  

\[ \square \]

3.3. Generically unobstructed deformations of subvarieties of \( Y \).

Definition 3.1. For the subscheme \( Z \subseteq Y \) as above, we call \( Z \) generically unobstructed if, at each generic point \( Z_{\text{gen.}} \), \( \mathcal{I}_{Z_{\text{gen.}},Y} \) is unobstructed in \( \mathcal{O}_{Y,Z_{\text{gen.}}} \), that is,

\[
\mathcal{I}_{Z_{\text{gen.}},Y} \subseteq \mathcal{O}_{Z_{\text{gen.}},Y}
\]

admits a (formal) family of flat deformations over the completion of the local ring at 0 of the tangent space consisting of all flat extensions of \( \mathcal{I}_{Z_{\text{gen.}},Y} \) over \( \frac{k[e]}{e^2} \).
Lemma 3.3. If each component of $Z$ meets the smooth locus of $Y$ and has multiplicity $1$, then $Z$ is generically unobstructed in $Y$.

Proof. Each component $Z_{\text{gen.}}$ is a complete intersection so that $I_{Z_{\text{gen.}},Y}$ has a Koszul resolution. That means that $\text{Hom}_{Y,O_{Z_{\text{gen.}}}}(\cdot,O_{Z_{\text{gen.}}})$ is exact on this resolution so that

$$\text{Ext}^1_{O_{Y,Z_{\text{gen.}}}}(I_{Z_{\text{gen.}},Y},O_{Z_{\text{gen.}}}) = 0.$$ 

□

Theorem 3.4. If $Z$ is generically unobstructed in $Y$, and if $Z$ has no embedded primes, nor does $Y$ (along $Z$), then

$$\text{Obst}(Z,Y) \subseteq \text{Ext}^1_{O_Z}(I_Z,O_Z) \subseteq \text{Ext}^1_{O_Y}(I_Z,O_Z).$$

Proof. First of all, we consider the diagram

$$
\begin{array}{c}
\text{Ext}^1_{O_Z}(I_Z,O_Z) \\
\downarrow \\
0 \rightarrow \text{Hom}_{O_Y}(I_Z,O_Z) \rightarrow \text{Ext}^1_{O_Y}(I_Z,O_Z) \rightarrow \text{Ext}^1_{O_Y}(I_Z,O_Z)
\end{array}
$$

in which the vertical map is an inclusion by the definition of $O$-module extension and the bottom row is exact since $\text{Hom}_{O_Y}(I_Z,O_Z) = \text{Hom}_{O_Y}(I_Z,O_Z)$.

Suppose $0 \neq E' \in \text{image}(\text{Hom}_{O_Y}(I_Z,O_Z)) \cap \text{image}(\text{Ext}^1_{O_Z}(I_Z,O_Z))$.

Since $E' \mapsto 0$ in $\text{Ext}^1_{O_Z}(I_Z,O_Z)$ we would have

$$
0 \rightarrow O_Z \rightarrow E' \rightarrow \frac{I_Z}{I_Z^2} \rightarrow 0
$$

and so to the conclusion that

$$E' = \frac{O_Z + I_Z}{\{(h(x),x) : x \in I_Z^2,p\}}$$

where $h$ is the restriction of $\varepsilon$ to $0 + I_Z^2$. On the other hand, $E'$ has the structure of an $O_Z$-module. But for this to happen for the quotient just above, we would need that

$$I_Z \cdot (O_Z + I_Z) = (0 + I_{Z,p}^2) \subseteq \{(h(x),x) : x \in I_{Z,p}^2\}$$

so that $h = 0$. Thus the natural map

$$\text{Ext}^1_{O_Z}(I_Z,O_Z) \rightarrow \text{Ext}^1_{O_Y}(I_Z,O_Z)$$

is an inclusion.
Next we show that $E \in \text{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}_Z, \mathcal{O}_Z \otimes J)$ goes to $E_2 = 0$ in $\text{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}_Z^2, \mathcal{O}_Z \otimes J)$. To see this, consider the diagram

\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_Z \otimes J & \to & \mathcal{E}_2 & \to & \mathcal{I}_Z^2 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_Z \otimes J & \to & \mathcal{E} & \to & \mathcal{I}_Z & \to & 0
\end{array}
\]

of extensions induced by $E \mapsto E_2$. The element $E$ induces an element $e \in H^0(\mathcal{E}_2, \mathcal{O}_Z \otimes J)$. The support $S$ of $e$ contains no generic point of $Z$ and no point of $Y - Z$. If $x \in Y - S$, then any two local splittings

\[s_x, s'_x : \mathcal{I}_Z|_x \to \mathcal{E}_2|_x,
\]

they differ by an element of $\text{Hom}_{\mathcal{O}_Y,x}(\mathcal{I}_Z|_x, \mathcal{O}_{Z,x} \otimes J)$ and so

\[s'_x - s_x|_{\mathcal{I}_Z^2|_x} = 0.
\]

Thus the $s_x|_{\mathcal{I}_Z^2}$ fit together to give a well-defined splitting

\[s : \mathcal{I}_Z^2|_{Y - S} \to \mathcal{E}_2|_{Y - S}.
\]

On the other hand, at any point $x \in S$ we always have a splitting

\[t_x : \mathcal{I}_Z|_x \to \mathcal{E}|_x.
\]

of $\mathcal{E}$ at the stalk level. Let $U$ be a Zariski open neighborhood of $x$ in $Y$ on which $t_x$ is defined. Then $t_x|_{\mathcal{I}_Z^2}$ and $s$ agree on $((Y - S) \cap U_x)$ and so extend to a splitting $t$ on $((Y - S) \cup U_x)$. If $t'_x$ is another choice of splitting at $x$, then we obtain another splitting $t'$ on $((Y - S) \cup U_x)$. Then

\[t - t' : \mathcal{I}_Z^2|_{((Y - S) \cup U_x)} \to \mathcal{E}_2|_{((Y - S) \cup U_x)}
\]

must be supported on an embedded prime of

\[Z \cap ((Y - S) \cup U_x).
\]

But, by hypothesis, there are no such primes. Thus our obstruction element $E$ is the image of an element $E_1$ in

\[\text{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}_Z^2, \mathcal{O}_Z \otimes J).
\]

Finally, let

\[0 \to \mathcal{O}_Z \otimes J \to \mathcal{E}_1 \to \mathcal{I}_Z^2 \to 0
\]

be the exact sequence of $\mathcal{O}_Y$-modules given by $E_1$. Then, as above,

\[\mathcal{I}_Z \cdot \mathcal{E}_1 \subseteq \mathcal{O}_Z \otimes J
\]

and is supported on embedded primes of $Z$. So, since there are none,

\[\mathcal{I}_Z \cdot \mathcal{E}_1 = 0
\]

and the proof is complete. \qed
3.4. An interesting example.

Exercise 3.1. Let $Y = \mathbb{C}P^3$ with coordinates $x, y, z, w$ and let

$$\mathcal{I}_Z = \{xy, z^2, -xz, -yz\}.$$

i) Show that the Hilbert polynomial $p_Z(d) = \dim H^0(\mathcal{O}_Z(d))$ of $Z$ is $2d + 2$.

ii) Show that $\text{Hilb}(Y)$ has at least two components at $\{Z\}$, one being made up of the closure of the scheme of varieties which are plane conics together with a (reduced) point in $Y - Z$, and another being the closure of the scheme of pairs of disjoint lines in $\mathbb{C}P^3$.

iii) Now restrict $\mathcal{O}_{\mathbb{C}P^3}$ and $\mathcal{I}_Z$ to their local rings at $[0, 0, 0, 1]$. To see what happens there, write the 2-parameter family

$$\mathcal{I}_{Z,(0,0,0)}(s,t) = \{xy - sy, z^2 - tz, -xz + tx, -yz\}.$$

By Nakayama’s lemma

$$r_0 = \begin{pmatrix} z & 0 & y & 0 \\ z & 0 & 0 & x \\ 0 & x & z & 0 \\ 0 & y & 0 & z \end{pmatrix}.$$

Compute

$$r_{(s,t)} = \begin{pmatrix} z - t & 0 & y & -s \\ z & 0 & 0 & x - s \\ 0 & x & z & 0 \\ 0 & y & 0 & z - t \end{pmatrix}$$

and, for $J = (s^2, st, t^2)$, compute the mapping

$$f : R \to \mathcal{O}_{Z,(0,0,0)} \otimes J.$$

iv) Show that the extension

$$0 \to \mathcal{O}_{Z,(0,0,0)} \to \frac{(\mathcal{O}_{Z,(0,0,0)} + G)}{-y, (z, 0, y, 0)} \to \mathcal{I}_{Z,(0,0,0)} \to 0$$

is not split. Hint: Look at the map

$$\{0\} + G \to \frac{\mathcal{O}_{Z,(0,0,0)}}{m_{Z,(0,0,0)}}$$

induced by a splitting homomorphism.

v) Conclude that the maximal unobstructed quotient $J_{\text{max}} = \frac{(s^2, st, t^2)}{(st)}$.

vi) Conclude that $\text{Ext}_Y^1(\mathcal{I}_Z, \mathcal{O}_Y) \neq 0$, since $\mathfrak{m}_{\mathcal{O}_Y}$ is a non-zero skyscraper sheaf, supported at $x = y = z = 0$ and so $H^0(\mathfrak{m}_{\mathcal{O}_Y}) (\mathcal{I}_Z, \mathcal{O}_Z) \neq 0$.

3.5. Local obstructions.

Proposition 3.5. 1) If $Z \subseteq Y$ is a local complete intersection, there are no local obstructions to deforming $Z$ in $Y$, that is, the image of $\text{Obst}_Y(Z)$ in $H^0(\mathfrak{m}_{\mathcal{O}_Y} (\mathcal{I}_Z, \mathcal{O}_Z))$ is zero.

2) If $Y$ is smooth and $Z$ is of codimension 2 and Cohen-Macaulay, then the image of $\text{Obst}_Y(Z)$ in $H^0(\mathfrak{m}_{\mathcal{O}_Y} (\mathcal{I}_Z, \mathcal{O}_Z))$ is zero.
Proof. 1) Locally along $Z$ we have
\[ O_Z = \mathcal{O}_Y \{ f_1, \ldots, f_c \} \]
and we can resolve $I_A$ via the Koszul resolution
\[
\bigwedge^{c-2} V \otimes \mathcal{O}_Y \wedge (\sum_i f_i dx_i) \rightarrow \bigwedge^{c-1} V \otimes \mathcal{O}_Y \wedge (\sum_i f_i dx_i) \rightarrow \bigwedge^c V \otimes I_A \rightarrow 0.
\]
Take any extension of the $f_i$ to functions over $B$. The resulting Koszul complex will still be exact.

2) Again locally at $x \in Z$, since $O_{Z,x}$ is Cohen-Macaulay at $x$, its homological dimension is $\dim Z$. (See [AK]. Chapter III.) Since $O_{Y,x}$ is a regular local ring
\[
\dim O_{Y,x} = \text{depth} (O_{Z,x}) + \text{proj. dim}_{O_{Y,x}} (O_{Z,x})
\]
so that the stalk of its ideal at any point has a 2-step projective (i.e. free) resolution
\[
0 \rightarrow \mathcal{R}_Y (r_{ij}^0) \rightarrow \mathcal{G}_Y (g_{ij}^0) \rightarrow I_{Z,x} \rightarrow 0.
\]
So if $n = \text{rank} (\mathcal{R}_Y)$,
\[
\text{rank} (\mathcal{G}_Y) = n + 1
\]
and $I_{Z,x}$ is determinantal, that is, given by the $(n \times n)$-minors of the $n \times (n + 1)$ matrix $(r_{ij}^0)$ in the regular local ring $O_{Y,x}$. \qed

Example 3.1. Suppose $Y$ is smooth and $Z$ is l.c.i. and reduced. Then $\text{Obst}_Y (Z)$ lies in the image of the injection
\[
H^1 \left( \text{Hom}_{O_Z} \left( \frac{I_Z}{I_Z^2}, \mathcal{O}_Z \right) \right) \rightarrow H^1 (\text{Hom}_{O_Y} (I_Z, \mathcal{O}_Z)) \rightarrow \text{Ext}^1_{O_Y} (I_Z, \mathcal{O}_Z),
\]
that is, in $H^1 (N_{Z,Y})$.

4. Deformations of maps

We next wish to study deformations of morphisms
\[
f : X \rightarrow Y
\]
of projective schemes.

Fact: If $X^{\text{flat}}/S$ and $Y/S$ are projective, then the deformation functor of morphisms from $X$ to $Y$ defined over $S$ is represented by an open subscheme of
\[
\text{Hilb} (X \times_S Y/S)
\]
which we will call
\[
\text{Hom}_S (X, Y).
\]

Example 4.1. Let $Y/S$ be the universal smooth quintic threefold, let $X$ be $\mathbb{P}^1 \times S$ and choose the appropriate Hilbert polynomial by specifying a degree $d$. 
Theorem 4.1. Let $X$ and $Y$ be projective over $k$ with no embedded points. Let $f : X \to Y$ be such that the image of each component of $X$ meets the smooth locus of $Y$. Then:

1) $T_{[f]} \text{Hom}(X, Y) = \text{Hom}_X \left(f^*\Omega^1_Y, \mathcal{O}_X\right)$,

2) $\dim_{[f]} \text{Hom}(X, Y) \geq \dim \text{Hom}_X \left(f^*\Omega^1_Y, \mathcal{O}_X\right) - \dim \text{Ext}^1_X \left(f^*\Omega^1_Y, \mathcal{O}_X\right)$.

Proof. Let $\Gamma = \text{graph}(f) \subseteq X \times Y$ and, for the ideal sheaf $\mathcal{I}$ of $\Gamma$ in $X \times Y$, consider the exact sequence

\[ \mathcal{I} \xrightarrow{\alpha} \Omega^1_{X \times Y}|_{\Gamma} \to \Omega^1_X \to 0. \]

Now $\alpha$ is injective in our case. To see this, begin with the fact that the sequence

\[ 0 \to \mathcal{I} \xrightarrow{\alpha} \Omega^1_{X \times Y} \xrightarrow{\mathcal{O}_X} \mathcal{O}_X \to 0 \]

is split by projection to $X$. Let $s := \text{pr}_X^* : \mathcal{O}_X \to \frac{\mathcal{O}_{X \times Y}}{\mathcal{I}^2}$ denote the splitting. Then define

\[ \varphi : \frac{\mathcal{O}_{X \times Y}}{\mathcal{I}^2} \to \frac{\mathcal{I}}{\mathcal{I}^2} \]

\[ b \mapsto b - s(b|_{\Gamma}). \]

Now

\[(b - s(b|_{\Gamma})) \cdot (b' - s(b'|_{\Gamma})) = 0\]

which gives, by some elementary algebra that $\varphi$ is a derivation, that is, that

\[ \frac{\mathcal{I}}{\mathcal{I}^2} \xrightarrow{\alpha} \frac{\mathcal{O}_{X \times Y}}{\mathcal{I}^2} \xrightarrow{\mathcal{O}_X} \frac{\mathcal{I}}{\mathcal{I}^2} \]

factors through

\[ \frac{\mathcal{I}}{\mathcal{I}^2} \xrightarrow{d} \frac{\mathcal{O}_{X \times Y}}{\mathcal{I}^2} \xrightarrow{\varphi} \frac{\mathcal{I}}{\mathcal{I}^2}. \]

Comparing this diagram to (9) we conclude that $\alpha$ is injective and the image of $\frac{\mathcal{I}}{\mathcal{I}^2}$ in $\Omega^1_{X \times Y}|_{\Gamma} = \text{pr}_X^* \Omega^1_X |_{\Gamma} + \text{pr}_Y^* \Omega^1_Y |_{\Gamma}$ projects isomorphically to $\text{pr}_Y^* \Omega^1_Y |_{\Gamma} = f^* \Omega^1_Y$.

So we can use

\[ \frac{\mathcal{I}}{\mathcal{I}^2} \xrightarrow{\alpha} f^* \Omega^1_Y \]

in Theorems 3.2 and 3.4 to conclude the proof of both 1) and 2) in the Proposition. \[ \square \]
Notice that this Proposition is especially appealing the the case in which \( \dim X = 1 \) and \( Y \) is smooth since then we have a lower bound on \( \dim \text{Hom} (X,Y) \), provided it is non-empty. Namely

\[
\dim [f] \text{Hom} (X,Y) \geq \chi (f^* T_Y).
\]

Part 2. Analytic theory

5. Differential graded Lie algebras

When we are deforming geometry objects in a situation in which the local deformations are all not only unobstructed but also trivial, there is a different formulation of the deformation problem which uses a different algebraic tool. This part of the notes will explore that tool and its uses. We will have to assume for this part that \( \text{char } (k) = 0 \).

Definition 5.1. A differential graded Lie algebra (DGLA) is a triple

\[
L^*, d, [\ , \ ]
\]

such that

1) \( L^* = \sum_{i=-\infty}^{\infty} L^i \) where each \( L^i \) is a \( k \)-vector space to whose non-zero elements \( a \) we assign the degree \( i \), which we write in the form \( \bar{a} = i \),

2) there is a \( k \)-linear homomorphism \( d : L^* \to L^{*+1} \) satisfying \( d \circ d = 0 \) (and \( dk = 0 \)),

3) there is a \( k \)-bilinear graded Lie bracket

\[
[\ , \ ] : L^i \times L^j \to L^{i+j},
\]

that is,

\[
[a,b] + (-1)^{\bar{a}\bar{b}} [b,a] = 0
\]

and the graded Jacobi identity is satisfied, that is

\[
[a, [b,c]] = [[a,b] , c] + (-1)^{\bar{a}\bar{b}} [b, [a,c]],
\]

4) \( d [a,b] = [da, b] + (-1)^{\bar{a}} [a, db] \).

Example 5.1. 1) Any \( k \)-Lie algebra considered as the \( k \)-vector space \( L^0 \) with \( d = 0 \).

2) Let \( X \) be a compact complex manifold and let

\[
L^i = A^{0,i}_X (T_X)
\]

\[
d = \bar{\partial}
\]

\[
[\phi d \bar{\partial} \bar{z}_{r_1} \ldots d \bar{z}_{r_i}, \psi d \bar{z}_{s_1} \ldots d \bar{z}_{s_j}] = [\phi, \psi] d \bar{z}_{r_1} \ldots d \bar{z}_{r_i} d \bar{z}_{s_1} \ldots d \bar{z}_{s_j}.
\]

3) Let \( E \) be a finite-rank holomorphic vector bundle on a compact complex manifold \( X \). Let

\[
L^i = A^{0,i}_X (\text{End}_X (E))
\]

\[
d = \bar{\partial}
\]

\[
[\phi d \bar{\partial} \bar{z}_{r_1} \ldots d \bar{z}_{r_i}, \psi d \bar{z}_{s_1} \ldots d \bar{z}_{s_j}] = [\phi, \psi] d \bar{z}_{r_1} \ldots d \bar{z}_{r_i} d \bar{z}_{s_1} \ldots d \bar{z}_{s_j}.
\]
Exercise 5.1. If $A = \sum_{i=-\infty}^{\infty} A^i$ is any associative, graded-commutative $k$-algebra (that is, $ba = (-1)^{\bar{a} \bar{b}} ab$), and $L$ is a DGLA, then so is $L \otimes A$

where
\[
d(x \otimes a) = dx \otimes a \\
[x \otimes a, y \otimes b] = [x, y] \otimes (-1)^{\bar{a} \bar{b}} ab.
\]

Example 5.2. (Hochschild DGLA) For an associative $k$-algebra $A$, define
\[
G^n = \text{Hom} \left( A^{\otimes (n+1)}, A \right)
\]
with $d\phi (a_0 \otimes \ldots \otimes a_{n+1})$ given by the formula
\[
a_0 \phi (a_1 \otimes \ldots \otimes a_{n+1}) + \sum_{i=0}^{n} \left( (-1)^i (a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \right)
\]
and $[\phi, \phi']$ defined in the following way. First define
\[
\phi \cup \phi'
\]
by setting $(\phi \cup \phi') (a_0 \otimes \ldots \otimes a_{n+n'})$ equal to
\[
\sum_{i=0}^{n} (-1)^{n_i} \phi (a_0 \otimes \ldots \otimes a_i \phi' (a_i \otimes \ldots \otimes a_{i+n'}) \otimes a_{i+1+n'} \otimes \ldots \otimes a_{n+n'})
\]
and then defining
\[
[\phi, \phi'] = \phi \cup \phi' - (-1)^{\bar{\phi} \bar{\phi}'} \phi' \cup \phi
\]

5.1. Cohomology of DGLA’s. In what follows
\[
Z^i (L) = \ker (d: L^i \rightarrow L^{i+1}) \\
B^i = \text{image} (d: L^{i-1} \rightarrow L^i) \\
H^i = \frac{Z^i}{B^i}
\]

Exercise 5.2. The Lie bracket on a DGLA induces a well-defined Lie bracket on its cohomology.

Definition 5.2. A morphism $f : L \rightarrow L'$ of DGLA’s is called a quasi-isomorphism if the induced map
\[
H^* (L, d) \rightarrow H^* (L', d)
\]
is an isomorphism. A DGLA is called formal if it is quasi-isomorphic to its cohomology with $d = 0$ and the induced Lie bracket.

We will be interested in the Maurer-Cartan equation on $L$, namely the differential equation
\[
dx + \frac{1}{2} [x, x] = 0
\]
on $L_1$. The interaction of this differential equation with the cohomology of $L \otimes A$ where $A$ is an Artinian $k$-algebra (concentrated in degree 0) will be the focus of our use of DGLA’s in deformation theory.
5.2. **The augmentation** $L_{(d)}$ of a DGLA $L$. Given any DGLA, we augment it by defining

$$L_{(d)}^i = L^i, \ i \neq 1,$$

$$L_{(d)}^1 = L^1 + k \cdot d$$

with

$$d_{(d)} (x + c \cdot d) = [d, x + c \cdot d] = dx$$

$$[x + c \cdot d, y + c' \cdot d] = [x, y] + [c \cdot d, y] + [x, c' \cdot d]$$

$$= [x, y] + c \cdot dy - (-1)^x c' \cdot dx.$$

Check that $L_{(d)}$ is indeed a DGLA and that the inclusion $L \rightarrow L_{(d)}$ is a morphism of DGLA’s. Notice that, for $x \in L^1$,

$$[d + x, d + x] = [x, x] + 2dx$$

so that the MC equation is equivalent to

(12) $$[d + x, d + x] = 0.$$  

5.3. **Gauge actions.** Notice that, if $L$ is a DGLA then we can define the associated (graded) enveloping algebra

$$U(L) := \sum_{n \in \mathbb{Z}} L^\otimes n \{ (xy + (-1)^{ij} yx) - [x, y] : x \in L^i, y \in L^j \}_{i,j \in \mathbb{Z}}$$

If $N$ is a nilpotent Lie algebra, we can define

$$\exp (N^0) \subseteq U^0 (N).$$

by formal power series with inverse

$$\log : \exp (N^0) \rightarrow N^0$$

again defined by the formal power series for log. For example we can take an arbitrary DGLA $L$ and tensor it with an Artinian ring $A$ with maximal ideal $m$ and let $N = L \otimes m$.

The Baker-Campbell-Hausdorff formula states that, for $a \in N^0, x \in L^1$,

(13) $$\exp (a) \cdot x \cdot \exp (-a) = \exp ([a, ]) (x)$$

where the left-hand side is a priori an element of $U^1 (N)$ and the right-hand side is an element of $L^1$. To see this, for variable $t \in k$ differentiate

$$\frac{\partial}{\partial t} (\exp (ta) \cdot x \cdot \exp (-ta)) = \exp (ta) \cdot a \cdot x \cdot \exp (-ta) - \exp (ta) \cdot x \cdot a \cdot \exp (-ta)$$

$$= \exp (ta) \cdot [a, x] \cdot \exp (-ta)$$

and

$$\frac{\partial}{\partial t} \exp ([ta, ]) (x) = \exp ([ta, ]) [a, x].$$

So

$$g(t) = \exp (ta) \cdot y \cdot \exp (-ta)$$

and

$$h(t) = \exp ([ta, ]) (y)$$
are both solutions to the differential equation
\[ \frac{\partial u}{\partial t} = u \cdot [a,] \]
on the space of analytic functions
\[ [0, 1] \to \text{Hom}_k \left( L^1, U(L) \right). \]
Also
\[ g(0) = h(0) \]
so
\[ g(1) = h(1). \]
Unless the Lie algebra is abelian, it is not true that
\[ \exp(a + b) = \exp(a) \cdot \exp(b) \in U^0(N), \]
in fact, there is no \textit{a priori} reason why one should expect that \( \exp(a) \cdot \exp(b) \in \exp(N^0) \). However, guided by the example \( L = A^n_X(T_X) \), (13) can be used to show that \( \exp(a) \cdot \exp(b) \in \exp(N^0) \) in general and that
\[ a \ast b := \log(\exp(a) \cdot \exp(b)) \]
is given by the (Baker-Campbell-Hausdorff) formula
\[ \sum_{n>0} \frac{(-1)^n}{n} \left( \sum_{r_i+s_i>0} C_{(r_i),(s_i)} [a,]^{r_i} \circ [b,]^{s_i} \circ \ldots \circ [a,]^{r_n} \circ [b,]^{s_n-1}(b) \right) \]
where, for each pair of \( n \)-tuples \((r_i),(s_i)\) of non-negative integers
\[ C_{(r),(s)} = \left( \frac{\sum_{i=1}^n (r_i+s_i)^{-1}}{r_1! \cdot s_1! \cdot \ldots \cdot r_n! \cdot s_n!} \right) \]
and, for terms with \( s_n = 0 \) the expression in the sum terminates with \( [a,]^{r_n-1}(a) \in \exp(N^0) \). Then
\[ \exp(a) \cdot \exp(b) = \exp(a \ast b) \]
so that, for \( G_L := \exp(L^0) \),
\[ (G_L, \ast) \]
is a group.
Now for any DGLA \( L \) such that \( L^0 \) is nilpotent, the action
\[ L^0 \times L^1 \rightarrow L^1 \]
\[ (a,x) \mapsto \exp([a,]x) \]
of the gauge group \( L^0 = G \) on \( L \) is well-defined since
\[ \left( \exp [b,] \circ \exp [a,] \right)(x) = \exp [b,] \circ \left( \exp(a) \cdot x \cdot \exp(-a) \right) = \exp(b) \cdot \exp(a) \cdot x \cdot \exp(-a) \cdot \exp(-b) = \exp(b \ast a) \cdot x \cdot \exp(- (b \ast a)). \]
5.4. Maurer-Cartan and the gauge group. As mentioned above, the Maurer-Cartan condition on \( x \in L^1 \) is
\[
[d + x, d + x] = 0.
\]
On the other hand, by the remark just above, the action of \( L^0 \) on \( L^1_{\{d\}} \) given by
\[
(a, d + x) \mapsto \exp \left([a, ](d + x)ight)
\]
can be rewritten
\[
\exp (a) \cdot (d + x) \cdot \exp (-a) = \exp \left([a, ](d + x) \right)
= \exp \left([a, ](d) + \exp ([a, ](x) \right)
= d - \frac{\exp \left([a, ] \right) - 1}{[a, ]} (da) + \exp \left([a, ] \right)(x)
\]
by the formula for differentiation of an exponential function. From the expression on the right-hand of this last equality, we see that we conclude that the action of \( L_0 \) on \( L^1_{\{d\}} \) preserves the affine space
\[
d + L^1
\]
and from the expression on the left-hand side and (12) we see that it also preserves the set
\[
MC_L := \{ x \in L^1 : [d + x, d + x] = 0 \}.
\]
Suppose now that I have a one-parameter family of gauge actions
\[
\exp (sa) \cdot (d + x) \cdot \exp (-sa).
\]
It is immediate from (15) that
\[
\frac{\partial}{\partial s} \left( \exp (sa) \cdot (d + x) \cdot \exp (-sa) \right) \bigg|_{s=0} = da + [a, x].
\]

6. Deformation functors associated to a DGLA

Given a DGLA \( L \) we can associate to it a functor which takes an Artinian scheme \( A \) over \( k \) with maximal ideal \( m \) to the set
\[
Def_L (A) = G_{L \otimes m} \setminus MC_{L \otimes m}.
\]
It is clear that every morphism
\[
L \to L'
\]
of DGLA’s induces a morphism of functors
\[
Def_L \to Def_{L'}.
\]
Now \( MC_{L \otimes m} \) has tangent space
\[
\{ x \in L^1 : dx = 0 \}
\]
since
\[
[L^1 \otimes m, L^1 \otimes m] \equiv C[[t]]/m^2 \neq 0.
\]
Also \( G_{L_{\{d\}}} \) has tangent space
\[
\{-da : a \in L^0\}.
\]
So, computing the action of $T_{GL_{L_1}}i_{id}$ on $T_{MC_{L_1}}x$ we get

$$T_{Def_L} = H^1(L, d).$$

6.1. **Obstructions.** Consider now any small extension of Artinian $k$-algebras

$$0 \to J \to B \to A \to 0$$

and let $x \in MC_{L_1}(A)$. We wish to define an obstruction class

$$\text{obst}(x) \in H^2(L, d) \otimes J.$$

To do this, lift $x$ arbitrarily to $\tilde{x} \in L_1 \otimes B$ and consider

$$h := d\tilde{x} + \frac{1}{2} [\tilde{x}, \tilde{x}] \in L_2 \otimes J.$$

Now

$$dh = \frac{1}{2} d[\tilde{x}, \tilde{x}] = [h, \tilde{x}] - \frac{1}{2} [[\tilde{x}, \tilde{x}], \tilde{x}].$$

But

$$[h, \tilde{x}] \in [L_2 \otimes J, L_1 \otimes m] = 0$$

and

$$[[\tilde{x}, \tilde{x}], \tilde{x}] = 0$$

by the Jacobi identity. So $dh = 0$. Now let $\tilde{x}'$ be another lifting of $x$. Then

$$y = \tilde{x} - \tilde{x}' \in L_1 \otimes J$$

and

$$h' - h = dy + [y, \tilde{x}] = dy$$

so that we obtain a well defined element

$$\text{obst}(x) \in H^2(L, d) \otimes J.$$

It is immediate from the definition that $x$ extends to $MC_{L_1}(B)$ if and only if this class is zero.

**Theorem 6.1.** Let

$$\phi : L \to L'$$

be a quasi-isomorphism of DGLA’s, that is, $H^i(\phi)$ is an isomorphism for each $i$. Then, for each Artinian scheme $A$, the induced map

$$\Phi_A : Def_L(A) \to Def_{L'}(A)$$

is an isomorphism. In fact, the conclusion still holds under the weaker hypotheses that $H^0(\phi)$ is surjective, $H^1(\phi)$ is bijective, and $H^2(\phi)$ is injective.

**Proof.** We prove the theorem by Artinian induction. Namely, for a small extension

$$0 \to J \to B \to A \to 0$$

we suppose $\Phi_A$ to be isomorphism. The extensions of $L_1 \otimes A$ to $B$ are a principal homogeneous bundle over $L_1 \otimes J$ and similarly for $(L')_1$. Suppose now that

$$(17) \quad d\tilde{x}' + \frac{1}{2} [\tilde{x}', \tilde{x}'] = 0$$

over $B$. Pick some $\tilde{x}$ such that the restriction of $\phi(\tilde{x})$ over $A$ equals that of $\tilde{x}'$. Then

$$y' := \phi(\tilde{x}) - \tilde{x}' \in (L')_1 \otimes J$$
and
\[ d (\tilde{x}' + y') + \frac{1}{2} [(\tilde{x}' + y'), (\tilde{x}' + y')] \]
\[ = dy' + [\tilde{x}', y'] = dy' \]
over \( B \). Thus
\[
\left\{ d (\tilde{x}' + y') + \frac{1}{2} [\tilde{x} + y', \tilde{x} + y'] \right\} = 0 \in H^2 (L') \otimes J.
\]
Since \( H^2 (\phi) \) is injective,
\[
\left\{ d \tilde{x} + \frac{1}{2} [\tilde{x}, \tilde{x}] \right\} = 0 \in H^2 (L) \otimes J
\]
so that
\[ d\tilde{x} + \frac{1}{2} [\tilde{x}, \tilde{x}] = dy \]
for some \( y \in L^1 \otimes J \) and, as above,
\[ d (\tilde{x} - y) + \frac{1}{2} [\tilde{x} - y, \tilde{x} - y] = d\tilde{x} + \frac{1}{2} [\tilde{x}, \tilde{x}] - dy = 0. \]
Then
\[ \phi (\tilde{x} - y) = \phi (\tilde{x}) - \phi (y) \]
also satisfies Maurer-Cartan over \( B \). But
\[ \phi (\tilde{x} - y) = \tilde{x}' + y' - \phi (y) \]
So, as above,
\[ d (y' - \phi (y)) = 0. \]
So \( \{ y' - \phi (y) \} \in H^1 (L') \otimes J \) has a preimage \( \{ y_1 \} \in H^1 (L) \otimes J \), that is, there is \( y_1 \in Z^1 (L) \otimes J \) so that
\[ \phi (y_1) + (y' - \phi (y)) = dz' \]
for some \( z' \in (L')^0 \otimes J \). Then, as above, \( \tilde{x} - y + y_1 \) also satisfies Maurer-Cartan and
\[
\phi (\tilde{x} - y + y_1) = \tilde{x}' + y' - \phi (y) + \phi (y_1) = \tilde{x}' + y' - \phi (y) + dz' - (y' - \phi (y))
\]
\[ \tilde{x}' + dz'. \]
Since this last element is in the same \( G_{L'} \) equivalence class as \( \tilde{x}' \), we conclude that \( \Phi_B \) is surjective.

On the other hand, assume \( \tilde{x}_1 \) and \( \tilde{x}_2 \) both satisfy Maurer-Cartan and
\[ \{ \phi (\tilde{x}_1) \} = \{ \phi (\tilde{x}_2) \} \in Def_{L'} (B), \]
and, for \( \alpha_1 = \{ \tilde{x}_1 \} \in Def_L (B) \) and \( \alpha_2 = \{ \tilde{x}_2 \} \in Def_L (B) \)
\[ \{ \tilde{x}_1 \} = \{ \tilde{x}_2 \} \in Def_L (A). \]

Therefore acting by an element of the gauge group in \( L \) we can achieve that
\[ \tilde{x}_1 \equiv_A \tilde{x}_2. \]
Also we have \( \tilde{z}' \in (L')^0 \otimes B \) such that
\[ \exp ([\tilde{z}', ]) (d + \phi (\tilde{x}_1)) = d + \phi (\tilde{x}_2). \]
We claim that we can choose
\[ z' \in (L')^0 \otimes J \]
To see this, suppose that, for \( A' \) a given quotient of \( A \) and \( B' \) a quotient of \( B \) for which there is a small extension
\[ 0 \to J' \to B' \to A' \to 0, \]
there are
\[ \{ \tilde{x}_1 \} = \alpha_1, \{ \tilde{x}_2 \} = \alpha_2 \]
and \( z'|_{B'} \in (L')^0 \) such that
\[ \tilde{x}_1 \equiv_{A'} \tilde{x}_2 \]
and \( \tilde{x}_1 \equiv_{A'} \tilde{x}_2 \) such that
\[ \exp ([z', ])(d + \phi(\tilde{x}_1)) = d + \phi(\tilde{x}_2). \]
Then for
\[ y_{B'} = \tilde{x}_1 - \tilde{x}_2 \in L^1 \otimes J' \]
we have
\[ 0 = d(\tilde{x}_2 + y_{B'}) + \frac{1}{2}[\tilde{x}_2 + y, \tilde{x}_2 + y_{B'}] = dy_{B'}. \]
Also
\[ dz_{B'} \equiv_{B'} \phi(\tilde{x}_1) - \phi(\tilde{x}_2) = \phi(y_{B'}). \]
So by the injectivity of \( H^1(\phi) \) there is a \( z_{B'} \in Z^0(L) \otimes J' \) such that
\[ dz_{B'} = y_{B'}, \]
so that
\[ d\phi(z_{B'}) \equiv_{B'} dz_{B'}. \]
So by the surjectivity of \( H^0(\phi) \) there exists \( I \) can in fact choose \( z_{B'} \) so that
\[ \phi(z_{B'}) \equiv_{B'} z'_{B'} + du'_{B'}. \]
for some \( u'_{B'} \) in \( (L')^{-1} \otimes J'. \) Then
\[ \exp ([z'_{B'}, ])(d + \phi(\tilde{x}_1)) \equiv_{B'} \exp ([z'_{B'}, + du'_{B'}, ])(d + \phi(\tilde{x}_1)) \equiv_{B'} \phi \exp ([z_{B'}, ])(d + \tilde{x}_1). \]
Let \( \tilde{z} \) denote an extension of \( z_{B'} \) to \( B \). Replace \( d + \tilde{x}_1 \) by
\[ \exp ([\tilde{z}, ])(d + \tilde{x}_1) \]
and \( z'_{B'} \) by
\[ z'_{B'} + du'_{B'}. \]
Then we achieve that we have the desired result for
\[ A'' = \frac{B'}{\text{image}(J)}, \]
namely
\[ \tilde{x}_1 \equiv_{A''} \tilde{x}_2 \]
and, for a small extension
\[ 0 \to J'' \to B'' \to A'' \to 0 \]
there is \( z'_{B''} \in (L'')^0 \otimes J' \) such that
\[ \exp ([z'_{B''}, ])(d + \phi(\tilde{x}_1)) = d + \phi(\tilde{x}_2). \]
So, by Artinian induction we conclude (18). Let 
\[ y = \tilde{x}_1 - \tilde{x}_2. \]
To finish the proof it will suffice to show that \( y \in d\tilde{z} \) for some \( \tilde{z} \in (L^0 \otimes J) \) since this implies that the gauge action given by \( z \) takes \( \tilde{x}_1 \) to \( \tilde{x}_2 \). Now just as above we have that 
\[ \phi(y) = d\tilde{z}' \].
Also 
\[ 0 = d(\tilde{x}_2 + y) + \frac{1}{2}[\tilde{x}_2 + y, \tilde{x}_2 + y] = dy. \]
So by injectivity of \( H^1(\phi) \) there exists \( \tilde{z} \in L^0 \otimes J \) such that 
\[ d\tilde{z} = y. \]
\[ \square \]

Part 3. Deformations of complex structures

7. Deformations of complex structure on a polydisk

7.1. Gauge transformations. Let \( D \) denote a complex polydisk with holomorphic coordinates \( x = (x_i) \). We fix a real analytic structure on \( D \) to be the one induced by the complex structure, that is, a complex-valued function is real analytic if it can be written locally as a power-series in the variables \( x_i \) and \( \pi_i \). In this chapter we wish to study deformations of the \( \overline{\partial} \)-operator on \( D \) with parameter a polydisk \( \Delta \) with coordinates \( t = (t_j) \). First, for fixed \( x \in D \), I can let \( x \) vary holomorphically with \( t \) by writing
\[ \Delta \to D \]
\[ t \mapsto x + \varphi(t) \]
with \( \varphi(0) = 0. \) For any real-analytic function \( f(x, \overline{x}) \) on \( D \) the formula for the pullback is
\[ (f \circ \varphi)(t) = \exp \left( L_{\beta(t) + \overline{\beta(t)}} \right) (f) \]
where
\[ \beta(t) = \sum_i \varphi_i(t) \cdot \frac{\partial}{\partial x_i} = \sum_{i,j,|J|>0} a_{i,j} \cdot t^J \cdot \frac{\partial}{\partial x_i} \]
is the vector field associated to the one-parameter group action
\[ s \mapsto \varphi(t, s) = x + s \cdot \varphi(t). \]
We put these trajectories together real-analytically by defining
\[ G : D \times \Delta \to D \times \Delta \]
\[ (x, t) \mapsto (x + \varphi(t), t) \]
where \( G \) is holomorphic in \( t \) and real-analytic in \( x \). Thus, for any real-analytic function \( f(x, \overline{x}; t) \) on \( D \times \Delta \) which is holomorphic in \( t \) we have
\[ G^* (f) = \exp \left( L_{\beta(t) + \overline{\beta(t)}} \right) (f) \]
for
\[ \beta(x, \overline{x}; t) = \sum_i \varphi_i(x, \overline{x}; t) \cdot \frac{\partial}{\partial x_i} = \sum_{i,j,|J|>0} \left( a_{i,j}(x, \overline{x}) \cdot \frac{\partial}{\partial x_i} \right) \cdot t^J. \]
For the sheaf $\mathcal{R}$ of complex-valued real analytic functions on $D$ define the sheaf of rings
\[
\mathcal{C} = \left\{ \sum_{J,K} f_{J,K} \cdot t^J \bar{t}^K : f_{J,K} \in \mathcal{R} \right\} = \mathcal{R}[[t]]
\]
where \(\{t\}\) is the ideal generated the the functions $t_J$. Then
\[
G^*: \mathcal{R}[[t]] \rightarrow \mathcal{R}[[t]]
\]
\[
f \mapsto \exp (L_{\beta(t)}) (f).
\]

7.2. Deformations of the $\bar{\partial}$-operator. For $G$ as above, denote
\[
F_{\beta} = G^{-1}.
\]
Then, for $f \in \mathcal{R}[[t]]$, $F_{\beta}^*(f)$ is holomorphic if and only if
\[
\exp (L_{\beta(t)}) \circ \bar{\partial} \circ \exp (L_{-\beta(t)}) (f)
\]
\[
= \left( (\bar{\partial} \circ \exp (L_{\beta(t)}) - [\bar{\partial}, \exp (L_{\beta(t)})]) \circ \exp (L_{-\beta(t)}) \right) (f)
\]
\[
= \bar{\partial} + \left( [\exp (L_{\beta(t)}), \bar{\partial}] \circ \exp (L_{-\beta(t)}) \right) (f) = 0.
\]

We rewrite this equation as
\[
(\bar{\partial} + L_{\xi(t)})(f) = 0
\]
where
\[
L_{\xi(t)} = \left[ \exp (L_{\beta(t)}), \bar{\partial} \right] \circ \exp (L_{-\beta(t)}).
\]

To explain the notation $L_{\xi(t)}$, we begin with the formula
\[
L_{\beta}(f) = \langle \alpha | df \rangle
\]
for any real analytic $(1,0)$-vector field $\alpha$ on $D$, where $d = \partial + \bar{\partial}$ is exterior differentiation on $\mathcal{R}$. We will also need the corresponding formula for the algebra of $d$-differential forms $\mathcal{A}^*_D$ which is
\[
L_{\beta}(\omega) = \langle \alpha | d\omega \rangle + d \langle \alpha | \omega \rangle.
\]

More generally, for any
\[
\zeta \in \mathcal{A}^* \otimes T_D,
\]
we expand $\zeta$ as a sum of terms
\[
dx^I \cdot \alpha,
\]
define
\[
L_{dx^I \cdot \alpha} = dx^I \cdot L_{\alpha}
\]
and extend linearly to define
\[
L_{\zeta}.
\]
Define the Lie bracket of $dx^I \cdot \alpha \in \mathcal{A}^i \otimes T_D$ and $dx^{I'} \cdot \alpha' \in \mathcal{A}^{i'} \otimes T_D$ as
\[
dx^I \wedge dx^{I'} \cdot [\alpha, \alpha']
\]
and extend linearly.

For $\zeta \in \mathcal{A}^i \otimes T_D$ and $\zeta' \in \mathcal{A}^{i'} \otimes T_D$ we define
\[
[L_{\zeta}, L_{\zeta'}] = L_{\zeta} \circ L_{\zeta'} - (-1)^{ii'} L_{\zeta'} \circ L_{\zeta'}.
\]
Exercise 7.1. 1) For $\varsigma \in \mathcal{A}^* \otimes T_D$ and $\varsigma' \in \mathcal{A}^* \otimes T_D$

$$[L_{\varsigma}, L_{\varsigma'}] = L_{[\varsigma, \varsigma']}.$$  

2) $L = \mathcal{A}^* \otimes T_D$ with $d = \overline{\partial}$ and $[,]$ as above is a DGLA.

7.3. Deformations of a vector bundle over a disk. If I have a complex vector bundle

$$D \times V$$

over the polydisk $D$ then I wish to characterize those

$$G^* : \mathcal{R}[[t]] \to \mathcal{R}[[t]]$$

$$f \mapsto \exp \left( L_{\beta(t)} \right) (f).$$

which correspond to maps

$$\psi_x : V_x \to D \times V$$

which are morphisms of vector bundles. To do this we consider analytic sections of $D \times V$ as analytic functions $f$ on $D \times V^\vee$ which satisfy the condition of linearity. That is, for fixed $x \in D$, I can let $x$ vary holomorphically with $t$ by writing

$$\Delta \to D$$

$$t \mapsto x + \varphi(t)$$

with $\varphi(0) = 0$, and let $V_x$ vary by writing a holomorphic family of complex-linear maps

$$V_x^\vee \to V_x^\vee + \varphi(t)$$
deforming the identity on $V_x$. This is given by a holomorphic map

$$A : \Delta \to GL(V^\vee)$$

with $A(0) = 0$ and so, via the inverse of the map

$$\exp : gl(V^\vee) \to GL(V^\vee),$$
to a holomorphic map

$$\tilde{\beta} = (\alpha, \tilde{\beta}) : \Delta \to gl(V^\vee) \oplus T_{D,x}$$

As above we let $x$ move in $D$ to obtain

$$\tilde{\beta}(x, t)$$

and let $\overline{\partial}$ and Lie bracket act coordinate-wise on $gl(V^\vee) \oplus T_D$.

Let

$$\chi \in \mathcal{A}^0 \otimes T_{V^\vee}$$
denote the Euler vector field, that is, the vector field associated to the action of the one-parameter group $\mathbb{C}^*$ on $V^\vee$. If $\text{rank}(V) = 1$, linearity is just the condition that $\psi_x$ commute with the $\mathbb{C}^*$-action, that is,

$$[\chi, \tilde{\beta}(t)] = 0,$$

in other words

$$\alpha(t) = c(t) \cdot \chi$$

for some holomorphic function $c(t)$. For the higher-rank case we will later us the construction

$$|\mathcal{O}(-1)| \to \mathbb{P}(D \times V^\vee)$$
and an analytic map
\[
|\mathcal{O}(-1)| \quad \rightarrow \quad |\mathcal{O}(-1)|
\]
\[
\mathbb{P}(D \times \Delta \times V^\vee) \quad \rightarrow \quad \mathbb{P}(D \times \Delta \times V^\vee).
\]
\[
D \times \Delta \quad \rightarrow \quad D \times \Delta.
\]

As above, form the deformation
\[
\left(\exp(L_{\tilde{g}(\tilde{x},t)}) \circ \overline{\partial} \circ \exp\left(L_{-\tilde{g}(x,t)}\right)\right)(f)
\]
of the $\overline{\partial}$-operator on sections $f$ of the vector bundle $D \times V$, that is functions on $D \times V$ which are linear on $\{x\} \times V$ for each $x \in D$. Holomorphicity of sections is then given by the condition
\[
\left(\partial + L_{\tilde{g}(t)}(f) = 0\right)
\]
where $\tilde{\xi} (t) \in A_{D}^{0,1} \otimes (gl(V^\vee) \oplus T_{D})$. As above we have a DGLA
\[
\tilde{L}^i = A_{D}^{0,i} \otimes (gl(V^\vee) \oplus T_{D}).
\]
In fact, we have an exact sequence of DGLA’s
\[
0 \rightarrow \tilde{L}_{vb} \rightarrow \tilde{L} \rightarrow L_{D} \rightarrow 0
\]
corresponding to deformations of the complex structure on just the vector bundle, on the pair (space, vector bundle) or just on the space $D$.

7.4. **Maurer-Cartan equations.** For a element
\[
\xi (t) \in (A_{1} \otimes T_{D}) \otimes m_{\mathbb{C}[{[t]}]}
\]
to be such that
\[
\overline{\partial} + L_{\xi(t)}
\]
is a (relative) $\overline{\partial}$-operator for a complex structure on $D \times \Delta$, it is clearly necessary that
\[
\left(\overline{\partial} + L_{\xi(t)}\right) \circ \left(\overline{\partial} + L_{\xi(t)}\right) = 0.
\]
(Similarly for $\tilde{\xi} (t)$.) In fact, if we replace $\Delta$ by an Artinian scheme
\[
\Delta_{T} := Spec \mathbb{C}[{[t]}]/T
\]
at $t = 0$, this condition will turn out to be sufficient. We will call $\overline{\partial} + L_{\xi(t)}$ integrable if it satisfies (19).

First notice that $\xi (t)$ determines an almost complex structure on $D \times \{t\}$ as follows. Let
\[
T(D)
\]
denote the real tangent bundle of the real differentiable manifold $D$ with standard decompositions
\[
T(D) \otimes \mathbb{C} = T_{D}^{1,0} \oplus T_{D}^{0,1}
\]
\[
(T(D) \otimes \mathbb{C})^\vee \rightarrow A_{D}^{1,0} \oplus A_{D}^{0,1}.
\]
Let
\[
pr_{D}^{0,1} : (T(D) \otimes \mathbb{C})^\vee \rightarrow A_{D}^{0,1}
\]
denote the standard projection, define
\[ \Omega_t = \ker \left( \langle (D) \otimes \mathbb{C} \rangle^\vee \to A_{D}^{0,1} \right), \]
and write
\[ \mathcal{T}_t = \Omega_t^\vee. \]
Then
\[ A_1^{0,1} D = \Omega_t \oplus \Omega_t, \]
and projection onto the summands of this direct sum decomposition gives
\[ d = \partial_t + \overline{\partial}_t. \]

**Exercise 7.2.** The “(0, 1)” tangent distribution \( \Omega_t = T_t \) is given by the image of
\[ T_0 \to T(D) \otimes \mathbb{C}[[t]] \]
\[ \varsigma \mapsto \varsigma + \langle \varsigma | \xi \rangle. \]

**Exercise 7.3.** Let
\[ pr_{0,1}^t \]
denote the isomorphism from \( \Omega_t \to \Omega_t \) given by applying to \( \Omega_t \) the projection associated with the decomposition (20). Then, for a function \( f \) on \( M_0 \),
\[ pr_{0,1}^t \circ (\overline{\partial} + L_{\xi(t)}) f = \overline{\partial}_t f. \]

**Lemma 7.1.** The condition (19) is the condition
\[ [T_t, T_t] \subseteq T_t, \]
that is, the condition
\[ \partial \Omega_t \subseteq \Omega_t \wedge (\langle D \rangle \otimes \mathbb{C} \rangle^\vee, \]
that is, the condition
\[ \overline{\partial}_t \circ \partial_t = 0. \]

**Proof.** We first show that the condition \([T_t, T_t] \subseteq T_t\), or, what is the same, \([\overline{T_t}, T_t] \subseteq T_t\), is equivalent to (19) by showing thatLet \( (v^j) \) be a system of local holomorphic coordinates on \( D \). Locally
\[ \xi = \sum_j t_j^l h_{J,k}^j \partial_{v^l} \cdot \]
Then the image of \( \iota \) is framed locally by the vector fields
\[ \left( \frac{\partial}{\partial v^k} + \sum_{J,|J|>0} h_{J,k}^j t_J^l \frac{\partial}{\partial v^l} \right) \]
Using a slight adaptation of the Einstein summation convention, we have
\[ \left[ \left( \frac{\partial}{\partial v^r} + t^J h_{J,k}^l \frac{\partial}{\partial v^k} \right), \left( \frac{\partial}{\partial v^s} + t^J h_{J,k}^m \frac{\partial}{\partial v^k} \right) \right] = \]
\[ t^r \frac{\partial h_{J,k}^m}{\partial v^k} \frac{\partial}{\partial v^s} - t^r \frac{\partial h_{J,k}^m}{\partial v^k} \frac{\partial}{\partial v^s} + t^J h_{J,k}^m \frac{\partial}{\partial v^k} \frac{\partial}{\partial v^l} \]
So integrability is checked by pairing the above vector fields with
\[ dv^r + t^{J''} h_{J''}^{r,s} dv^s. \]
We get that integrability is equivalent to the identical vanishing of
\[ t^J \frac{\partial h^{r}_{r',k}}{\partial v^j} - t^J + t^{J'} \left( h^{l}_{l,j} \frac{\partial h^{r}_{r',k}}{\partial v^l} - h^{m}_{m,j} \frac{\partial h^{r}_{r',k}}{\partial v^m} \right), \]
that is, of
\[ t^J \frac{\partial h^{r}_{r',k}}{\partial v^j} d v^k d v^j - t^J + t^{J'} \left( h^{l}_{l,j} \frac{\partial h^{r}_{r',k}}{\partial v^l} - h^{m}_{m,j} \frac{\partial h^{r}_{r',k}}{\partial v^m} \right) d v^k d v^j, \]
which becomes the system of equations
\[ 0 = 2 d v^k d v^j \otimes \left( \frac{\partial h^{r}_{r',k}}{\partial v^j} \frac{\partial}{\partial v^j} \right) + \sum_{J'} \frac{\partial h^{r}_{r',k}}{\partial v^l} d v^j \otimes \left( h^{l}_{l,j} \frac{\partial h^{r}_{r',k}}{\partial v^l} - h^{m}_{m,j} \frac{\partial h^{r}_{r',k}}{\partial v^m} \right) \]
\[ = 2 d v^k d v^j \otimes \left( \frac{\partial h^{r}_{r',k}}{\partial v^j} \frac{\partial}{\partial v^j} \right) + \sum_{J'} \frac{\partial h^{r}_{r',k}}{\partial v^l} d v^j \otimes \left[ h^{l}_{l,j} \frac{\partial}{\partial v^l}, h^{m}_{m,j} \frac{\partial}{\partial v^m} \right]. \]
The passage from \([T_t, T_t] \subseteq T_t \otimes (T (D) \otimes \mathbb{C}) \) is a standard fact about distributions from differential geometry. But I can rewrite this last condition in the form
\[ \overline{\partial}_t \overline{\partial} = 0, \]
which is the same as the condition \( \overline{\partial}_t \circ \overline{\partial}_t = 0. \]

In the case \( D \times V^\vee \) everything is the same. If we have, for
\[ \tilde{\xi} = (\eta, \xi) \]
that
\[ \xi = 0 \]
then
\[ \eta (t) \in \left( T^0_{0,1} \right)^{\vee} \otimes gl (V^\vee) \]
gives rise, by the above process to a deformation
\[ d = D_t^{1,0} + D_t^{0,1} \]
of the standard decomposition
\[ d = \partial + \overline{\partial} \]
of the standard flat connection on \( D \times V \) induced by \( d \) on \( D \) and the identity map on \( V \). Notice that, even though we “compensate” for the deformation \( \overline{\partial} + L_{\eta (t)} \) of \( \overline{\partial} \) by defining the deformation of \( \partial \) as \( \partial - L_{\eta (t)} \) so that the deformation of \( d = \partial + \overline{\partial} \) is trivial, the condition
\[ (\overline{\partial} + L_{\eta (t)}) \cdot (\overline{\partial} + L_{\eta (t)}) = 0 \]
is not vacuous, even over \( \Delta_{m^2}. \)

There it becomes the condition
\[ \overline{\partial} \eta \equiv \Delta_{m^2} 0. \]

More generally we could take any family of connections
\[ D_t : V \rightarrow V \otimes (T (D) \otimes \mathbb{C}) \]
and decompose $D_t$ into $D_t^{1,0} + D_t^{0,1}$ using (20). The Maurer-Cartan condition then becomes the condition that the $(0, 2)$-component of the curvature of $D_t$ vanishes.

8. Globalization

8.1. Stein manifolds. To globalize the above local computations, we first need to recall some fundamental facts about Stein varieties, the analytic equivalent of affine varieties in algebraic geometry.

**Definition 8.1.** Let $\Omega$ be a complex domain, that is, a possibly non-compact complex manifold. A function $\rho : \Omega \to \mathbb{R}$ is called **pluri-subharmonic** ($psh$) if at each point $p \in \Omega$ with local holomorphic coordinates $(z_j)$, the hermitian form

$$
\sum_{j, k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} dz_j \otimes d\bar{z}_k
$$

is non-negative. If in fact this form is positive definite, $f$ is called **strictly pluri-subharmonic**.

We let

- $\mathcal{P}(\Omega) = \{\rho : \Omega \to \mathbb{R}, \rho \text{ psh}\}$
- $\mathcal{O}(\Omega) = \{f : \Omega \to \mathbb{C}, f \text{ holomorphic}\}$
- $\mathcal{C}^\infty(\Omega) = \{f : \Omega \to \mathbb{C}, f \in \mathcal{C}^\infty\}$
- $\mathcal{A}^{p,q}(\Omega) = \{\omega : \Omega \to T^{p,q}_\Omega, \text{ $\omega$ a } \mathcal{C}^\infty \text{-section}\}$.

**Definition 8.2.** $\Omega$ is called **holomorphically convex** if, for each compact subset $K \subseteq \Omega$, the holomorphically convex hull of $K$, that is,

$$K := \left\{p \in \Omega : \forall f \in \mathcal{O}(\Omega), \ |f(p)| \leq \sup_{k \in K} |f(k)| \right\}
$$

is also compact.

$\Omega$ is called **psh-convex** if, for each compact subset $K \subseteq \Omega$, the **psh-convex hull** of $K$, that is,

$$\hat{K} := \left\{p \in \Omega : \forall \rho \in \mathcal{P}(\Omega), \ \rho(p) \leq \sup_{k \in K} \rho(k) \right\}
$$

is also compact.

Suppose now that $\Omega$ admits a single pluri-subharmonic function $\rho$ such that such that, for every $a \in \mathbb{R}$, the set

$$K_{\rho, a} := \rho^{-1}((\mathbb{R}, a]) = \{p \in \Omega : \rho(p) \leq a\}
$$

is compact. We call $\rho$ a **pluri-subharmonic exhaustion function** for $\Omega$. If $\Omega \subseteq \mathbb{C}^N$ as a closed submanifold, then it is easy to see that

$$\rho(x) = |x|^2
$$

is a strictly pluri-subharmonic exhaustion function on $\Omega$. It is easy to see that if $\Omega$ has a pluri-subharmonic exhaustion function, then $\Omega$ is **psh-convex**.
**Definition 8.3.** $\Omega$ is called **Stein** if
1) $\Omega$ is holomorphically convex,
2) $\mathcal{O}(\Omega)$ separates points on $\Omega$, 
3) $\forall p \in \Omega$ and corresponding maximal ideal $m_p \leq \mathcal{O}(\Omega)$, 
   \[ \dim \frac{m_p}{m_p^2} = \dim \Omega. \]

Clearly, if $\Omega$ can be embedded as a closed submanifold of $\mathbb{C}^N$, then $\Omega$ is Stein.

Given a measure $dV$ on $\Omega$, let 
$L^p,q_{2,\text{loc.}}(\Omega)$ denote the space of $(p,q)$-forms on $\Omega$ which are locally square-integrable with respect to $dV$.

We next quote four important theorems from complex analysis:

**Theorem 8.1.** (Solution to $\overline{\partial}$). Suppose that $\Omega$ with hermitian metric $h$ has a pluri-subharmonic exhaustion function $\rho$. Then there exists a convex increasing function 
$\chi : \mathbb{R} \to \mathbb{R}$
with associated norm 
$\|g\|_{\chi}^2 := \int_{\Omega} |g|^2 e^{-\chi \rho} dV$
such that the $\overline{\partial}$-equation has a weak solution on $\Omega$. That is, given $\omega \in L^{p,q+1}_{2,\text{loc.}}(\Omega)$ such that, as a distribution on $\Omega$, 
$\overline{\partial}\omega = 0$, 
then there exists $\eta \in L^{p,q}_{2,\text{loc.}}(\Omega)$ with 
$\|\eta\|_{\chi} \leq \|\omega\|_{\chi}$
such that, as distributions on $\Omega$, 
$\overline{\partial}\eta = \omega$.

**Theorem 8.2.** (Regularity) If, in the previous Theorem, $\omega \in A^{p,q+1}(\Omega)$, then $\eta \in A^{p,q}(\Omega)$.

**Theorem 8.3.** (Holomorphic pointwise bounds) Let $K^{\text{cpt.}} \subseteq \Omega' \subseteq \Omega$ and let $f$ be holomorphic on $\Omega$. Then 
$\sup_{K} |\partial^j f| \leq c_{\Omega',1} \|f\|_{L^2(\Omega')}$ where $c_{\Omega',1}$ is independent of $f$.

**Theorem 8.4.** (Approximation) Suppose a complex domain $\Omega$ admits a strictly-psh exhaustion function $\rho$. If $f$ is a function which is holomorphic in a neighborhood of $K_{\rho,a} \subseteq \Omega$, then for each $\varepsilon > 0$ there is a $g \in \mathcal{O}(\Omega)$ such that 
$\sup_{k \in K} |g(k) - f(k)| < \varepsilon$.

**Proof.** (Outline) Show approximation first for compact $K \subseteq \mathbb{C}^N$ and then use a finite cover of a general $K_{\rho,a}$ and a patching argument. \hfill \Box

These theorems are used to show the following:

**Theorem 8.5.** Suppose a complex domain $\Omega$ admits a strictly-psh exhaustion function $\rho$. Then $\Omega$ can be embedded as a closed submanifold of $\mathbb{C}^N$. 

To prove this theorem, one begins with the following lemma.

**Lemma 8.6.** There exists a neighborhood $W_p$ of $p$ in $\Omega$ and a polynomial function $u_p \in m_p$ which is a polynomial of degree $\leq 2$ in local coordinates $(x_j)$ in $m_p$ such that, for all $q \in (W_p - \{p\})$,

$$ \text{Re} \left( u_p(q) \right) < (\rho(q) - \rho(p)). $$

**Proof.** Use the Taylor expansion

$$ \rho(q) - \rho(p) = \sum_j \frac{\partial \rho}{\partial x_j}(p) x_j + \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(p) x_j x_k + \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(p) x_j x_k + \ldots $$

Now the fact that $\rho$ is psh implies that the term $\left( \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(p) x_j x_k \right) > 0. \quad \square$

**Proof.** (of Theorem 8.5, outline) The proof of Theorem 8.5 reduces to the proving the following three properties:

1') for any $p \in \Omega$ and $a < \rho(p)$, then $p \notin \overline{K_{p,a}}$, that is, there exists an $f \in \mathcal{O}(\Omega)$ such that $|f(p)| > \sup_{k \in K_{p,a}} |f(k)|$. Thus

$$ \overline{K_{p,a}} = K_{p,a}. $$

2') Given $p$ as in 1') and $q \in (K_{p,\rho(p)} - \{p\})$ then $\mathcal{O}(\Omega)$ separates $p$ and $q$.

3') $\forall p \in \Omega$ and corresponding maximal ideal $m_p \subset \mathcal{O}(\Omega)$,

$$ \dim \frac{m_p}{m_p^2} = \dim \Omega. $$

To prove 1'), for given $a' < \rho(p)$ choose a system

$$ W''_p \subset W' \subset W_p \subset W'' $$

of neighborhoods of $p$ with compact closure such that

$$ W'_p \cap K_{p,a'} = \emptyset. $$

Then for $q$ in the compact set

$$ (W'_p - W''_p) \cap K_{p,\rho(p)} $$

we must have

(22) $$ \text{Re} \left( u_p(q) \right) < 0 $$

on the compact set, so that there is some $a > \rho(p)$ such that the same is true for the compact set

$$ (W'_p - W''_p) \cap K_{p,a}. $$

Thus there is an $\varepsilon > 0$ such that, for $q \in (W'_p - W''_p) \cap K_{p,a}$, $\text{Re} \left( u_p(q) \right) < -\varepsilon$.

Next let

$$ \psi : \Omega \to [0, 1] $$

be a $C^\infty$ cut-off function such that $\overline{W'_p} \subseteq \psi^{-1}(1)$ and $(\Omega - W'_p) \subseteq \psi^{-1}(0)$. Now choose $a > \rho(p)$ and $\varepsilon > 0$ as above. That is, for all $q \in \text{supp}(\overline{\partial}\psi)$ for which $\rho(q) < a$, (22) holds. Pick $u \in \mathcal{O}(W_p)$ and extend $g$ to a $C^\infty$-function on $\Omega$. Let $\Omega_a = \rho^{-1}((-\infty, a))$. Then on $\Omega_a$ define

$$ \omega_t = \overline{\partial} (\psi \cdot u \cdot e^{tu}) = g \cdot e^{tu} \cdot \overline{\partial}\psi. $$
for $t \gg 0$ and let

$$\rho_a = \frac{1}{a - \rho}.$$  

Using the Solution-to-$\overline{\partial}$ Theorem for $(\Omega_a, \rho_a)$ and the Regularity Theorem, there is a $C^\infty$-solution $u_t$ on $\Omega_a$ to the equation

$$\omega_t = \overline{\partial} u_t.$$  

such that

$$\|u_t\|_{\chi_a} \leq \|\omega_t\|_{\chi_a} = \int_{\Omega_a} |\overline{\partial} (\psi \cdot g \cdot e^{t\psi} u_t)|^2 e^{-\chi_a} dV < \text{const.} \cdot e^{-2\varepsilon t}.$$  

Thus

$$\int_{\Omega_a} |u_t|^2 e^{-\chi_a} dV < \text{const.} \cdot e^{-2\varepsilon t}.$$  

Define

$$v_t := (\psi \cdot u \cdot e^{t\psi} - u_t) \in \mathcal{O}(\Omega_a).$$  

On the other hand, for $a' < \rho(p)$ we have that $u_t$ is holomorphic on a neighborhood of $\Omega_{a'}$ so one can use the Holomorphic Pointwise Bound Theorem above to conclude that $u_t$ converges uniformly to zero on $\Omega_{a'}$. So $v_t$ also converges uniformly to 0 on $\Omega_{a'}$. On the other hand, the same theorem implies $u_t(p)$ converges to zero so that $v_t(p)$ converges to $u(p)$ which can be chosen arbitrarily. So taking $t \gg 0$ we have the analogue of 1') for $\Omega_a$ in place of $\Omega$. But now use the Approximation Theorem to approximate the holomorphic function $v_t$ on $(K_{\rho, a/2^{j+1}} \subseteq \Omega$ by an element of $\mathcal{O}(\Omega)$, thereby completing the proof of 1').

To prove 2') we simply take $W_p$ small enough around $p$ that it excludes $q$. Then use the same argument as in the proof of 1') to show that $u_t$ converges to 0 at both $p$ and $q$.

For 3') choose for $g$ in the proof of 1') a set $u^i$ of local holomorphic coordinates centered at $p$. Since, for any such coordinate, $u(p) = 0$, we have

$$\partial u_t(p) = \partial u(p) - \partial v_t(p)$$

since $u(p) = 0$. But, proceeding as in the proof of 1') the Holomorphic Pointwise Boundedness Theorem gives that $\partial u_t(p)$ converges to zero as $t \to +\infty$. Applying this reasoning to $u = u^i$ for each $i$ we conclude that

$$\det \left( \frac{\partial v_t^i}{\partial u^j} \right) \to 1$$

as $t \to +\infty$. Thus for $t \gg 0$ the $(v_t^i)$ give local coordinates at $p$ which are holomorphic on $\Omega_a$. But again use the Approximation Theorem to approximate the holomorphic functions $v_t^i$ on $(K_{\rho, a/2^{j+1}} \subseteq \Omega$ by elements of $\mathcal{O}(\Omega)$. These approximate functions must still have Jacobian determinant near 1 with respect to the $(u^i)$-coordinate system, thereby completing the proof of 3'). The proof of Theorem 8.5 is therefore complete.

Thus any Stein manifold can be embedded as a closed submanifold of complex euclidean space.

**Theorem 8.7.** (Grauert) A paracompact real-analytic manifold $M_R$ can be real-analytically embedded in $\mathbb{R}^N$.  

**DEFORMATION THEORY**
Proof. Form a locally finite covering of \( M_{R} \) with real-analytic (relatively compact) coordinate polydisks \( U_{\alpha,R} \). For the real power series giving the transition functions

\[
\varphi_{\alpha\beta,R}: U_{\alpha\beta,R} \rightarrow U_{\beta\alpha,R}
\]

complexify the variables in domain and range to get complex-valued transition functions

\[
\varphi_{\alpha\beta}: U_{\alpha\beta} \rightarrow U_{\beta\alpha}
\]

which fit together to give a complex manifold \( T \) with locally finite covering \( U_{\alpha} \) in which \( M_{R} \) sits as a closed real-analytic submanifold with \( T \) a tubular neighborhood of \( M_{R} \) with regular boundary. On each \( U_{\alpha} \) denote the coordinates as

\[
z_{j}^{\alpha} = x_{2j-1} + ix_{2j}.
\]

We show that some subdomain \( T' \subseteq T \) with closure can be chosen to admit a pluri-subharmonic exhaustion function and therefore by the previous theorem \( T' \) is a Stein manifold which can be realized as a closed submanifold of \( \mathbb{C}^{N} \). To construct the requisite \( psh \) exhaustion function, form the function

\[
p_{\alpha}(z^{\alpha}) = 2 \sum_{j} (x_{2j}^{\alpha})^{2} = -\sum_{j} (\overline{z}_{j}^{\alpha} - z_{j}^{\alpha})^{2}
\]
on \( U_{\alpha} \). The Levi form

\[
L_{p_{\alpha}} = \sum_{j,k} \frac{\partial^{2}p_{\alpha}}{\partial z_{j}^{\alpha} \partial \overline{z}_{k}^{\alpha}} dz_{j}^{\alpha} \otimes d\overline{z}_{k}^{\alpha} = 2 \sum_{j} dz_{j}^{\alpha} \otimes d\overline{z}_{j}^{\alpha}
\]
is strictly \( psh \). Notice that \( p_{\alpha}(z^{\alpha}) = 0 \) iff \( dp_{\alpha}(z^{\alpha}) = 0 \) iff \( z^{\alpha} \in U_{\alpha,R} \). Also notice that, for \( \sum_{j} (z_{j}^{\alpha} + \overline{z}_{j}^{\alpha})^{2} = 4 \sum_{j} (x_{2j}^{\alpha})^{2} \) we have

\[
\sum_{j,k} \frac{\partial^{2}}{\partial z_{j}^{\alpha} \partial \overline{z}_{k}^{\alpha}} \sum_{j} (z_{j}^{\alpha} + \overline{z}_{j}^{\alpha})^{2} dz_{j}^{\alpha} \otimes d\overline{z}_{k}^{\alpha} = 2 \sum_{j,k} \frac{\partial^{2}}{\partial z_{j}^{\alpha} \partial \overline{z}_{j}^{\alpha}} (z_{j}^{\alpha} \cdot \overline{z}_{j}^{\alpha}) dz_{j}^{\alpha} \otimes d\overline{z}_{j}^{\alpha} = 2 \sum_{j,k} dz_{j}^{\alpha} \otimes d\overline{z}_{j}^{\alpha}.
\]

We define, for an open set \( U \) in \( T \), the notion of a (strong) \( p \)-function, that is, a non-negative real-valued function which vanishes exactly on \( M_{R} \cap U \), whose differential \( dp \) vanishes exactly on \( M_{R} \cap U \), and whose Levi form is non-negative (positive) at points of \( M_{R} \cap U \). So \( p_{\alpha} \) above is such a \( p \)-function on the open set \( U_{\alpha} \). One easily checks that, if \( r \) is a non-negative real-valued \( C^\infty \)-function of \( U \) and \( f \) and \( g \) are \( p \)-functions, then \( f + g \) and \( r \cdot f \) are also \( p \)-functions.

We make the following claim:

Claim 1: Given any real-valued function \( R \) on \( T' \) there is a strong \( p \)-function on \( T' \) such that \( p + R \) is strictly pluri-subharmonic on \( T' \).

For this we will need an appropriate partition-of-unity which we construct as follows. For a \( C^\infty \)-projection

\[
\pi: T \rightarrow M_{R}
\]
let \( T_{\alpha} = \pi^{-1}(U_{\alpha}) \) and assume, as we always can, that we have chosen \( V_{\alpha} \subseteq \overline{V_{\alpha}} \subseteq T_{\alpha} \) such that the \( V_{\alpha} \) form a locally finite, countable cover of \( M_{R} \) by open sets in \( T \) such that each \( \overline{V_{\alpha}} \) is compact and such that \( \bigcup V_{\alpha} \supseteq T' \) for some tubular neighborhood \( T' \) of \( M_{R} \). We construct a \( C^\infty \)-partition-of-unity \( \{\rho_{\alpha}\} \) such that the support of \( \rho_{\alpha} \) is a compact subset of \( T_{\alpha} \), such that

\[
\rho_{\alpha}|_{T'} = \rho_{\alpha}|_{M_{R}} \circ \pi|_{T'}
\]
such that the support of $\rho_\alpha$ contains

$$T' \cap \pi^{-1}(V_\alpha).$$

Then, for any positive real constant $c_\alpha$, we have at points of $V_\alpha \cap M_\alpha$ that

$$L(c_\alpha \rho_\alpha p_\alpha) = c_\alpha \rho_\alpha L(p_\alpha)$$

so that, by choosing $c_\alpha$ large enough, we can make $L(c_\alpha \rho_\alpha p_\alpha)$ as positive as we want on any compact subset of the interior of the support of $\rho_\alpha$. Setting

$$p = \sum_\alpha c_\alpha \rho_\alpha p_\alpha$$

gives Claim 1.

We want $R$ appearing in Claim 1 to be an exhaustion function on some neighborhood $U'$ of $T'$ in $T$. We construct $R$ as follows. For the partition-of-unity $\{\rho_\alpha\}$ constructed above, if we assume as we can that the indices $\{\alpha\}$ are a subset of the positive integers, then the function

$$R = \sum_\alpha \alpha \cdot \rho_\alpha + \sum_\alpha \left(\frac{(\rho_\alpha \circ \pi) - \rho_\alpha}{1 - \sum_\alpha (\rho_\alpha \circ \pi) - \rho_\alpha}\right)$$

is a positive-valued $C^\infty$ exhaustion function on the open neighborhood $U'$ of $T'$ in $T$ given by

$$\sum_\alpha (\rho_\alpha \circ \pi) - \rho_\alpha < 1.$$

That is, $R^{-1}(r)$ is compact, for each $r \in \mathbb{R}^+$. We denote the spsh function given by Claim 1 for this $R$ as $p_1$.

Next we have:

Claim 2: There exists a spsh-function $p'$ on $T'$ such that $p'$ is sphs except at points of the set where it is 0.

To see this claim, let

$$z_j^\alpha = x_{2j}^\alpha + ix_{2j}^\alpha$$

be local holomorphic coordinates for $U_\alpha$. Pick a later to-be-determined set

$$a^\nu = (a_j^\nu) \in V_\alpha(\nu) \cap M_\mathbb{R}$$

such that, for each $\alpha$, $\{\nu: \alpha(\nu) = \alpha\}$ is finite and define

$$\tilde{p}_{a^\nu} = 2 \sum_j (z_{2j}^\alpha)^2 - \sum_j |x_{2j-1}^\alpha - a_j^\nu|^2$$

and let

$$K_{a^\nu}^+ = \left\{z^\alpha(\nu): \tilde{p}_{a^\nu}\left(z^\alpha(\nu)\right) > 0\right\}$$

$$K_{a^\nu}^- = \left\{z^\alpha(\nu): \tilde{p}_{a^\nu}\left(z^\alpha(\nu)\right) < 0\right\}.$$ 

Then, for all $z^\alpha(\nu) \in T_\alpha(\nu) \cap (M_\mathbb{R} - \{a^\nu\})$ we have that $\tilde{p}_{a^\nu}\left(z^\alpha(\nu)\right) < 0$ but by making the shape of the $V_\alpha$ sufficiently small in the imaginary direction we can achieve that

$$V_{a^\nu} \cap \partial K_{a^\nu}^+ = V_{a^\nu} \cap \partial K_{a^\nu}^-$$

exits $V_{a^\nu}$ outside $T'$. Also

$$L \tilde{p}_{a^\nu} = \frac{1}{2} \sum_j dz_j^\alpha \otimes d\overline{z}_j^\alpha.$$
So \( \bar{p}_{\nu} \) is sesh on \( V_{\nu} \cap \overline{T'} \). We next define

\[
p_{\nu'} = \begin{cases} 
0 & \text{outside } K_{\nu}^+ \\
1 & \text{on } K_{\nu}^+ 
\end{cases}.
\]

So \( p_{\nu'} \) is \( C^\infty \) on \( \overline{T'} \) and vanishes on \( M_\mathbb{R} \). Now choose enough \( a_{\nu} \) so that some \( p_{a_{\nu}} \) is positive at each point of \( \partial T' \). Then for \( c_{\nu} \gg 0 \) the function

\[
p_2 = \sum_{\nu} c_{\nu} p_{a_{\nu}}
\]

will be \( \approx 1 \) on \( \partial T' \) and \( V' = \{ x \in T' : p_2 (x) < 1 \} \) will be an open neighborhood of \( M_\mathbb{R} \) inside \( T' \) such that \( \partial T' \cap \partial V' = \emptyset \). Then, referring to the function \( p_1 \) constructed in Claim 1, we have that

\[
p_1 + \frac{1}{1 - p_2 (x)}
\]

is the desired psh exhaustion function on \( V' \). \( \square \)

8.2. Real-analytic structure. Suppose now that I have a holomorphic family of compact complex manifolds

\[
(23) \quad \pi : M \to \Delta.
\]

The complex structure on \( M \) induces a real-analytic structure on \( M \) whose functions are all power series in \( x, \bar{x}, t, \bar{t} \).

**Theorem 8.8.** (Nash) Let \( \Delta_\mathbb{R} \) be a small real polydisk. Let

\[
\pi : M_\mathbb{R} \to \Delta_\mathbb{R}
\]

be a real-analytic family of compact real-analytic manifolds. Then this deformation is trivial, that is, there is a real-analytic isomorphism

\[
F_\mathbb{R} : M_\mathbb{R} \to M_0 \times \Delta_\mathbb{R}
\]

defined over \( \Delta_\mathbb{R} \).

**Proof.** By the previous theorem of Grauert \( M_\mathbb{R} \) is realized as a closed, real-analytic submanifold of \( \mathbb{R}^{2N} \). Restricting the Euclidean metric of \( \mathbb{R}^{2N} \) to \( M_\mathbb{R} \) endows the latter with a real-analytic metric.

Let

\[
V \subseteq M_0 \times \mathbb{R}^{2N}
\]

be the set of points

\[
(p, x)
\]

such that

\[
(x - p) \perp T_{M_0, p}.
\]

Now \( M_0 \) is compact. An easy calculus exercise shows that, for \( x \) in a sufficiently small neighborhood of \( M_0 \), the fiber of the projection \( pr_{M_0} : V \to M_0 \) is exactly those points \((p, x) \in V\) such that the square-distance of \( x \) from \( M_0 \) is exactly \( ||x - p||^2 \). Since the equations defining \( V \) are real analytic, \( V \) is a real-analytic manifold. Again the map

\[
pr_{\mathbb{R}^{2N}} : V \to \mathbb{R}^{2N}
\]

is immersive by the Jacobian criterion and therefore, shrinking \( V \) if necessary, is a real-analytic diffeomorphism onto its image, which is an open set \( V' \) in \( \mathbb{R}^{2N} \) with smooth projection \( \pi' : V' \to M_0 \). Now, \( V' \) is open and \( M_0 \) is compact, then
for $t \in \Delta_R$ sufficiently small $M_t \subseteq V'$ and $\pi'|_{M_t} : M_t \rightarrow M_0$ is a real-analytic diffeomorphism by the Jacobian criterion. Shrinking $\Delta_R$ if necessary, we have a real-analytic map

$$M_R \rightarrow V' \xrightarrow{\pi', \pi} M_0 \times \Delta_R$$

is then seen to be a diffeomorphism by the Jacobian criterion.

In fact the above theorem can be extended to hold when $\Delta_R$ is replaced by any base $B$ which is a real-analytic manifold admitting a real-analytic vector field $\tau$ whose associated family of diffeomorphisms is parametrized by $R$ and all of whose flow lines meet at a single fixpoint 0. The ideal is again to embed $M_R$ as a closed, real-analytic submanifold of $R^{2N}$ and lift the vector field $\tau$ uniquely to a vector field $\tilde{\tau}$ on $M_R$ such that, at each point $p \in M_t$, $\tilde{\tau}(p) \perp T_{M_t, p}$ and $\pi^*(\tilde{\tau}(p)) = \tau(\pi(p))$. The family of diffeomorphisms associated to $\tilde{\tau}$ gives the desired real-analytic trivialization.

**Theorem 8.9.** Referring to (23) there is a real-analytic isomorphism

$$F : M \rightarrow M_0 \times \Delta$$

defined over $\Delta$ which is transversely holomorphic, that is, for any point $x \in M$,

$$F^{-1}(\{x\} \times \Delta)$$

is an analytic submanifold of $M$ which maps isomorphically to $\Delta$ under the mapping $\pi$. The same result holds if $\Delta$ is replaced by an Artinian scheme $\Delta_T$.

**Proof.** Let $M_R$ denote $M$ considered as a real-analytic manifold and let

$$M \xrightarrow{(G, \varphi)} M_0 \times \Delta$$

be the real-analytic trivialization given by the previous theorem. We consider the complex polydisk $\Delta$ as the product

$$\Delta = \Delta_R \times \sqrt{-1} \cdot \Delta_R$$

of its real and imaginary parts and let $N \subseteq M$ be the preimage of $\Delta_R$. Let

$$F = (G|_N)^{-1} : M_0 \times \Delta_R \rightarrow N.$$ 

On a small neighborhood $U_p$ of $p \in M_0$ we let $((z_i), (t_j + \sqrt{-1} \cdot u_j))$ be holomorphic coordinates for $M$. Write $F$ as a power series

$$z_i' = F_i(w_i, \overline{w_i}, t_j)$$

where

$$w_i = z_i|_{U_p \cap M_0}.$$ 

These power series patch together on the intersection of coordinate neighborhoods because $F$ is globally defined. That means that the power series for $F_i^{U_p}$ and for $F_i^{U_p'}$, when written in the same coordinate system, are the same. (Notice that the functions $t_j$ do not change when passing for $U_p$ to $U_p'$.) Now extend scalars by letting the variables $(t_j)$ take complexified values $(t_j + \sqrt{-1} \cdot u_j)$. The extended power series converge for $(t_j + \sqrt{-1} \cdot u_j)$ small and patch together on intersections because the power series for $F_i^{U_p}$ and for $F_i^{U_p'}$, when written in the same coordinate system, are the same. The changes of coordinates from $U_p$ to $U_p'$ are holomorphic.
so that the complex structure on $M$ is not changed, just the maps $G^{-1}$, and these only at points of $M$ not on $N$. By construction the new $F_i'$ are holomorphic in $(t_j + \sqrt{-1} \cdot u_j)$. □

**Corollary 8.10.** The family $\pi : M \to \Delta$, together with its transversely holomorphic trivialization, defines an integrable deformation

$$\overline{\Omega} + L_{\xi(t)}$$

of the $\overline{\Omega}$-operator on $M_0$. The same result holds if $\Delta$ is replaced by an Artinian scheme $\Delta_I$.

**Corollary 8.11.** Two deformations $M/\Delta$ and $M'/\Delta$ of the same central fiber $M_0$ with respective transversely holomorphic trivializations $F$ and $F'$ are equivalent, that is, there is a commutative diagram

$\begin{array}{ccc}
M & \xrightarrow{\pi} & M_0 \times \Delta \\
\downarrow H & & \downarrow \cong \\
M' & \xrightarrow{\pi'} & M_0 \times \Delta
\end{array}$

with $H$ holomorphic if and only if

$$\xi = \xi',$$

that is, they give the same deformation of $\overline{\Omega}_{M_0}$. The same result holds if $\Delta$ is replaced by an Artinian scheme $\Delta_I$.

**Proof.** $H$ is holomorphic if and only if $H_* \left( T_{M}^{1,0} \right) \subseteq T_{M'}^{1,0}$ if and only if $T_i = T'_i$ if and only if $\ker \left( pr_D^{0,1} - \langle \xi(t) \mid \rangle \right) = \ker \left( pr_D^{0,1} - \langle \xi'(t) \mid \rangle \right)$ if and only if $\xi = \xi'$. □

### 8.3. Estimates for convergence proof.

In this section we will need a long list of estimates on the Sobolev space of sections of a vector bundle $E$ on a compact hermitian manifold $M_0$. We fix a hermitian metric $h$ on $M_0$ on the holomorphic tangent bundle $T_{M_0}$. Standard harmonic theory allows us to define an (elliptic) Laplacian

$$\Box = \partial \overline{\partial} + H \overline{\partial} : A^{0,q}_{M_0} (T_{M_0}) \to A^{0,q}_{M_0} (T_{M_0})$$

and an orthogonal decomposition

$$A^{0,q}_{M_0} (T_{M_0}) = H^{0,q} (T_{M_0}) + \Box \left( A^{0,q}_{M_0} (T_{M_0}) \right).$$

We will denote projection onto the first or ‘harmonic’ summand by $H$.

Also there is a Green’s operator $G : A^{0,q}_{M_0} (T_{M_0}) \to A^{0,q}_{M_0} (T_{M_0})$ with

$$\Box \circ G = id \cdot \Box \left( A^{0,q}_{M_0} (T_{M_0}) \right).$$

Applying $\Box$ to both sides of the above decomposition and using that $\Box H^{0,q} \left( T_{M_0}^{1,0} \right) = 0$ we can refine the decomposition to read

$$A^{0,q}_{M_0} (T_{M_0}) = H^{0,q} (T_{M_0}) + \Box \left( A^{0,q}_{M_0} (T_{M_0}) \right).$$

Also

$$[G, \overline{\partial}] = \left[ G, \overline{\partial} \right] = 0.$$
We fix a finite open cover $U$ of $M_0$ by disks $U$ with compact closure in a neighborhood of which we have local holomorphic coordinates

$$z^j_U = x^{2j-1}_U + i x^{2j}_U$$

over which the bundle $E$ is trivial with local coordinates

$$h_{U,j}.$$

For $0 < \alpha < 1$ we denote the $m$-norm on sections as

$$\|\psi\|_{m,\alpha} = \max_{U, U' \in U} \max_{j,k} \|h_{U,j}\|_{m,\alpha}$$

where the (Hölder) norm on a $C^k$-function $f$ on a set $V$ is given by

$$\|f\|_{m,\alpha} = \sum_{|J| \leq m} \sup_{x \in V} \left| D^I f(x) \right| + \sum_{|J| = m} \sup_{(x,y) \in V \times V} \left| D^I f(x) - D^I f(y) \right| |x-y|^\alpha.$$

There is a long list of estimates for norms on sections of hermitian vector bundles on compact complex hermitian manifolds. We list several we will need below:

(24) $|\langle \psi, \varphi \rangle| \leq C \cdot \|\psi\|_{0,\alpha} \cdot \|\varphi\|_{0,\alpha}$

(25) $\|\psi\|_{m+2,\alpha} \leq C_1 \left( \|\Box \psi\|_{m,\alpha} + \|\psi\|_{0,\alpha} \right)$

(26) $\|G \psi\|_{m+2,\alpha} \leq C_2 \left( \|\psi\|_{m,\alpha} + \|H \psi\|_{m,\alpha} \right)$

(27) $\left\| \partial^* \psi \right\|_{m,\alpha} \leq C_3 \left( \|\psi\|_{m+1,\alpha} \right)$

(28) $\|\langle \psi, \varphi \rangle\|_{m,\alpha} \leq C_4 \left( \|\psi\|_{m+1,\alpha} \cdot \|\varphi\|_{m+1,\alpha} \right)$

(29) $\|H \psi\|_{m,\alpha} \leq C_5 \cdot \|H \psi\|_{0,\alpha}.$

The proofs of most of these estimates follow a similar pattern. For example, to prove (26) use (25) to get

$$\|G \psi\|_{m+2,\alpha} \leq C_1 \left( \|\Box G \psi\|_{m,\alpha} + \|G \psi\|_{0,\alpha} \right).$$

Then substitute

$$\psi = H \psi + \Box G \psi$$

to obtain

$$\|G \psi\|_{m+2,\alpha} \leq C_1 \left( \|\psi\|_{m,\alpha} + \|H \psi\|_{m,\alpha} + \|G \psi\|_{0,\alpha} \right).$$

So we will be done if we can show

$$\|G \psi\|_{0,\alpha} \leq C \left( \|\psi\|_{m,\alpha} + \|H \psi\|_{m,\alpha} \right).$$

To see this use

$$G \psi = G (\psi - H \psi)$$

so we can replace $\psi$ by $\psi - H \psi$ and show, if $H \psi = 0$ then

$$\|G \psi\|_{0,\alpha} \leq \|\psi\|_{m,\alpha}.$$
Suppose not. Then we can find a sequence $\psi_n$ with $H\psi_n = 0$ such that $\|G\psi_n\|_0 = 1$ but $\|\psi_n\|_{m,\alpha} \to 0$. Let $\varphi_n = G\psi_n$. Then by (25)
\[
\|\varphi_n\|_{m+2,\alpha} \leq C_1 \left(\|\Box \varphi_n\|_{m,\alpha} + \|\varphi_n\|_{0,\alpha}\right) \leq C_1 \left(1 + \frac{1}{n}\right).
\]
Since the set $\{\varphi_n\}$ is uniformly bounded and equicontinuous, the Arzela-Ascoli theorem gives the existence of a sub-sequence converging to a $C^{m+2}$-section $\varphi$ with $\|\varphi\|_{0,\alpha} = 1$
but with $\psi_n = \Box G\psi_n$
converging to $\Box \varphi$ in the norm $\|\|_{m,\alpha}$. Since $\|\psi_n\|_{m,\alpha} \to 0$ we have $\Box \varphi = 0$ so $\varphi$ is harmonic. But $(\varphi, \varphi) = \lim (\varphi_n, \varphi) = \lim (G\psi_n, \varphi) = 0$
since $G\psi_n$ is a orthogonal to $\varphi$. So $\varphi = 0$ which gives the desired contradiction.

We also have:

**Theorem 8.12.** Let 
\[L = L_d + \ldots + L_0\]
be a pseudo-differential operator acting on sections of $E/M_0$. Suppose that the principal part $L_d$ of this operator is elliptic. Then there exists a constant $C_{m,\alpha,L}$ such that 
\[
\|\psi\|_{m,\alpha} \leq C_{m,\alpha,L} \left(\|L\psi\|_{m,\alpha} + \|\psi\|_{0,\alpha}\right).
\]

8.4. Convergence. The purpose of this section is to prove the following:

**Theorem 8.13.** Let $M_0$ be a compact complex manifold and suppose that $H^2(M; T_{M_0}) = 0$.

Then there is an analytic neighborhood $\Delta_\varepsilon$ of 0 in $H^1(M; T_{M_0})$ with local coordinates $\tau = (t_1 \ldots, t_r)$
and power series
\[
\xi = \sum_{|I| > 0} \xi_I t^I
\]
with
\[
\xi_I \in A^{0,1}_{M_0}(T_{M_0})
\]
such that:
1) the natural map
\[
T_{H^1(M; T_{M_0}), 0} \rightarrow A^{0,1}_{M_0}(T_{M_0})
\]
induces the identity on $H^1(M; T_{M_0})$,
2) $\xi$ converges uniformly on $M_0 \times \Delta_\varepsilon$ to a smooth function $M_0 \times \Delta_\varepsilon \rightarrow A^{0,1}_{M_0}(T_{M_0})$
whose restriction
\[
\{p\} \times \Delta_\varepsilon \rightarrow A^{0,1}_{M_0}(T_{M_0})_p
\]
is holomorphic for each $p \in M_0$, 

3) \[ \partial M_0 (\xi) + \frac{1}{2} [\xi, \xi] = 0. \]

Thus the functions annihilated by the operator \( (\partial M_0 + \xi) + \partial \Delta \) give \( M_0 \times \Delta \) the structure of a complex manifold, proper and smooth over \( \Delta \) thereby determining a local isomorphism \( \Delta \rightarrow \text{Def} (M_0) \).

Proof. For \( \overline{\partial} = \overline{\partial} M_0 \) in what follows, to solve the differential equation \( \overline{\partial} \psi = \phi \) for \( \psi \) when \( \phi \in A^{0,q}_{M_0} (T_{M_0}) \) is given, \( \overline{\partial} \phi = 0 \) and \( H^{0,q} (T_{M_0}) \), we will use the solution

\[
\psi = \square \circ G (\phi) = \overline{\partial} \overline{\partial}^* G (\phi) + G (\overline{\partial}^* \overline{\partial} \phi) = \overline{\partial}^* G (\phi).
\]

We begin by choosing the harmonic representatives \( \xi_i \) for a basis for \( H^1 (T_{M_0}) \) and writing

\[
\Xi_1 = \sum_{i=1}^r t_i \xi_i.
\]

We then form the Maurer-Cartan expression

\[
\overline{\partial} \Xi_1 + \frac{1}{2} [\Xi_1, \Xi_1] = \sum_{|I|=2} \Phi_I \cdot t_I = \frac{1}{2} [\Xi_1, \Xi_1]
\]

where the \( \Phi_I \) are \( \overline{\partial} \)-closed elements of \( A^{0,2}_{M_0} (T_{M_0}) \). By hypothesis we can write \( \Phi_I = \overline{\partial} (\overline{\partial}^* G \Phi_I) + H \Phi_I \) so we define

\[
\Xi_2 = -\sum_{|I|=2} \left( \overline{\partial}^* G \Phi_I \right) \cdot t_I - \frac{1}{2} \overline{\partial} G [\Xi_1, \Xi_1]
\]

\[
\Xi_2 = \Xi_1 + \Xi_2.
\]

We then have that

\[
\overline{\partial} \Xi_2 + \frac{1}{2} [\Xi_2, \Xi_2] = \sum_{|I|=2} H \Phi_I \cdot t_I + \sum_{3 \leq |I| \leq 4} \Phi_I \cdot t_I = \frac{1}{2} \sum_{k+l=3} [\Xi_k, \Xi_l] + h.p.
\]

such that, when \( |I| = 3 \),

\[
\overline{\partial} \Phi_I = 0.
\]

Again we define

\[
\Xi_3 = -\sum_{|I|=3} \left( \overline{\partial}^* G \Phi_I \right) \cdot t_I - \frac{1}{2} \overline{\partial} G \sum_{k+l=3} [\Xi_k, \Xi_l]
\]

\[
\Xi_3 = \Xi_2 + \Xi_3.
\]

to achieve that

\[
\overline{\partial} \Xi_3 + \frac{1}{2} [\Xi_3, \Xi_3] = \sum_{2 \leq |I| \leq 3} H \Phi_I \cdot t_I + \sum_{4 \leq |I| \leq 7} \Phi_I \cdot t_I.
\]
Inductively we will define
\[ \Xi_s = -\bar{\partial}^* G \sum_{k+l=s} \frac{1}{2} [\Xi_k, \Xi_l] \]
to achieve a formal sum
\[ \Xi = \sum_{s=1}^{\infty} \Xi_s \]
for which formally
\[ \bar{\partial} \Xi + \frac{1}{2} [\Xi, \Xi] = -\frac{1}{2} H ([\Xi, \Xi]) \]
Notice that \(\Xi\) satisfies the relation
\[ \Xi = \Xi_1 - \frac{1}{2} \bar{\partial}^* G [\Xi, \Xi] \]
which we can work into a differential equation for \(\Xi\) itself as follows. Since \(\Xi_1\) is harmonic, \(\bar{\partial} \Xi_1 = 0\) so \(\bar{\partial} \Xi = 0\). Then
\[ \Box \Xi = \bar{\partial}^* \bar{\partial} \Xi = -\frac{1}{2} \bar{\partial}^* \bar{\partial} \bar{\partial}^* G [\Xi, \Xi] = -\frac{1}{2} \bar{\partial}^* \Box G [\Xi, \Xi] \]
but, since \(\bar{\partial}^* H = 0\), this reduces to
\[ (32) \quad \Box \Xi = -\frac{1}{2} \bar{\partial}^* [\Xi, \Xi]. \]
The rest of the proof will be devoted to the convergence of this sequence on a small neighborhood of 0. To bound
\[ \| \Xi_s \|_{m, \alpha} = \left\| \bar{\partial}^* G \sum_{k+l=s} \frac{1}{2} [\Xi_k, \Xi_l] \right\|_{m, \alpha} \]
we can use (26) and (27) to conclude
\[ \| \Xi_s \|_{m, \alpha} \leq \frac{C_2 \cdot C_3}{2} \sum_{k+l=s} \| [\Xi_k, \Xi_l] \|_{m-1, \alpha} \]
and, using (28) on the right-hand side of this last inequality and interchanging sum and norms we obtain
\[ (33) \quad \| \Xi_s \|_{m, \alpha} \leq C \sum_{k+l=s} \| [\Xi_k] \|_{m, \alpha} \cdot \| \Xi_l \|_{m, \alpha} \]
where
\[ C = \frac{C_2 \cdot C_3 \cdot C_4}{2}. \]
For convergence of \(\Xi_{[s]}\) as \(s \to \infty\) we will find a series
\[ A = \sum_{s \geq 0} A_s \]
where each \(A_s\) is a positive homogeneous real form of degree \(s\) in the variables \((|t_1|, \ldots, |t_r|)\) such that
\[ \| \Xi_s \|_{m, \alpha} \leq_{\text{strong}} A_s, \]
meaning that, for every term
\[ \Xi_l t^l \]
in \(\Xi_s\) we have
\[ \| \Xi_l \|_{m, \alpha} |t^l| \leq A_l |t^l|. \]
Using (33) and induction on $s$, in order to achieve this last inequality it will suffice to have
\[ C \sum_{k+l=s} A_k \cdot A_l \leq_{\text{strong}} A_s. \]

To find such an $A$ we form
\[ A = \frac{u}{v} \sum_{s>0} \frac{v^s}{s^2} (|t_1| + \ldots + |t_r|)^s \]
where $u$ and $v$ are two positive constants to be chosen later. Indeed we claim:
(34) \[ A^2 \leq_{\text{strong}} \frac{4\pi^2}{3} \frac{u}{v} \cdot A \]

To establish (34), write
\[ A = \frac{u}{v} B (v (|t_1| + \ldots + |t_r|)) \]
where
\[ B (x) = \sum_{s>0} \frac{x^s}{s^2}. \]

Then
\[ B^2 (x) = \sum_{s,s'>0} \frac{x^{s+s'}}{s^2 (s')^2} \]
\[ = \sum_{s \geq 2} x^s \sum_{s',s''>0, s'+s''=s} \frac{1}{(s's'')^2}. \]

But
\[ \sum_{s',s''>0, s'+s''=s} \frac{1}{(s's'')^2} \leq \frac{2}{(s/2)^2} \sum_{s''=1} \frac{1}{(s'')^2} = \frac{8}{\pi^2} \cdot \frac{\pi^2}{6} \]
so that
\[ B^2 (x) \leq \frac{4\pi^2}{3} B (x). \]
This completes the proof of (34).

Notice that the radius of convergence of the series $B (v \cdot x)$ is $\frac{1}{v}$. Next we need to figure out the values of $u$ and $v$ in the formula for $A$ so that $\frac{u}{v}$ is big enough that $A$ strongly bounds $\Xi$. Begin by choosing
\[ u = \max \{ \| \beta_j \| \}_{m,a} \]
Then
\[ \| \Xi_1 \| \leq_{\text{strong}} A_1. \]

We have that
\[ A^2 \leq_{\text{strong}} \frac{4\pi^2}{3} \frac{u}{v} \cdot A \]
and need to achieve that
\[ C \cdot A^2 \leq_{\text{strong}} A \]
so need $v$ such that
\[ \frac{4\pi^2}{3} \frac{u}{v} \leq \frac{1}{C}, \]
that is
\[ v \geq \frac{4\pi^2}{3} \cdot C \cdot u. \]
Thus \( \Xi \) converges uniformly and absolutely to a \( C^m \)-section inside the set
\[
\sum_{j=1}^{r} |t_j| < \frac{1}{v} =: \varepsilon_m.
\]

The final difficulty is that the value of \( C \) in this last calculation depends, \( a \ priore, \) on \( m \) so that we still need to show that \( \Xi \) is in fact \( C^\infty \). To see this, split (32) locally putting all the second-order terms \( S \) of \( \partial^*[\Xi, \Xi] \) to the left-hand side to get
\[
(\Box - S) \Xi = F \Xi
\]
where \( F \) is a first-order differential operator and \( S \) is a second-order operator which vanishes at \( t = 0 \). So there will be some small \( \varepsilon > 0 \) such that \( \Xi \in C^m \) for \( m \geq 2 \), then \( \Xi \in C^{m+1} \). For this we use the estimate
\[
\|\Xi\|_{m+1, \alpha} \leq C' \left( \|L\Xi\|_{m-1, \alpha} + \|\Xi\|_{0, \alpha} \right)
\]
given in Theorem 8.12 for elliptic operators \( L \) to show boundedness of the \( (m+1) \)-norm. A standard Sobolev-type argument using approximation by smooth functions then shows that \( \Xi \in C^{m+1} \).

When
\[
H^2 (M; T_{M_0}) \neq 0
\]
one can argue along the same lines to prove the following.

**Theorem 8.14.** Let \( M_0 \) be a compact complex manifold. Then there is an analytic neighborhood \( \Delta_\varepsilon \) of 0 in \( H^1 (M; T_{M_0}) \) with local coordinates
\[
t = (t_1, \ldots, t_r)
\]
and power series
\[
\xi = \sum_{|I| > 0} \xi_I t^I
\]
with
\[
\xi_I \in A^{0,1}_{M_0} (T_{M_0})
\]
such that:

1) the natural map
\[
T_{H^1 (M; T_{M_0})} \rightarrow A^{0,1}_{M_0} (T_{M_0})
\]
induces the identity on \( H^1 (M; T_{M_0}) \),

2) \( \xi \) converges uniformly on \( M_0 \times \Delta_\varepsilon \) to a smooth function
\[
M_0 \times \Delta_\varepsilon \rightarrow A^{0,1}_{M_0} (T_{M_0})
\]
whose restriction
\[
\{p\} \times \Delta_\varepsilon \rightarrow A^{0,1}_{M_0} (T_{M_0}) |_p
\]
is holomorphic for each \( p \in M_0 \),

3) the equation
\[
\bar{\partial}_{M_0} (\xi) + \frac{1}{2} [\xi, \xi] = 0
\]
defines a closed analytic scheme \( Z \subseteq \Delta_\varepsilon \).
4) the functions annihilated by the operator
\[(\overline{\partial}_{M_0} + \xi) + \overline{\partial}_{\Delta}
\]
give \(M_0 \times Z\) the structure of a complex analytic scheme, proper and smooth over \(Z\) thereby determining a local isomorphism
\[Z(I) \rightarrow \text{Def}(M_0).
\]
The proof proceeds just as in the proof of the convergence theorem in the unobstructed case. The only new ingredient is the emergence of the term
\[H\left(\sum_{k+l=s} \frac{1}{2} [\Xi_k, \Xi_l]\right)
\]
in the equation
\[\overline{\partial}\Xi_s = -H\left(\sum_{k+l=s} \frac{1}{2} [\Xi_k, \Xi_l]\right) - \overline{\partial}^* G \sum_{k+l=s} \frac{1}{2} [\Xi_k, \Xi_l]
\]
at the moment when the deformation of \(M_0\) is obstructed. So, to achieve convergence of \(\Xi\) the effect of the harmonic summand will have to be bounded with the remaining terms from that point in the induction forward. That is achieved using the estimate (29).

9. Kähler manifolds

If \(X_0\) is a compact Kähler manifold, it turns out that the obstructions
\[\text{Obst}(X_0) \subseteq H^2(T_{X_0})
\]
to deforming \(X_0\) can be bounded, or, in some cases, shown to be 0 by the following theorem.

**Theorem 9.1.** [C1] Let \(X_0\) be a compact Kähler manifold. Then \(\text{Obst}(X_0)\) lies in the intersection of the kernels of all the Yoneda product mappings
\[H^2(T_{X_0}) \rightarrow \text{Hom}\left(H^q(\Omega^p_{X_0}), H^{q+2}(\Omega^{p-1}_{X_0})\right).
\]

This theorem gives a very quick proof of the fact that, if the canonical bundle of \(X_0\) is trivial, then \(X_0\) is unobstructed.

10. Deformation theory of a pair

Let \(X_0\) be a compact, complex manifold and \(Z_0 \subseteq X_0\) a compact complex submanifold. We would like to study the space
\[\text{Def}(x_0, z_0)
\]
which locally parametrizes a maximal flat family
\[Z \hookrightarrow X \xrightarrow{\text{Def}(x_0, z_0)}
\]
of deformations of \((X_0, Z_0)\). The DGLA that controls this problem is the one associated to the morphism
\[(35)\]
\[T_{X_0} \rightarrow N_{Z_0\setminus X_0},
\]
Namely, let
\[T_{Z_0\setminus X_0}
\]
denote the kernel of (35).
Exercise 10.1.
\[
(A^{0,*} (T_{\partial_0}|_{X_0}), \overline{\partial}, \lbrack \cdot , \cdot \rbrack)
\]
is a DGLA whose cohomology is the hypercohomology of the complex of (35).

Exercise 10.2. For any family
\[
\begin{array}{ccc}
Z & \hookrightarrow & X \\
\downarrow & & \downarrow \\
\Delta & & \\
\end{array}
\]
a transversely holomorphic trivialization of \(X/\Delta\) is compatible with \(Z/\Delta\) if and only if the associated \(\xi\) takes values in \(A^{0,1} (T_{\partial_0}|_{X_0})\).

11. 2-CONNECTED THREEFOLDS WITH TRIVIAL CANONICAL BUNDLE

We end these notes with an interesting example, namely a class of compact complex threefolds that have to exist because of deformation theory and somehow cannot be studied except through their degenerations to known objects. Namely let \(X \subseteq \mathbb{CP}^4\) be a generic quintic threefold. \(X\) has 2875 lines on it, and these are are mutually disjoint. Take any two of these lines, call them \(L\) and \(L'\).

Exercise 11.1. There are analytic neighborhoods \(U\) and \(U'\) of \(L\) and \(L'\) respectively such that each is isomorphic to a neighborhood of the 0-section of the rank-2 vector bundle
\[
\mathcal{O}_{\mathbb{P}^1} (-1) \oplus \mathcal{O}_{\mathbb{P}^1} (-1).
\]

Exercise 11.2. There are mappings of \(U\) and \(U'\) respectively to a neighborhood \(N\) of 0 in
\[
f (z) = \sum_{j=1}^{4} z_j^2 = 0
\]
such that the restrictions to \(U - L\) and \(U' - L'\) are isomorphisms onto \(N - \{0\}\).

Let \(Y_0\) denote the analytic variety obtained by glueing a two copies of \(N\) only \(X - (L \cup L')\) via the two isomorphisms in the previous exercise. We wish to study flat deformations of \(Y_0\). Local deformations of \(N\) are given by deformations
\[
f + u
\]
of \(f\), but two are equivalent if there is an analytic diffeomorphism in \(\mathbb{C}^4\)
\[
w (z) = (w_j (z))
\]
such that
\[
f (w (z)) = f (z) + u (z).
\]

Exercise 11.3. Show that the set of equivalence classes has tangent space
\[
\mathcal{O}_{\mathbb{C}^4} \left\{ \frac{\partial f}{\partial z_j} \right\}_{j=1, \ldots, 4} = \mathfrak{g} \mathfrak{t} \mathfrak{l}_N (\Omega_N^1, \mathcal{O}_N).
\]

This last exercise leads, as in the first part of these notes, to the fact that the deformation theory of \(Y_0\) is controlled by \(\text{Ext}^*_Y (\Omega_{Y_0}^1, \mathcal{O}_{Y_0})\), that is, the tangent space to the deformation space is
\[
\text{Ext}^1_{Y_0} (\Omega_{Y_0}^1, \mathcal{O}_{Y_0})
\]
and the obstruction space lies in
\[
\text{Ext}^2_{Y_0} (\Omega_{Y_0}^1, \mathcal{O}_{Y_0}).
\]
**Exercise 11.4.** Use the local-to-global spectral sequence to show that there must be at least a formal deformation of $Y_0$ which smooths its two nodes. (In fact, convergence can be proved so that there is actually a smooth geometric deformation.)

A smooth deformation $Y$ of $Y_0$ is simply connected and so $\pi_2(Y) = H_2(Y;\mathbb{Z}) = 0$. Also, by semi-continuity $H^1(Y;\mathcal{O}_Y) = 0$ for $Y$ near $Y_0$ since $H^1(Y_0;\mathcal{O}_{Y_0}) = 0$ so $Y$ admits no non-trivial line bundles so $\Omega^3_Y = \mathcal{O}_Y$.

**Exercise 11.5.** Show that, for all the 2873 other lines in $Y_0$, these lines continue to live in $Y$ for $Y$ near $Y_0$. (In fact, it can be shown that $Y$ contains an infinite number of rational curves.)

**References**


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